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Geometry of flocks and n-ary groups

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Abstract. Semiabelinan flocks and n-ary groups are characterized by the properties of parallelograms and vectors of the affine geometry defined by these flocks and n-ary groups.

1. Introduction

If in the standard (affine) geometry is fixed point O, then any point P of this geometry is uniquely determined by the vector $\vec{p} = \overrightarrow{OP}$, and conversely, any vector $\overrightarrow{OP}$ uniquely determines the point P. Moreover, any interval $\overline{AB}$ is interpreted as the vector $\vec{a} - \vec{b}$ or as the vector $\vec{b} - \vec{a}$. In the first case,

$$\overline{AB} = \overline{CD} \quad \Longleftrightarrow \quad \vec{a} - \vec{b} + \vec{d} = \vec{c},$$

or, in the other words

$$\overline{AB} = \overline{CD} \quad \Longleftrightarrow \quad f(a, b, d) = c,$$

where each vector $\vec{v}$ is treated as an element v of a commutative group $(G, +)$. The operation f has the form $f(x, y, z) = x - y + z$. Groups (also non-commutative) with a ternary operation defined in such a way were considered by J. Certaine [3] as a special case of *ternary heaps* investigated by H. Prüfer [18]. Ternary heaps have interesting applications to projective geometry [1], affine geometry [2], theory of nets (webs), theory of knots and even to the differential geometry [24], [25].

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All affine geometries may be treated as geometries defined by some n-ary relations (see, for example, [23]). The class of affine geometries defined by n-ary groups, which are a natural generalization of the notion of groups, was introduced by S.A. Rusakov (see [20], [21]) and in detail described by Yu.I. Kulazhenko.

Below, using methods proposed by W.A. Dudek in his fundamental paper [7], we give very short and elegant proofs of various Kulazhenko's results.

2. Preliminaries

We will use the standard notation: the sequence $x_i, \ldots, x_j$ will be denoted as x_i^j (for $j < i$ it is the empty symbol). In the case $x_{i+1} = \ldots = x_{i+k} = x$ instead of x_{i+1}^{i+k} we will write $\overset{(k)}{x}$. Obviously $\overset{(0)}{x}$ is the empty symbol. In this notation the formula

$$f(x_1, \ldots, x_i, y_{i+1}, \ldots, y_{i+k}, x_{i+k+1}, \ldots, x_n),$$

where $y_{i+1} = \ldots = y_{i+k} = y$, will be written as $f(x_1^i, \overset{(k)}{y}, x_{i+k+1}^n)$.

By an *n-ary group* (G, f) we mean (see [4]) a non-empty set G together with one n-ary operation $f : G^n \to G$ satisfying for all $i = 1, 2, \ldots, n$ the following two conditions:

1^0 *the associative law*:

$$f(f(x_1^n), x_{n+1}^{2n-1}) = f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1})$$

2^0 for all $x_1, x_2, \ldots, x_n, b \in G$ there exits a unique $x_i \in G$ such that

$$f(x_1^{i-1}, x_i, x_{i+1}^n) = b.$$

Such n-ary group may be considered also as an algebra (G, f, g) with one associative n-ary operation f and one unary operation g satisfying some identities (see, for example, [5], [6], [8] or [9]). In particular, an n-ary group may be treated as an algebra $(G, f, {}^{[-2]})$ with one associative n-ary operation f and one unary operation ${}^{[-2]} : x \mapsto x^{[-2]}$ such that

$$f(x^{[-2]}, \overset{(n-2)}{x}, f(\overset{(n-1)}{x}, y)) = f(f(y, \overset{(n-1)}{x}), \overset{(n-2)}{x}, x^{[-2]}) = y, \qquad (1)$$

is true for all $x, y \in G$ (see [19]).

Applying associativity to (1) we obtain

$$f(f(x^{[-2]}, \overset{(n-1)}{x}), \overset{(n-2)}{x}, y) = f(y, \overset{(n-2)}{x}, f(\overset{(n-1)}{x}, x^{[-2]})) = y, \qquad (2)$$

which together with results proved in [9] and [5] shows that

$$f(x^{[-2]}, \overset{(n-1)}{x}) = f(\overset{(n-1)}{x}, x^{[-2]}) = \overline{x}, \tag{3}$$

where $\overline{x}$ denotes the *skew element to* x (see [4], [5] or [9]). In general $\overline{x} \neq x$, but the situation when $\overline{x} = x$ or $\overline{x} = \overline{y}$ for $x \neq y$ also is possible (see [8]). Moreover, in any n-ary group (G, f) with $n \geqslant 3$ we have

$$f(x, \overset{(i-3)}{y}, \overline{y}, \overset{(n-i)}{y}, z) = f(x, \overset{(j-3)}{y}, \overline{y}, \overset{(n-j)}{y}, z) \tag{4}$$

for all $x, y, z \in G$ and $3 \leqslant i, j \leqslant n$.

An n-ary operation f defined on G is *semiabelian* if

$$f(x_1, x_2^{n-1}, x_n) = f(x_n, x_2^{n-1}, x_1)$$

for all $x_1, \ldots, x_n \in G$.

One can prove (for details see [5]) that for $n \geqslant 3$ an n-ary group (G, f) is semiabelian if and only if there exists $a \in G$ such that for all $x, y \in G$ holds $f(z, \overset{(n-2)}{a}, y) = f(z, \overset{(n-2)}{a}, y)$, or equivalently,

$$f(z, \overline{a}, \overset{(n-3)}{a}, y) = f(z, \overline{a}, \overset{(n-3)}{a}, y). \tag{5}$$

A nonempty set G with one ternary operation $[\cdot, \cdot, \cdot]$ satisfying the *para-associative law*

$$[[x, y, z], u, w] = [x, [u, z, y], w] = [x, y, [z, u, w]]$$

and such that for all $a, b, c \in G$ there are uniquely determined $x, y, z \in G$ such that

$$[x, a, b] = [a, y, b] = [a, b, z] = c \tag{6}$$

is called a *flock* (see [7] or [10]). Obviously, a semiabelian flock is a semiabelian ternary group. So, a flock $(G, [\cdot, \cdot, \cdot])$ is semiabelian if and only if there exists $a \in G$ such that $[x, a, z] = [z, a, x]$ for all $x, z \in G$.

Properties of flocks are similar to properties of ternary groups.

Further, we will use the following lemmas proved in [7].

Lemma 2.1. *In any flock* $(G, [\cdot, \cdot, \cdot])$ *for each* $x \in G$ *there exists* $\overline{x}$ *such that*

$$[x, \overline{x}, y] = [\overline{x}, x, y] = [y, x, \overline{x}] = [y, \overline{x}, x] = y$$

for all $y \in G$.

Lemma 2.2. *In any flock* $\overline{\overline{x}} = x$ *and* $\overline{[x, y, z]} = [\overline{x}, \overline{y}, \overline{z}]$.

By the Post's Coset Theorem (see [17]), for any n-ary group (G,f) there exists a binary group $(G^{\#},\cdot)$ such that $G\subset G^{\#}$ and $f(x_1^n)=x_1\cdot x_2\cdot\ldots\cdot x_n$ for $x_1,\ldots,x_n\in G$. Since in this group $\overline{x}=x^{2-n}$ for all $x\in G$, then

$$[x,y,z]=f(x,\overline{y},\overset{(n-3)}{y},z) \tag{7}$$

is an idempotent para-associative ternary operation. So, if (G,f) is an n-ary group then $(G,[\cdot,\cdot,\cdot])$ with an operation defined by (7) is an idempotent ternary flock. We will say that this flock is *induced* by an n-group (G,f). From Post's Theorem it follows that the operation of this flock can be presented in the form $[x,y,z]=x\cdot y^{-1}\cdot z$, where $(G^{\#},\cdot)$ is the covering group of the corresponding an n-ary group (G,f).

The following obvious lemma plays an important role in the proofs of our results presented in this paper.

Lemma 2.3. *An n-ary group (G,f) is semiabelian if and only if the flock $(G,[\cdot,\cdot,\cdot])$ defined by* (7) *is semiabelian.*

Further, for simplicity, instead of $[\ldots[[x_1,x_2,x_3],x_4,x_5],\ldots,x_{2k},x_{2k+1}]$ we will write $[x_1,x_2,\ldots,x_{2k+1}]$. Since the operation $[\cdot,\cdot,\cdot]$ is para-associative we also have

$$[x_1,x_2,\ldots,x_{2k+1}]=[x_1,x_2,[x_3x_4,[\ldots[x_{2k-1},x_{2k},x_{2k+1}]\ldots]]].$$

3. Parallelograms

Generalizing the idea presented by W. Szmielew (see [23]) S.A. Rusakov considered in [22] the affine geometry as the geometry induced by n-ary groups. In his generalization elements of an n-ary group (G,f) are *points*. The ordered pair of two points $a,b\in G$ is called an *interval* and is denoted by $\langle a,b\rangle$. The set of four points $a,b,c,d\in G$ such that $\langle a,b\rangle,\langle b,c\rangle,\langle c,d\rangle$ and $\langle d,a\rangle$ are intervals is called a *quadrangle*. Intervals $\langle a,b\rangle,\langle b,c\rangle,\langle c,d\rangle$ and $\langle d,a\rangle$ are *sides* of this quadrangle. Intervals $\langle a,c\rangle$ and $\langle b,d\rangle$ are its *diagonals*.

It is easy to see that the relation $\equiv$ defined on the set of all intervals by

$$\langle a,b\rangle\equiv\langle c,d\rangle \iff f(a,b^{[-2]},\overset{(2n-4)}{b},d)=c \tag{8}$$

is an equivalence. In view of (3) this relation can be rewritten in the form

$$\langle a,b\rangle\equiv\langle c,d\rangle \iff f(a,\overline{b},\overset{(n-3)}{b},d)=c.$$

The equivalence class of $\langle a,b\rangle$ is interpreted as a *vector* $\overrightarrow{ab}$. Such defined vectors form some vector space (see [22]), where the addition of vectors can be defined (see [11]) by

$$\overrightarrow{ab}+\overrightarrow{cd}=\overrightarrow{ag}=\overrightarrow{hd},$$

where $g=f(b,c^{[-2]},\overset{(2n-4)}{c},d)$ and $h=f(c,b^{[-2]},\overset{(2n-4)}{b},a)$, or equivalently (see [13]) $g=f(b,\overline{c},\overset{(n-3)}{c},d)$ and $h=f(c,\overline{b},\overset{(n-3)}{b},a)$.

According to [22] four points $a,b,c,d\in G$ form a *parallelogram* if

$$f(a,b^{[-2]},\overset{(2n-4)}{b},c)=d. \tag{9}$$

As a simple consequence of (4) and (7) we obtain the following two lemmas.

Lemma 3.1. *For an n-ary group (G,f), where $n\geqslant 3$, the following conditions are equivalent:*

(i) *elements $a,b,c,d\in G$ form a parallelogram,*

(ii) $f(a,\overline{b},\overset{(n-3)}{b},c)=d$,

(iii) $[a,b,c]=d$.

Lemma 3.2. *For an n-ary group (G,f), where $n\geqslant 3$, the following conditions are equivalent:*

(i) *intervals $\langle a,b\rangle$ and $\langle c,d\rangle$ are equivalent,*

(ii) $f(a,\overline{b},\overset{(n-3)}{b},d)=c$,

(iii) $[a,b,d]=c$.

Now let $(G,[\cdot,\cdot,\cdot])$ be an arbitrary flock and $a,b,c,d\in G$. Then the relation

$$\langle a,b\rangle\equiv\langle c,d\rangle \quad\Longleftrightarrow\quad [a,\overline{b},d]=c$$

is an equivalence. Similarly as in the case of n-ary groups, the equivalence class of $\langle a,b\rangle$ can be interpreted as a *vector* $\overrightarrow{ab}$. Consequently,

$$\overrightarrow{ab}=\overrightarrow{cd} \quad\Longleftrightarrow\quad [a,\overline{b},d]=c. \tag{10}$$

The addition of such vectors is defined by

$$\overrightarrow{ab}+\overrightarrow{cd}=\overrightarrow{a[b,\overline{c},d]}=\overrightarrow{[c,\overline{b},a]d}. \tag{11}$$

Thus points $a,b,c,d\in G$ form a *parallelogram* if $[a,\overline{b},c]=d$.

Therefore for $n > 2$ the affine geometry introduced by Rusakov and investigated by Kulazhenko is a special case of the affine geometry induced by flocks (see [7]). Namely, for $n > 2$, the affine geometry induced by an n-ary group (G, f) coincides with the affine geometry induced by an idempotent flock defined by (7). So, all Kulazhenko's results on parallelograms proved in [11] and [13] are a simple consequence of Dudek's results from [7].

4. Vectors of semiabelian flocks

In this section we characterize semialelian flocks by the properties of vectors of the corresponding geometry.

Lemma 4.1. *A flock* $(G, [\cdot, \cdot, \cdot])$ *is semiabelian if and only if*

$$z = [x, \overline{u}, z, \overline{u}, y, \overline{x}, u, \overline{y}, u]$$

is true for all $x, y, z, u \in G$.

Proof. A semiabelian flock is a ternary group, hence the operation $[\cdot, \cdot, \cdot]$ is associative. Thus, by Lemma 2.1

$$\begin{aligned}[x, \overline{u}, z, \overline{u}, y, \overline{x}, u, \overline{y}, u] &= [x, \overline{u}, z, \overline{u}, u, \overline{x}, y, \overline{y}, u] = [x, \overline{u}, z, \overline{x}, u] \\ &= [x, \overline{x}, z, \overline{u}, u] = z.\end{aligned}$$

Conversely, if $z = [x, \overline{u}, z, \overline{u}, y, \overline{x}, u, \overline{y}, u]$ for all $x, y, z, u \in G$, then multiplying this equation on the right by $\overline{u}, y, \overline{u}, x$ we obtain

$$[z, \overline{u}, y, \overline{u}, x] = [x, \overline{u}, z, \overline{u}, y],$$

which for $z = u$ gives $[y, \overline{u}, x] = [x, \overline{u}, y]$. This, by Lemma 2.2, means that $[y, v, x] = [x, v, y]$ for all $x, y, v \in G$. So, $(G, [\cdot, \cdot, \cdot])$ is a semiabelian flock. □

Lemma 4.2. *A flock* $(G, [\cdot, \cdot, \cdot])$ *is semiabelian if and only if*

$$z = [x, \overline{y}, u, \overline{x}, z, \overline{v}, y, \overline{u}, v]$$

holds for all $x, y, z, u, v \in G$.

Proof. The proof of the necessity is the same as in the previous lemma. To prove the sufficiency it is sufficient to multiple this equation on the right by $\overline{v}, u, \overline{y}, v$. Then, after reduction, putting $x = z = v$, we can see that $(G, [\cdot, \cdot, \cdot])$ is semiabelian. □

In the same way we can prove

Lemma 4.3. *A flock* $(G, [\cdot, \cdot, \cdot])$ *is semiabelian if and only if*

$$u = [x, \overline{y}, z, \overline{x}, y, \overline{x}, u, \overline{z}, x],$$

or equivalently

$$u = [x, \overline{y}, z, \overline{x}, u, \overline{z}, y]$$

for all $x, y, z, u \in G$.

In the case when a flock is defined by (7), as a simple consequence of the above lemmas we obtain the Kulazhenko's result proved in [16].

Theorem 4.4. (Kulazhenko) *For* $n > 2$ *an* n*-ary group* (G, f) *is semiabelian if and only if one of the following equivalent identities is satisfied*

(a) $z = f(x, u^{[-2]}, \overset{(2n-4)}{u}, z, u^{[-2]}, \overset{(2n-4)}{u}, y, x^{[-2]}, \overset{(2n-4)}{x}\overline{x}, u, y^{[-2]}, \overset{(2n-4)}{y}, u)$,

(b) $z = f(x, y^{[-2]}, \overset{(2n-4)}{y}, u, x^{[-2]}, \overset{(2n-4)}{x}, z, v^{[-2]}, \overset{(2n-4)}{v}\overline{v}, y, u^{[-2]}, \overset{(2n-4)}{u}, v)$.

Theorem 4.5. *A flock* $(G, [\cdot, \cdot, \cdot])$ *is semiabelian if and only if for any pairs* (x_i, y_i) *of elements of* G*, where* $1 \leqslant i \leqslant t$ *and* $t > 2$ *is an odd natural number, the identity*

$$\begin{aligned} &[x_1, \overline{x}_2, x_3, \overline{x}_4, \ldots, x_{t-2}, \overline{x}_{t-1}, x_t, \overline{x}_1, y_1, \overline{y}_2, x_2, \overline{x}_3, y_3, \\ &\quad \ldots, \overline{y}_{t-1}, x_{t-1}, \overline{x}_t, y_{t-1}, \overline{y}_{t-2}, \ldots, y_6, \overline{y}_5, y_4, \overline{y}_3, y_2] = y_1, \end{aligned} \tag{12}$$

is valid.

Proof. Applying Lemma 2.1 we can see that (12) holds in any semiabelian flock.

Conversely, if (12) holds in the flock $(G, [\cdot, \cdot, \cdot])$, then putting $y_1 = [x, y, z]$, $x_t = z$ and x for other x_i and y_j we obtain

$$\begin{aligned} &[x, \overline{x}, x, \overline{x}, \ldots, x, \overline{x}, z, \overline{x}, [x, y, z], \overline{x}, x, \overline{x}, x, \overline{x}, x, \overline{z}, x, \overline{x}, \ldots, x, \overline{x}, x, \overline{x}, x] \\ &\quad = [x, y, z]. \end{aligned}$$

The left side of this equation, after application of Lemma 2.1, can be reducedto the form $[z, \overline{x}, [x, y, z], \overline{z}, x]$. Later, applying the para-associativity of the operation $[\cdot, \cdot, \cdot]$ and Lemma 2.1, we obtain

$$\begin{aligned} [z, \overline{x}, [x, y, z], \overline{z}, x] &= [[z, \overline{x}, [x, y, z]], \overline{z}, x] = [[[z, \overline{x}, x,]y, z], \overline{z}, x] \\ &= [[z, y, z], \overline{z}, x] = [z, y, [z, \overline{z}, x]] = [z, y, x]. \end{aligned}$$

This proves that a flock $(G, [\cdot, \cdot, \cdot])$ is semiabelian. □

Theorem 4.6. *A flock $(G,[\cdot,\cdot,\cdot])$ is semiabelian if and only if*

$$\overrightarrow{[x,\overline{y},z][u,\overline{v},w]} = \overrightarrow{xu} - \overrightarrow{yv} + \overrightarrow{zw} \tag{13}$$

holds for all $x,y,z,u,v,w \in G$.

Proof. Let $(G,[\cdot,\cdot,\cdot])$ be a semiabelian flock. Then it is a ternary group and

$$\overrightarrow{xu} - \overrightarrow{yv} + \overrightarrow{zw} = \overrightarrow{xu} + \overrightarrow{vy} + \overrightarrow{zw} = \overrightarrow{x[u,\overline{v},y]} + \overrightarrow{zw} = \overrightarrow{x[u,\overline{v},y,\overline{z},w]}.$$

Consider the quadrangle $\langle [x,\overline{y},z], x, [u,\overline{v},y,\overline{z},w], [u,\overline{v},w]\rangle$. Since, as it is not difficult to verify, $[[x,\overline{y},z],\overline{x},[u,\overline{v},y,\overline{z},w]] = [u,\overline{v},w]$, this quadrangle is a parallelogram (Lemma 3.1). Thus $\overrightarrow{[x,\overline{y},z][u,\overline{v},w]} = \overrightarrow{x[u,\overline{v},y,\overline{z},w]}$. This proves (13).

Conversely, if (13) holds for all $x,y,z,u,v,w \in G$, then

$$\overrightarrow{[x,\overline{y},z][u,\overline{v},w]} = \overrightarrow{xu} - \overrightarrow{yv} + \overrightarrow{zw} = \overrightarrow{x[u,\overline{v},y,\overline{z},w]},$$

i.e., the quadrangle $\langle [x,\overline{y},z], x, [u,\overline{v},y,\overline{z},w], [u,\overline{v},w]\rangle$ is a parallelogram. Thus $[[x,\overline{y},z],\overline{x},[u,\overline{v},y,\overline{z},w]] = [u,\overline{v},w]$. Multiplying this identity on the right by $\overline{w},u$ we obtain the identity from Lemma 4.2. Hence this flock is semiabelian. □

Theorem 4.7. *A flock $(G,[\cdot,\cdot,\cdot])$ is semiabelian if and only if for any pairs (x_i,y_i) of elements of G, where $1 \leqslant i \leqslant t$ and $t > 2$ is an odd natural number, the identity*

$$\begin{aligned}\overrightarrow{[x_1,\overline{x}_2,x_3,\overline{x}_4,\ldots,\overline{x}_{t-1},x_t][y_1,\overline{y}_2,y_3,\overline{y}_4,\ldots,\overline{y}_{t-1},y_t]}\\ = \overrightarrow{x_1y_1} - \overrightarrow{x_2y_2} + \overrightarrow{x_3y_3} - \ldots - \overrightarrow{x_{t-1}y_{t-1}} + \overrightarrow{x_ty_t}\end{aligned} \tag{14}$$

is satisfied.

Proof. By Theorem 4.6 in a flock $(G,[\cdot,\cdot,\cdot])$ we have

$$\overrightarrow{x_1y_1} - \overrightarrow{x_2y_2} + \overrightarrow{x_3y_3} = \overrightarrow{[x_1,\overline{x}_2,x_3][y_1,\overline{y}_2,y_3]}.$$

Thus

$$\begin{aligned}&\overrightarrow{x_1y_1} - \overrightarrow{x_2y_2} + \overrightarrow{x_3y_3} - \overrightarrow{x_4y_4} + \overrightarrow{x_5y_4}\\ &\quad= \overrightarrow{[x_1,\overline{x}_2,x_3][y_1,\overline{y}_2,y_3]} - \overrightarrow{x_4y_4} + \overrightarrow{x_5y_5}\\ &\quad= \overrightarrow{[x_1,\overline{x}_2,x_3,\overline{x}_4,x_5][y_1,\overline{y}_2,y_3,\overline{y}_4,y_5]},\end{aligned}$$

and so on. This proves (14).

Conversely, if (14) holds in a flock $(G, [\cdot, \cdot, \cdot])$, then

$$\begin{aligned}
&\overrightarrow{[x_1, \overline{x}_2, x_3, \overline{x}_4, \dots, x_t][y_1, \overline{y}_2, y_3, \overline{y}_4, \dots, y_t]} \\
&\quad = \overrightarrow{x_1 y_1} - \overrightarrow{x_2 y_2} + \overrightarrow{x_3 y_3} - \dots - \overrightarrow{x_{t-1} y_{t-1}} + \overrightarrow{x_t y_t} \\
&\quad = \overrightarrow{x_1 y_1} + \overrightarrow{y_2 x_2} + \overrightarrow{x_3 y_3} - \dots + \overrightarrow{y_{t-1} x_{t-1}} + \overrightarrow{x_t y_t} \\
&\quad = \overrightarrow{x_1 [y_1, \overline{y}_2, x_2]} + \overrightarrow{x_3 y_3} + \dots + \overrightarrow{y_{t-1} x_{t-1}} + \overrightarrow{x_t y_t} \\
&\quad = \overrightarrow{x_1 [y_1, \overline{y}_2, x_2, \overline{x}_3, y_3]} + \dots + \overrightarrow{y_{t-1} x_{t-1}} + \overrightarrow{x_t y_t} \\
&\quad = \overrightarrow{x_1 [y_1, \overline{y}_2, x_2, \overline{x}_3, y_3, \overline{y}_4, x_4]} + \dots + \overrightarrow{y_{t-1} x_{t-1}} + \overrightarrow{x_t y_t} \\
&\quad = \dots = \overrightarrow{x_1 [y_1, \overline{y}_2, x_2, \overline{x}_3, y_3, \overline{y}_4, x_4, \dots, \overline{x}_t, y_t]}.
\end{aligned}$$

This means that

$$\begin{aligned}
&\langle [x_1, \overline{x}_2, x_3, \dots, x_t], x_1, [y_1, \overline{y}_2, x_2, \overline{x}_3, y_3, \overline{y}_4, x_4 \dots, \overline{x}_t, y_t], \\
&\quad [y_1, \overline{y}_2, y_3. \dots, y_t] \rangle
\end{aligned}$$

is a parallelogram. So, by Lemma 3.1,

$$\begin{aligned}
&[[x_1, \overline{x}_2, x_3, \dots, x_t], \overline{x}_1, [y_1, \overline{y}_2, x_2, \overline{x}_3, y_3, \overline{y}_4, x_4, \dots, \overline{x}_t, y_t]] \\
&\quad = [y_1, \overline{y}_2, y_3, \dots, y_t].
\end{aligned}$$

Multiplying this identity by $\overline{y}_t, y_{t-1}, \overline{y}_{t-2}, y_{t-3}, \dots, \overline{y}_3, y_2$ and applying Lemma 2.1, we obtain (12). Hence, by Theorem 4.5, this flock is semiabelian. □

In the case when $x_i = x$ (resp. $y_i = y$) for all $i = 1, 2, \dots, t$ we obtain

Corollary 4.8. *A flock $(G, [\cdot, \cdot, \cdot])$ is semiabelian if and only if for all elements $x, y_1, y_2, \dots, y_t \in G$, where $t > 2$ is an odd natural number, the identity*

$$\overrightarrow{x[y_1, \overline{y}_2, y_3, \overline{y}_4, \dots, \overline{y}_{t-1}, y_t]} = \overrightarrow{x y_1} - \overrightarrow{x y_2} + \overrightarrow{x y_3} - \dots - \overrightarrow{x y_{t-1}} + \overrightarrow{x y_t}$$

is satisfied.

Corollary 4.9. *A flock $(G, [\cdot, \cdot, \cdot])$ is semiabelian if and only if for all elements $x_1, x_2, \dots, x_t, y \in G$, where $t > 2$ is an odd natural number, the identity*

$$\overrightarrow{[x_1, \overline{x}_2, x_3, \overline{x}_4, \dots, \overline{x}_{t-1}, x_t] y} = \overrightarrow{x_1 y} - \overrightarrow{x_2 y} + \overrightarrow{x_3 y} - \dots - \overrightarrow{x_{t-1} y} + \overrightarrow{x_t y}$$

is satisfied.

5. Symmetry and semiabelianism

According to Rusakov (see [20] or [22]) two elements a and c of an n-ary group (G,f) are called *symmetric* if and only if there exists a uniquely determined point $x\in G$ such that

$$f(f(a,x^{[-2]},\overset{(n-2)}{x}),\overset{(n-2)}{x},c)=x.$$

Thus, in view of the above results, for $n\geqslant 3$ this definition can be formulated in the form:

Definition 5.1. Two elements a and c of an n-ary group (G,f) are *symmetric* if and only if there exists one and only one $x\in G$ such that

$$f(a,\overline{x},\overset{(n-3)}{x},c)=x. \tag{15}$$

Thus for symmetric elements a and c there exists uniquely determined element $x\in G$ and the symmetry S_x such that $S_x(a)=c$. Since in (15) the element c is uniquely determined by a and x, then using the same method as in [5] and [9] one can prove that the symmetry S_x has the form:

$$S_x(a)=f(x,\overline{a},\overset{(n-3)}{a},x).$$

In the case of flocks (see [7]) points $a,c\in G$ are *symmetric* if and only if there exists a uniquely determined $x\in G$ such that

$$[a,\overline{x},c]=x.$$

In this case $S_x(a)=[x,\overline{a},x]$.

Theorem 5.2. *A flock $(G,[\cdot,\cdot,\cdot])$ is semiabelian if and only if*

$$\overrightarrow{ux}+\overrightarrow{S_x(u)y}+\overrightarrow{S_yS_x(u)z}+\overrightarrow{S_zS_yS_x(u)w}=\overrightarrow{0} \tag{16}$$

for any points $x,y,z,u,w\in G$ such that $\langle x,y,z,w\rangle$ is a parallelogram.

Proof. Observe first that $\overrightarrow{0}=\overrightarrow{xx}$ for any $x\in G$. By Lemma 2.2 we also have

$$\overline{S_yS_x(u)}=\overline{[y,\overline{x},u,\overline{x},y]}=[\overline{y},x,\overline{u},x,\overline{y}]$$

and

$$\overline{S_zS_yS_x(u)}=\overline{[z,\overline{y},x,\overline{u},x,\overline{y},z]}=[\overline{z},y,\overline{x},u,\overline{x},y,\overline{z}].$$

Thus

$$\begin{aligned}
&\overrightarrow{ux} + \overrightarrow{S_x(u)y} + \overrightarrow{S_yS_x(u)z} + \overrightarrow{S_zS_yS_x(u)w} \\
&\quad = \overrightarrow{u[x,\overline{S_x(u)},y]} + \overrightarrow{S_yS_x(u)z} + \overrightarrow{S_zS_yS_x(u)w} \\
&\quad = \overrightarrow{u[x,[\overline{x},u,\overline{x}],y]} + \overrightarrow{S_yS_x(u)z} + \overrightarrow{S_zS_yS_x(u)w} \\
&\quad = \overrightarrow{u[u,\overline{x},y]} + \overrightarrow{S_yS_x(u)z} + \overrightarrow{S_zS_yS_x(u)w} \\
&\quad = \overrightarrow{u[x,\overline{y},z]} + \overrightarrow{S_zS_yS_x(u)w} \\
&\quad = \overrightarrow{u[x,\overline{y},z]} + \overrightarrow{[z,\overline{y},x,\overline{u},x,\overline{y},z]w} = \overrightarrow{u[u,\overline{x},y,\overline{z},w]}.
\end{aligned}$$

So,

$$\overrightarrow{ux} + \overrightarrow{S_x(u)y} + \overrightarrow{S_yS_x(u)z} + \overrightarrow{S_zS_yS_x(u)w} = \overrightarrow{u[u,\overline{x},y,\overline{z},w]}.$$

But $w = [x,\overline{y},z]$ because the quadrangle $\langle x,y,z,w\rangle$ is a parallelogram. Hence

$$\overrightarrow{ux} + \overrightarrow{S_x(u)y} + \overrightarrow{S_yS_x(u)z} + \overrightarrow{S_zS_yS_x(u)w} = \overrightarrow{u[u,\overline{x},y,\overline{z},x,\overline{y},z]}.$$

This means that the condition (16) can be written in the form

$$\overrightarrow{u[u,\overline{x},y,\overline{z},x,\overline{y},z]} = \overrightarrow{zz},$$

which, by (10), is equivalent to

$$[x,\overline{y},z,\overline{x},y] = z,$$

and consequently, to $[x,\overline{y},z] = [z,\overline{y},x]$. This completes the proof. □

Theorem 5.3. *A flock* $(G,[\cdot,\cdot,\cdot])$ *is semiabelian if and only if*

$$\begin{aligned}
&\overrightarrow{ux} + \overrightarrow{S_x(u)y} + \overrightarrow{S_yS_x(u)z} + \overrightarrow{S_zS_yS_x(u)w} + \overrightarrow{S_wS_zS_yS_x(u)v} \\
&\quad + \overrightarrow{S_vS_wS_zS_yS_x(u)x} = \overrightarrow{0}
\end{aligned} \tag{17}$$

for any points $x,y,z,u,w \in G$ *and* $v = [w,\overline{z},y]$.

Proof. As in the previous proof,

$$\overrightarrow{ux} + \overrightarrow{S_x(u)y} + \overrightarrow{S_yS_x(u)z} + \overrightarrow{S_zS_yS_x(u)w} = \overrightarrow{u[u,\overline{x},y,\overline{z},w]}.$$

Since

$$\overrightarrow{S_wS_zS_yS_x(u)v} = \overrightarrow{[w,\overline{z},y,\overline{x},u,\overline{x},y,\overline{z},w]v},$$

we have

$$\overrightarrow{ux} + \overrightarrow{S_x(u)y} + \overrightarrow{S_yS_x(u)z} + \overrightarrow{S_zS_yS_x(u)w} + \overrightarrow{S_wS_zS_yS_x(u)v} = \overrightarrow{ux}$$

because

$$\overrightarrow{u[u,\overline{x},y,\overline{z},w]} + \overrightarrow{[w,\overline{z},y,\overline{x},u,\overline{x},y,\overline{z},w]v}$$
$$= \overrightarrow{u[[u,\overline{x},y,\overline{z},w],\overline{[w,\overline{z},y,\overline{x},u,\overline{x},y,\overline{z},w]},v]}$$

and

$$\begin{aligned}
&[[u,\overline{x},y,\overline{z},w],\overline{[w,\overline{z},y,\overline{x},u,\overline{x},y,\overline{z},w]},v]\\
&\quad= [[u,\overline{x},y,\overline{z},w],[\overline{w},z,\overline{y},x,\overline{u},x,\overline{y},z,\overline{w}],v]\\
&\quad= [[[u,\overline{x},y,\overline{z},w]\overline{w},z],[\overline{w},z,\overline{y},x,\overline{u},x,\overline{y}],v]\\
&\quad= [[u,\overline{x},y,],[\overline{w},z,\overline{y},x,\overline{u},x,\overline{y}],v]\\
&\quad= [[[u,\overline{x},y,]\overline{y},x],[\overline{w},z,\overline{y},x,\overline{u}],v] = [u,[\overline{w},z,\overline{y},x,\overline{u}],v]\\
&\quad= [[u,\overline{u},x],[\overline{w},z,\overline{y}],v] = [x,[\overline{w},z,\overline{y}],v] = x
\end{aligned}$$

for $v = [w,\overline{z},y]$ (Lemma 2.1).

Similarly,

$$\overrightarrow{S_vS_wS_zS_yS_x(u)x} = \overrightarrow{[v,\overline{w},z,\overline{y},x,\overline{u},x,\overline{y},z,\overline{w},v]x}$$
$$= \overrightarrow{[w,\overline{z},y,\overline{w},z,\overline{y},x,\overline{u},x]x},$$

because

$$\begin{aligned}
&[v,\overline{w},z,\overline{y},x,\overline{u},x,\overline{y},z,\overline{w},v] = [[w,\overline{z},y],\overline{w},z,\overline{y},x,\overline{u},x,\overline{y},z,\overline{w},[w,\overline{z},y]]\\
&\quad= [[w,\overline{z},y],\overline{w},z,\overline{y},x,\overline{u},x,\overline{y},[z,\overline{w},[w,\overline{z},y]]] = [w,\overline{z},y,\overline{w},z,\overline{y},x,\overline{u},x].
\end{aligned}$$

Consequently, $\overrightarrow{ux} + \overrightarrow{S_vS_wS_zS_yS_x(u)x} = \overrightarrow{[w,\overline{z},y,\overline{w},z,\overline{y},x]x}$, by (11).

Thus (17) has the form

$$\overrightarrow{[w,\overline{z},y,\overline{w},z,\overline{y},x]x} = \overrightarrow{xx},$$

which, by (10), is equivalent to

$$[w,\overline{z},y,\overline{w},z,\overline{y},x] = x,$$

i.e., to $[w,\overline{z},y] = [y,\overline{z},w]$, which completes the proof. □

Theorem 5.4. *A flock $(G,[\cdot,\cdot,\cdot])$ is semiabelian if and only if*

$$2\overrightarrow{xy} = \overrightarrow{zu} + \overrightarrow{S_x(z)S_y(u)}$$

for any points $x,y,z,u \in G$.

Proof. According to (11) we have

$$2\overrightarrow{xy} = \overrightarrow{xy} + \overrightarrow{xy} = \overrightarrow{x[y,\overline{x},y]}$$

and

$$\overrightarrow{zu} + \overrightarrow{S_x(z)S_y(u)} = \overrightarrow{xu} + \overrightarrow{[x,\overline{z},x][y,\overline{u},y]} = \overrightarrow{[x,\overline{z},x,\overline{u},z][y,\overline{u},y]}.$$

So, by (10), the equation mentioned in our theorem can be written as

$$[x,\overline{[\overline{y},x,\overline{y}]},[y,\overline{u},y]] = [x,\overline{z},x,\overline{u},z],$$

i.e., as

$$[x,\overline{y},x,\overline{u},y] = [x,\overline{z},x,\overline{u},z].$$

The last equation is valid in semiabelian flocks only. □

Theorem 5.5. *A flock* $(G,[\cdot,\cdot,\cdot])$ *is semiabelian if and only if*

$$2(\overrightarrow{xy} + \overrightarrow{zu}) = 2\overrightarrow{xy} + 2\overrightarrow{zu}$$

for any points $x,y,z,u \in G$.

Proof. Since

$$2(\overrightarrow{xy} + \overrightarrow{zu}) = \overrightarrow{x[y,\overline{z},u,\overline{x},y,\overline{z},u]}$$

and

$$2\overrightarrow{xy} + 2\overrightarrow{zu} = \overrightarrow{x[y,\overline{x},u,\overline{z},u,\overline{z},u]},$$

the equation given in the above theorem is equivalent to

$$\overrightarrow{x[y,\overline{z},u,\overline{x},y,\overline{z},u]} = \overrightarrow{x[y,\overline{x},u,\overline{z},u,\overline{z},u]},$$

i.e., to

$$[x[\overline{y},z,\overline{u},x,\overline{y},z,\overline{u}][[y,\overline{x},y],\overline{z},[u,\overline{z},u]]] = x.$$

The last equation can be written as

$$[x[\overline{y},z,\overline{u},x,\overline{y},z,\overline{u}][y,\overline{x},y,\overline{z},u,\overline{z},u]] = x,$$

which, in view of Lemma 2.1 and (6), means that

$$[\overline{y},z,\overline{u},x,\overline{y},z,\overline{u}] = [\overline{y},x,\overline{y},z,\overline{u},z,\overline{u}],$$

i.e., to $[\overline{y},z,\overline{u},x,\overline{y}] = [\overline{y},x,\overline{y},z,\overline{u}]$. This equation holds only in semiabelian flocks. □

6. Conclusion

Our results are valid for arbitrary flocks and generalize various results proved by S.A. Rusakov and Yu.I. Kulazhenko for n-ary groups with $n \geqslant 3$. Moreover, in the case idempotent flocks, i.e., flocks with the property $\overline{x} = x$, our results coincide with the corresponding results proved for n-ary groups. It is a consequence of (7) and Lemma 2.3. Namely, in the case of idempotent flocks, our Lemma 4.1 coincides with the Kulazhenko's Proposition from [16], Lemma 4.2 with Lemma from [16], and Theorem 4.6 with the main theorem of [16]. His results are presented in very complicated form (see our Theorem 4.4). Theorems 4.5, 4.7 and Corollary 4.8 (also 4.9) generalize Kulazhenko's results from [15]. In the case of idempotent flocks these results are identical. Results of Section 5 generalize Kulazhenko's results from [12].

Another consequence of our results are short and more elegant proofs. For example, the original proof of Theorem 5.2 (for n-ary groups) has in [12] three printed pages of rather complicated transformations; the proof of Theorem 5.3 has four pages. Also the original proofs of Theorems 5.4 and 5.5 presented in [13] are much longer.

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