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Solutions of the matrix linear bilateral polynomial equation and their structure*

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Abstract. We investigate the row and column structure of solutions of the matrix polynomial equation

$$A(\lambda)X(\lambda) + Y(\lambda)B(\lambda) = C(\lambda),$$

where $A(\lambda), B(\lambda)$ and $C(\lambda)$ are the matrices over the ring of polynomials $\mathcal{F}[\lambda]$ with coefficients in field $\mathcal{F}$. We establish the bounds for degrees of the rows and columns which depend on degrees of the corresponding invariant factors of matrices $A(\lambda)$ and $B(\lambda)$. A criterion for uniqueness of such solutions is pointed out. A method for construction of such solutions is suggested. We also established the existence of solutions of this matrix polynomial equation whose degrees are less than degrees of the Smith normal forms of matrices $A(\lambda)$ and $B(\lambda)$.

Introduction and preliminary results

Let $\mathcal{F}$ be a field, and $\mathcal{F}[\lambda]$ be a ring of polynomials over $\mathcal{F}$. We denote by $M(n, \mathcal{F}[\lambda])$ and $M(n, m, \mathcal{F}[\lambda])$ a ring of $n \times n$ matrices and a set of $n \times m$ matrices over $\mathcal{F}[\lambda]$, respectively, and by $GL(n, \mathcal{F})$ and $GL(n, \mathcal{F}[\lambda])$ the groups of invertible $n \times n$ matrices over $\mathcal{F}$ and $\mathcal{F}[\lambda]$, respectively.

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We investigate solutions of the matrix linear bilateral polynomial equation

$$A(\lambda)X(\lambda) + Y(\lambda)B(\lambda) = C(\lambda), \qquad (1)$$

where $A(\lambda), B(\lambda), C(\lambda) \in M(n, \mathcal{F}[\lambda])$ are known matrices, and $X(\lambda)$, $Y(\lambda) \in M(n, \mathcal{F}[\lambda])$ are unknown ones. Matrix Sylvester-type equations over different domains [2,8], in particular, the matrix polynomial equation (1), appear in various branches of mathematics. Such equations play fundamental role in many problems of control theory and dynamical systems theory [5,7].

If the matrix polynomial equation (1) is solvable, it obviously has solutions of unbounded above degrees. So, there arises the following natural question: what is minimal degree of solutions of the matrix polynomial equation (1)? This problem was solved only in particular cases. The estimations of degrees of the solutions of the matrix polynomial equation (1) with regular polynomial matrix coefficients, and conditions for uniqueness of such solutions were obtained in [4] and [9].

We recall that collections of polynomial matrices $A_1(\lambda), \ldots, A_k(\lambda)$ and $B_1(\lambda), \ldots, B_k(\lambda)$, where $A_i(\lambda), B_i(\lambda) \in M(n, m_i, \mathcal{F}[\lambda])$, are called *semiscalar equivalent* if there exists a scalar matrix $Q \in GL(n, \mathcal{F})$ and invertible matrices $R_i(\lambda) \in GL(m_i, \mathcal{F}[\lambda])$ such that $B_i(\lambda) = QA_i(\lambda)R_i(\lambda)$, $i = 1, \ldots, k$.

It was shown in [6] that the collection of polynomial matrices $A_1(\lambda), \ldots, A_k(\lambda)$ with maximal ranks over algebraically closed field $\mathcal{F}$ of characteristic zero is semiscalar equivalent to the collection of the lower triangular polynomial matrices $T^{A_1}(\lambda), \ldots, T^{A_k}(\lambda)$ with invariant factors on main diagonals. These results were generalized for polynomial matrices over an arbitrary field $\mathcal{F}$ [10,13]. Triangular matrices $T^{A_i}(\lambda)$ are called *standard forms* of polynomial matrices $A_i(\lambda)$, $i = 1, \ldots, k$.

Note that similar form for one polynomial matrix over infinite field with respect to right semiscalar equivalence of matrices was obtained in [1].

In [2] the solutions of the matrix linear unilateral and bilateral equations over some other rings were investigated. These results were based on the standard form of the pair of matrices with respect to generalized equivalence [11,12]. The conditions for uniqueness of solutions with some properties were also proposed in [2].

In this paper we study solutions of the matrix polynomial equation (1) where both matrix coefficients $A(\lambda)$ and $B(\lambda)$ may be nonregular matrices, unlike to [4,9]. We describe the structure of solutions of this equation by using standard forms of the collection of polynomial matrices with respect to semiscalar equivalence. We establish the bounds for degrees of solutions of this matrix polynomial equation. A criterion for uniqueness of such solutions and a method for their construction are pointed out. These results are a generalization of results [3].

1. Structure of solutions of the matrix polynomial equation

According to results of [10], a pair of nonsingular polynomial matrices $A(\lambda)$, $B(\lambda) \in M(n, \mathcal{F}[\lambda])$ from the matrix polynomial equation (1) is semiscalar equivalent to the pair of triangular matrices $T^A(\lambda), T^B(\lambda)$ in standard form, i.e., there exists an upper unitriangular matrix $Q \in GL(n, \mathcal{F})$ and invertible matrices $R^A(\lambda)$ and $R^B(\lambda) \in GL(n, \mathcal{F}[\lambda])$ such that

$$\begin{aligned} T^A(\lambda) &= QA(\lambda)R^A(\lambda) \\ &= \begin{bmatrix} \mu_1^A(\lambda) & 0 & \cdots & 0 \\ \widetilde{a}_{21}(\lambda)\mu_1^A(\lambda) & \mu_2^A(\lambda) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \widetilde{a}_{n1}(\lambda)\mu_1^A(\lambda) & \widetilde{a}_{n2}(\lambda)\mu_2^A(\lambda) & \cdots & \mu_n^A(\lambda) \end{bmatrix}, \end{aligned} \tag{2}$$

where $\deg \widetilde{a}_{ij}(\lambda) < \deg \mu_i^A(\lambda) - \deg \mu_j^A(\lambda)$ if $\deg \mu_i^A(\lambda) > \deg \mu_j^A(\lambda)$, and $\widetilde{a}_{ij}(\lambda) \equiv 0$ if $\mu_i^A(\lambda) = \mu_j^A(\lambda)$, $i, j = 1, \ldots, n-1$, $i > j$,

$$\begin{aligned} T^B(\lambda) &= QB(\lambda)R^B(\lambda) \\ &= \begin{bmatrix} \mu_1^B(\lambda) & 0 & \cdots & 0 \\ \widetilde{b}_{21}(\lambda)\mu_1^B(\lambda) & \mu_2^B(\lambda) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \widetilde{b}_{n1}(\lambda)\mu_1^B(\lambda) & \widetilde{b}_{n2}(\lambda)\mu_2^B(\lambda) & \cdots & \mu_n^B(\lambda) \end{bmatrix}, \end{aligned} \tag{3}$$

where $\deg \widetilde{b}_{ij}(\lambda) < \deg \mu_i^B(\lambda) - \deg \mu_j^B(\lambda)$ if $\deg \mu_i^B(\lambda) > \deg \mu_j^B(\lambda)$, and $\widetilde{b}_{ij}(\lambda) \equiv 0$ if $\mu_i^B(\lambda) = \mu_j^B(\lambda)$, $i, j = 1, \ldots, n-1$, $i > j$, $\mu_i^A(\lambda)$ and $\mu_i^B(\lambda)$, $i = 1, \ldots, n$ are invariant factors of $A(\lambda)$ and $B(\lambda)$, respectively. Triangular matrices $T^A(\lambda), T^B(\lambda)$ can be written in form $T^A(\lambda) = T_1(\lambda)S^A(\lambda)$, $T^B(\lambda) = T_2(\lambda)S^B(\lambda)$, where $T_1(\lambda)$ and $T_2(\lambda)$ are lower unitriangular matrices, $S^A(\lambda)$ and $S^B(\lambda)$ are the Smith normal forms of matrices $A(\lambda)$ and $B(\lambda)$.

Thus, from equation (1) we obtain the following matrix polynomial equation

$$T^A(\lambda)\widetilde{X}(\lambda) + \widetilde{Y}(\lambda)T^B(\lambda) = \widetilde{C}(\lambda), \tag{4}$$

where $\widetilde{X}(\lambda) = (R^A(\lambda))^{-1}X(\lambda)R^B(\lambda)$, $\widetilde{Y}(\lambda) = QY(\lambda)Q^{-1}$, and $\widetilde{C}(\lambda) = QC(\lambda)R^B(\lambda)$. It is easy to verify that the matrix polynomial equation (1) is solvable if and only if the matrix polynomial equation (4) is solvable. Then the following equalities are valid:

$$X(\lambda) = R^A(\lambda)\widetilde{X}(\lambda)(R^B(\lambda))^{-1} \quad \text{and} \quad Y(\lambda) = Q^{-1}\widetilde{Y}(\lambda)Q \tag{5}$$

for solution $X(\lambda), Y(\lambda)$ of (1) and solution $\widetilde{X}(\lambda), \widetilde{Y}(\lambda)$ of (4). So, solving of the matrix polynomial equation (1) reduced to solving of the matrix polynomial equation (4).

The criterion for solvability of matrix Sylvester-type equations was established by Roth [14].

We denote by $\mathrm{row}_i(A)$ and $\mathrm{col}_j(A)$ the i-th row and j-th column of matrix A, respectively. Further we assume that $\deg 0 = -\infty$.

Theorem 1. *Let*

$$\begin{aligned} S^A(\lambda) = \operatorname{diag}(&\mu_1^A(\lambda), \dots, \mu_p^A(\lambda), \mu_{p+1}^A(\lambda), \\ &\dots, \mu_{p+q}^A(\lambda), \mu_{p+q+1}^A(\lambda), \dots, \mu_n^A(\lambda)), \quad p \geqslant 0,\ q \geqslant 0, \end{aligned} \tag{6}$$

be the Smith normal form of matrix $A(\lambda)$ from equation (1), *where*

$$\deg \mu_i^A(\lambda) = 0, \quad \text{that is,} \quad \mu_i^A = 1, \quad i = 1, \dots, p, \tag{7}$$

$$\deg \mu_i^A(\lambda) = 1, \quad i = p+1, \dots, p+q, \tag{8}$$

$$\deg \mu_i^A(\lambda) > 1, \quad i = p+q+1, \dots, n. \tag{9}$$

If the matrix polynomial equation (4) *be solvable, then this equation has solutions in the form*

$$\widetilde{X}_1(\lambda) = [\widetilde{x}_{ij}(\lambda)]_{i,j=1}^n, \quad \widetilde{Y}_1(\lambda) = [\widetilde{y}_{ij}(\lambda)]_{i,j=1}^n = \begin{bmatrix} \widetilde{Y}^{(p)}(\lambda) \\ \widetilde{Y}^{(q)}(\lambda) \\ \widetilde{Y}^{(n-(p+q))}(\lambda) \end{bmatrix}, \tag{10}$$

where

$$\widetilde{Y}^{(p)}(\lambda) = \begin{bmatrix} \operatorname{row}_1(\widetilde{Y}_1(\lambda)) \\ \vdots \\ \operatorname{row}_p(\widetilde{Y}_1(\lambda)) \end{bmatrix}, \qquad \widetilde{Y}^{(q)}(\lambda) = \begin{bmatrix} \operatorname{row}_{p+1}(\widetilde{Y}_1(\lambda)) \\ \vdots \\ \operatorname{row}_{p+q}(\widetilde{Y}_1(\lambda)) \end{bmatrix},$$

$$\widetilde{Y}^{(n-(p+q))}(\lambda) = \begin{bmatrix} \operatorname{row}_{p+q+1}(\widetilde{Y}_1(\lambda)) \\ \vdots \\ \operatorname{row}_n(\widetilde{Y}_1(\lambda)) \end{bmatrix}$$

and

$$\widetilde{Y}^{(p)}(\lambda) = \mathbf{0}, \tag{11}$$

$$\widetilde{Y}^{(q)}(\lambda) \text{ is scalar matrix, i.e., } \widetilde{Y}^{(q)} \in M(q, n, \mathcal{F}), \tag{12}$$

$$\deg \operatorname{row}_i(\widetilde{Y}^{(n-(p+q))}(\lambda)) < \deg \mu_i^A(\lambda) - \deg(\mu_i^A(\lambda), \mu_1^B(\lambda)), \tag{13}$$

where $i = p + q + 1, \dots, n$.

Proof. From (4), we obtain the system of linear polynomial equations

$$\sum_{l=1}^{i} \mu_l^A(\lambda)\widetilde{a}_{il}(\lambda)\widetilde{x}_{lj}(\lambda) + \sum_{k=j}^{n} \mu_j^B(\lambda)\widetilde{b}_{kj}(\lambda)\widetilde{y}_{ik}(\lambda) = \widetilde{c}_{ij}(\lambda), \tag{14}$$

where $\widetilde{a}_{ii} = \widetilde{b}_{ii} = 1$, $\widetilde{a}_{ij} = \widetilde{b}_{ij} = 0$ if $i < j$, and $\widetilde{C}(\lambda) = [\widetilde{c}_{ij}(\lambda)]_{i,j=1}^{n}$, $i, j = 1, \dots, n$. It is obvious that the matrix equation (4) has a solution if and only if the system of linear polynomial equations (14) has a solution.

Let system (14) be solvable and let

$$\widetilde{x}_{ij}(\lambda) = u_{ij}(\lambda), \widetilde{y}_{ij}(\lambda) = v_{ij}(\lambda), \quad i, j = 1, \dots, n, \tag{15}$$

be solutions of system (14).

We split the system (14) into $2n - 1$ subsystems in the following way: every t-th subsystem for $t \leqslant n$ consists of t equations. We obtain it from system (14) by assuming that $i = 1, 2, \dots, t$ and $j = n - (t - i)$, respectively, that is, $j = n - (t - 1), n - (t - 2), \dots, n$. If $t > n$, namely $t = n + q$, $q = 1, 2, \dots, n - 1$, then t-th subsystem consists of equations of system (14) for $i = q + 1, \dots, n$, $j = 1, \dots, n - q$.

The first subsystem of system (14), i.e., $t = 1$, and therefore $i = 1$, $j = n$, consists of one equation:

$$\mu_1^A(\lambda)\widetilde{x}_{1n}(\lambda) + \mu_n^B(\lambda)\widetilde{y}_{1n}(\lambda) = \widetilde{c}_{1n}(\lambda). \tag{16}$$

We denote $d_{i,j}^{(A,B)}(\lambda) = (\mu_i^A(\lambda), \mu_j^B(\lambda))$.

Since equation (16) is solvable, then $d_{1,n}^{(A,B)}(\lambda) \mid \widetilde{c}_{1n}(\lambda)$. Therefore, we obtain equation:

$$\frac{\mu_1^A(\lambda)}{d_{1,n}^{(A,B)}(\lambda)}\widetilde{x}_{1n}(\lambda) + \frac{\mu_n^B(\lambda)}{d_{1,n}^{(A,B)}(\lambda)}\widetilde{y}_{1n}(\lambda) = \frac{\widetilde{c}_{1n}(\lambda)}{d_{1,n}^{(A,B)}(\lambda)}. \tag{17}$$

Then $\widetilde{x}_{1n}(\lambda) = u_{1n}(\lambda), \widetilde{y}_{1n}(\lambda) = v_{1n}(\lambda)$ is the solution of (17).

We divide $v_{1n}(\lambda)$ by $\frac{\mu_1^A(\lambda)}{d_{1,n}^{(A,B)}(\lambda)}$: $v_{1n}(\lambda) = \frac{\mu_1^A(\lambda)}{d_{1,n}^{(A,B)}(\lambda)} s_{1n}(\lambda) + \widetilde{y}_{1n}^{(1)}(\lambda)$, where $\deg \widetilde{y}_{1n}^{(1)}(\lambda) < \deg \frac{\mu_1^A(\lambda)}{d_{1,n}^{(A,B)}(\lambda)}$. Then

$$\widetilde{x}_{1n}^{(1)}(\lambda) = u_{1n}(\lambda) + \frac{\mu_n^B(\lambda)}{d_{1,n}^{(A,B)}(\lambda)} s_{1n}(\lambda), \widetilde{y}_{1n}^{(1)}(\lambda) = v_{1n}(\lambda) - \frac{\mu_1^A(\lambda)}{d_{1,n}^{(A,B)}(\lambda)} s_{1n}(\lambda)$$

is the solution of (17) and (16), where

$$\deg \widetilde{y}_{1n}^{(1)}(\lambda) < \deg \mu_1^A(\lambda) - \deg(\mu_1^A(\lambda), \mu_n^B(\lambda)).$$

We obtain the second subsystem, that is, $t = 2$, from (14) by assuming that $i = 1, 2$ and $j = n-1, n$, respectively. This subsystem consists of the following two equations:

$$\begin{aligned} &\mu_1^A(\lambda)\widetilde{x}_{1,n-1}(\lambda) + \mu_{n-1}^B(\lambda)\widetilde{y}_{1,n-1}(\lambda) \\ &\qquad + \widetilde{b}_{n,n-1}(\lambda)\mu_{n-1}^B(\lambda)\widetilde{y}_{1n}(\lambda) = \widetilde{c}_{1,n-1}(\lambda), \end{aligned} \tag{18}$$

$$\widetilde{a}_{n-1,1}(\lambda)\mu_1^A(\lambda)\widetilde{x}_{1n}(\lambda) + \mu_2^A(\lambda)\widetilde{x}_{2n}(\lambda) + \mu_n^B(\lambda)\widetilde{y}_{2n}(\lambda) = \widetilde{c}_{2n}(\lambda). \tag{19}$$

Using the same procedure as in the previous case, we obtain the following solution of equations (18) and (19):

$$\widetilde{x}_{1,n-1}^{(1)}(\lambda),\ \widetilde{y}_{1,n-1}^{(1)}(\lambda), \quad \deg \widetilde{y}_{1,n-1}^{(1)}(\lambda) < \deg \mu_1^A(\lambda) - \deg(\mu_1^A(\lambda), \mu_{n-1}^B(\lambda)),$$

and

$$\widetilde{x}_{2n}^{(1)}(\lambda), \widetilde{y}_{2n}^{(1)}(\lambda), \quad \deg \widetilde{y}_{2n}^{(1)}(\lambda) < \deg \mu_2^A(\lambda) - \deg(\mu_2^A(\lambda), \mu_n^B(\lambda)).$$

Further, we consider the next subsystem, and so on. Thus, by similar considerations, we obtain solutions $\widetilde{x}_{ij}(\lambda), \widetilde{y}_{ij}(\lambda)$ of system (14) such that $\deg \widetilde{y}_{ij}(\lambda) < \deg \mu_i^A(\lambda) - \deg(\mu_i^A(\lambda), \mu_1^B(\lambda))$, $i, j = 1, \ldots, n$. From these solutions of system (14) we construct solution $\widetilde{X}(\lambda), \widetilde{Y}(\lambda)$ with conditions (11), (12) and (13) of the matrix polynomial equation (4). This completes the proof. □

Theorem 2. *The solution* $\widetilde{X}_1(\lambda), \widetilde{Y}_1(\lambda)$ *in form* (10) *with conditions* (11), (12) *and* (13) *of the matrix equation* (4) *is unique if and only if* $(\mu_n^A(\lambda), \mu_n^B(\lambda)) = 1$.

Proof. It is clear that the matrix polynomial equation (4) has unique solution $\widetilde{X}_1(\lambda), \widetilde{Y}_1(\lambda)$ with conditions (11), (12) and (13) if and only if system (14) has the corresponding unique solution. As in the proof of Theorem 1, the solving of system (14) is reduced to the solving of linear polynomial equations. It is known that linear polynomial equation in form (16) has unique solution with bounded degree $\deg \widetilde{y}_{1n}^{(1)}(\lambda) < \deg \mu_1^A(\lambda) - \deg(\mu_1^A(\lambda), \mu_n^B(\lambda))$ if and only if $(\mu_1^A(\lambda), \mu_n^B(\lambda)) = 1$. Then system (14) has unique solution with corresponding bounded degree if and only if $(\mu_i^A(\lambda), \mu_j^B(\lambda)) = 1$ for all $i, j = 1, \ldots, n$. That is true if and only if $(\mu_n^A(\lambda), \mu_n^B(\lambda)) = 1$. □

Analogously we can describe the column structure of the first component $\widetilde{X}(\lambda)$ of solution $\widetilde{X}(\lambda), \widetilde{Y}(\lambda)$ of equation (4).

Corollary 1. *Let the matrix polynomial equation* (4) *be solvable. Then it has solutions* $\widetilde{X}_1(\lambda) = [\widetilde{x}_{ij}^{(1)}(\lambda)]_{i,j=1}^n$, $\widetilde{Y}_1(\lambda) = [\widetilde{y}_{ij}^{(1)}(\lambda)]_{i,j=1}^n$ *such that*

$$\deg \operatorname{row}_i(\widetilde{Y}_1(\lambda)) < \deg \mu_i^A(\lambda) - \deg(\mu_i^A(\lambda), \mu_1^B(\lambda)), \quad i = 1, \ldots, n,$$

and $\widetilde{X}_2(\lambda) = [\widetilde{x}_{ij}^{(2)}(\lambda)]_{i,j=1}^n$, $\widetilde{Y}_2(\lambda) = [\widetilde{y}_{ij}^{(2)}(\lambda)]_{i,j=1}^n$ *such that*

$$\deg \operatorname{col}_j(\widetilde{X}_2(\lambda)) < \deg \mu_j^B(\lambda) - \deg(\mu_1^A(\lambda), \mu_j^B(\lambda)), \quad j = 1, \ldots, n.$$

Corollary 2. *Let the matrix polynomial equation* (1) *be solvable. Then it has the following solutions:*

(i) $X_1(\lambda), Y_1(\lambda)$ *such that* $\deg Y_1(\lambda) < \deg S^A(\lambda) - \deg(S^A(\lambda), S^B(\lambda))$,

(ii) $X_2(\lambda), Y_2(\lambda)$ *such that* $\deg X_2(\lambda) < \deg S^B(\lambda) - \deg(S^A(\lambda), S^B(\lambda))$.

Indeed, taking into account the equality (5) between solutions of equations (1) and (4) and a fact that Q is scalar invertible matrix, we obtain solution (i). To obtain solution (ii) we use standard forms $\widetilde{T}^A(\lambda) = \widetilde{R}^A(\lambda)A(\lambda)\widetilde{Q}$ and $\widetilde{T}^B(\lambda) = \widetilde{R}^B(\lambda)B(\lambda)\widetilde{Q}$ of matrices $A(\lambda)$ and $B(\lambda)$ with respect to the right semiscalar equivalence.

Theorem 3. *If* $(\det A(\lambda), \det B(\lambda)) = 1$ *and* $\deg \widetilde{c}_{ij}(\lambda) < \deg \mu_i^A(\lambda) + \deg \mu_n^B(\lambda)$, $i, j = 1, \ldots, n$, *then the matrix polynomial equation* (4) *has the solution* $\widetilde{X}(\lambda) = [\widetilde{x}_{ij}(\lambda)]_{i,j=1}^n$, $\widetilde{Y}(\lambda) = [\widetilde{y}_{ij}(\lambda)]_{i,j=1}^n$ *such that*

$$\deg \operatorname{col}_j \widetilde{X}(\lambda) < \deg \mu_n^B(\lambda) \quad \text{and} \quad \deg \operatorname{row}_i(\widetilde{Y}(\lambda)) < \deg \mu_i^A(\lambda), \quad (20)$$

where $i, j = 1, \ldots, n$, *and this solution is unique.*

Proof. As in the proof of Theorem 1, we consider the subsystems of system (14). Since $(\mu_1^A(\lambda), \mu_n^B(\lambda)) = 1$, there exists the following solution of (16):

$$\widetilde{x}_{1n}^{(1)}(\lambda), \quad \widetilde{y}_{1n}^{(1)}(\lambda), \qquad \deg \widetilde{x}_{1n}^{(1)}(\lambda) < \deg \mu_n^B(\lambda), \quad \deg \widetilde{y}_{1n}^{(1)}(\lambda) < \deg \mu_1^A(\lambda)$$

and this solution is unique.

By substituting $\widetilde{y}_{1n}^{(1)}(\lambda)$ in (18) instead $\widetilde{y}_{1n}(\lambda)$, we obtain

$$\mu_1^A(\lambda)\widetilde{x}_{1,n-1}(\lambda) + \mu_{n-1}^B(\lambda)\widetilde{y}_{1,n-1}(\lambda) \\ = \widetilde{c}_{1,n-1}(\lambda) - \widetilde{b}_{n,n-1}(\lambda)\mu_{n-1}^B(\lambda)\widetilde{y}_{1n}^{(1)}(\lambda).$$

This equation has solution $\widetilde{x}_{1,n-1}^{(1)}(\lambda), \widetilde{y}_{1,n-1}^{(1)}(\lambda)$ such that $\deg \widetilde{y}_{1,n-1}^{(1)}(\lambda) < \deg \mu_1^A(\lambda)$. By comparing the degrees on left and on right parts of the same equation, we obtain $\deg \widetilde{x}_{1,n-1}^{(1)}(\lambda) < \deg \mu_n^B(\lambda)$. This solution $\widetilde{x}_{1,n-1}^{(1)}(\lambda)$, $\widetilde{y}_{1,n-1}^{(1)}(\lambda)$ of equation (18) is also unique, because

$$(\mu_1^A(\lambda), \mu_{n-1}^B(\lambda)) = 1.$$

In the same way, we obtain unique solution

$$\widetilde{x}_{2n}^{(1)}(\lambda), \quad \widetilde{y}_{2n}^{(1)}(\lambda), \qquad \deg \widetilde{x}_{2n}^{(1)}(\lambda) < \deg \mu_n^B(\lambda), \quad \deg \widetilde{y}_{2n}^{(1)}(\lambda) < \deg \mu_2^A(\lambda)$$

from equation (19).

Further, we consider the next subsystem, and so on. In this way, we obtain solutions $\widetilde{x}_{ij}^{(1)}(\lambda), \widetilde{y}_{ij}^{(1)}(\lambda), i, j = 1, \ldots, n$, of system (14) and construct solution $\widetilde{X_1}(\lambda) = [\widetilde{x}_{ij}^{(1)}(\lambda)]_{i,j=1}^n$, $\widetilde{Y_1}(\lambda) = [\widetilde{y}_{ij}^{(1)}(\lambda)]_{i,j=1}^n$ of the matrix polynomial equation (4) with conditions (20). This completes the proof. □

Corollary 3. *The matrix polynomial equation* (1)*, where determinants of matrices* $A(\lambda)$ *and* $B(\lambda)$ *are relatively prime, has solutions:*

(i) $X_1(\lambda), Y_1(\lambda)$ *such that* $\deg Y_1(\lambda) < \deg S^A(\lambda)$;
(ii) $X_2(\lambda), Y_2(\lambda)$ *such that* $\deg X_2(\lambda) < \deg S^B(\lambda)$.

This Corollary is a generalization of results [4, 9] for the matrix polynomial equation (1) where the matrix coefficients $A(\lambda)$ and $B(\lambda)$ can be nonregular.

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