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A family of doubly stochastic matrices involving Chebyshev polynomials

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Abstract. A *doubly stochastic matrix* is a square matrix $A = (a_{ij})$ of non-negative real numbers such that $\sum_i a_{ij} = \sum_j a_{ij} = 1$. The *Chebyshev polynomial of the first kind* is defined by the recurrence relation $T_0(x) = 1, T_1(x) = x$, and

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

In this paper, we show a $2^k \times 2^k$ (for each integer $k \geqslant 1$) doubly stochastic matrix whose characteristic polynomial is $x^2 - 1$ times a product of irreducible Chebyshev polynomials of the first kind (upto rescaling by rational numbers).

1. Introduction

Chebyshev polynomial of the first kind is defined by

$$T_n(x) = \cos(n \cdot \arccos(x)).$$

The fact that roots of $T_{2^k}(x)$ are $\cos\left(\frac{2j-1}{2^{k+1}}\pi\right)$, for $1 \leqslant j \leqslant 2^k$ together with the trigonometric identity $2 + 2\cos\left(\frac{\theta}{2^{k+1}}\right) = 2 \pm \sqrt{2 + 2\cos\left(\frac{\theta}{2^k}\right)}$ make $T_{2^k}(x)$ a remarkable subsequence. For example, Kimberling [2] used

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these facts in order to obtain a Gray code by means of the numbers

$$2+c_1\sqrt{2+c_2\sqrt{2+c_3\sqrt{2+c_4\sqrt{\cdots+c_n\sqrt{2}}}}},$$

where each $c_j \in \{-1,1\}$ and generalized this result to a wider class of polynomials. Another reason why $T_n(x)$ indexed by powers of 2 are special is that, as a consequence of Esenstein's irreducibility criterion, $T_n(x)$ is irreducible over $\mathbb{Q}[x]$ if and only if n is a power of 2. A *normalized Chebyshev polynomial* is a Chebyshev polynomial divided by the coefficient of its leading term. So, the leading term of a normalized Chebyshev polynomial is always 1.

We represent the matrix in context as a self-similar structure. A self-similar algebra (see Bartholdi [1]) $(\mathfrak{A},\psi)$ is an associative algebra $\mathfrak{A}$ endowed with a morphism of algebras $\psi : \mathfrak{A} \longrightarrow M_d(\mathfrak{A})$, where $M_d(\mathfrak{A})$ is the set of $d\times d$ matrices with coefficients from $\mathfrak{A}$. Given $s \in \mathfrak{A}$ and integers $a \geqslant 0$ and $b \geqslant 0$, the 2×2 matrix $\psi_{a,b}(s)$ is obtained using the mapping $x \mapsto \begin{pmatrix} 0 & x^a \\ y^a & 0 \end{pmatrix}$, $\quad y \mapsto \begin{pmatrix} 0 & x^b \\ y^b & 0 \end{pmatrix}$. We write $\psi_{1,0}(s)$ as $\psi(s)$. Given a self-similar algebra $(\mathfrak{A},\psi)$, with $\psi : \mathfrak{A} \longrightarrow M_2(\mathfrak{A})$, we define $\left(\mathfrak{A},\psi^{(k)}\right)$ (for $k \geqslant 0$) with $\psi^{(k)} : \mathfrak{A} \longrightarrow M_{2^k}(\mathfrak{A})$ given by $\psi^{(0)}(s) := s$ and $\psi^{(k+1)}(s) := \left(\psi^{(k)}(s_{i,j})\right)_{0\leqslant i,j\leqslant 2^k-1}$, where $\psi(s) = (s_{i,j})_{0\leqslant i,j\leqslant 2^k-1}$. For $k \geqslant 1$, we consider the following doubly stochastic matrix

$$\mathfrak{M}_k(a,b) := \psi_{a,b}^{(k)}\left(\frac{1}{2}x+\frac{1}{2}y\right)\bigg|_{(x,y)=(1,1)} \in M_{2^k}(\mathbb{Q}).$$

We show that the characteristic polynomial of $\mathfrak{M}_k(1,0)$ is x^2-1 times a product of irreducible Chebyshev polynomials of the first kind (upto rescaling by rational numbers).

2. Preliminaries

In this section, we discuss some preliminaries from linear algebra. We use Newton identities for the characteristic polynomial of a matrix.

Lemma 2.1. Let M and N be two $k\times k$ square matrices. For each integer $n \geqslant 1$,

$$\operatorname{tr}\left[\begin{pmatrix} M & M \\ N & N \end{pmatrix}^n\right] = \operatorname{tr}\left[\begin{pmatrix} N & M \\ N & M \end{pmatrix}^n\right] = \operatorname{tr}\left[(M+N)^n\right].$$

Proof. The following identity can be checked by complete induction,

$$\begin{pmatrix} M & M \\ N & N \end{pmatrix}^n = \begin{pmatrix} M(M+N)^{n-1} & M(M+N)^{n-1} \\ N(M+N)^{n-1} & N(M+N)^{n-1} \end{pmatrix}.$$

Using the properties of the trace, we conclude that

$$\begin{aligned} \operatorname{tr}\left[\begin{pmatrix} M & M \\ N & N \end{pmatrix}^n\right] &= \operatorname{tr}\left[M(M+N)^{n-1}\right] + \operatorname{tr}\left[N(M+N)^{n-1}\right] \\ &= \operatorname{tr}\left[(M+N)^n\right]. \end{aligned}$$

The proof of $\operatorname{tr}\left[\begin{pmatrix} N & M \\ N & M \end{pmatrix}^n\right] = \operatorname{tr}\left[(M+N)^n\right]$ is analogous. □

Lemma 2.2. Let M and N be two $k \times k$ square matrices. The characteristic polynomials of $\begin{pmatrix} M & M \\ N & N \end{pmatrix}$ and $\begin{pmatrix} N & M \\ N & M \end{pmatrix}$ are the same.

Proof. By Lemma 2.1, for all $n \geqslant 1$,

$$\operatorname{tr}\left[\begin{pmatrix} M & M \\ N & N \end{pmatrix}^n\right] = \operatorname{tr}\left[\begin{pmatrix} N & M \\ N & M \end{pmatrix}^n\right].$$

Using Newton identities and the fact that both matrices have the same dimensions, we derive that both matrices have the same characteristic polynomials. □

Lemma 2.3. Let M and N be $k \times k$ square matrices. Let $p(x)$ and $q(x)$ be the characteristic polynomials of $\begin{pmatrix} M & M \\ N & N \end{pmatrix}$ and $M+N$, respectively. Then $p(x) = x^k q(x)$.

Proof. By Lemma 2.1, for all $n \geqslant 1$, $\operatorname{tr}\left[\begin{pmatrix} M & M \\ N & N \end{pmatrix}^n\right] = \operatorname{tr}\left[(M+N)^n\right]$. Using Newton identities, we derive that the coefficients of $p(x)$ and $q(x)$ coincide but the degree of $p(x)$ exceeds the degree of $q(x)$ by k. Hence, $p(x) = x^k q(x)$. □

Lemma 2.4. Let M be a $k \times k$ square matrix. Let $p(x)$ and $q(x)$ be the characteristic polynomials of M and $-M$, respectively. If $\operatorname{tr}(M^n) = 0$ for each odd non-negative integer n, then $p(x)$ and $q(x)$ coincide.

Proof. For each even integer $n \geqslant 0$, it follows that $\mathrm{tr}\,(M^n) = \mathrm{tr}\,((-M)^n)$. On the other hand, for any odd integer $n \geqslant 0$, we have $\mathrm{tr}\,(M^n) = \mathrm{tr}\,((-M)^n) = 0$, by hypothesis. So $\mathrm{tr}\,(M^n) = \mathrm{tr}\,((-M)^n)$ for any integer $n \geqslant 0$ regardless of its parity. Using Newton identities, we conclude that $p(x)$ is the same as $q(x)$. □

Lemma 2.5. Let M be a $k \times k$ square matrix. Let $p(x)$ and $q(x)$ be the characteristic polynomials of M and M^2, respectively. If $\mathrm{tr}(M^n) = 0$ for each odd non-negative integer n, then $(p(x))^2 = q(x^2)$.

Proof. By hypothesis and by Lemma 2.4, we have $|xI - M| = |xI + M|$, where I is the $k \times k$ identity matrix. Then,

$$|xI - M|^2 = |(xI - M)(xI + M)| = |x^2 I - M^2|,$$

and hence $(p(x))^2 = q(x^2)$. □

3. Eigenvalues of $\mathfrak{M}_k(a, b)$

Proposition 3.1. For all $a \equiv b\,(mod\ 2)$, if $\lambda \in \mathbb{C}$ is an eigenvalue of the matrix $\mathfrak{M}_k(a, b)$ then either $\lambda = -1$ or $\lambda = 1$.

Proof. We shall consider the following cases.

(i) If $a \equiv b \equiv 1\,(mod\ 2)$ then $\mathfrak{M}_k(a, b)$ is the exchange matrix, i.e. the matrix $J = (J_{i,j})_{0 \leqslant i,j \leqslant 2^k - 1}$, where

$$J_{i,j} = \begin{cases} 1 & \textit{if } j = 2^k - 1 - i, \\ 0 & \textit{if } j \neq 2^k - 1 - i. \end{cases}$$

(ii) If $a \equiv b \equiv 0\,(mod\ 2)$ then $\mathfrak{M}_k(a, b)$ is the block matrix $\begin{pmatrix} I_{2^{k-1}} & 0_{2^{k-1}} \\ 0_{2^{k-1}} & I_{2^{k-1}} \end{pmatrix}$. where I_n and 0_n are the $n \times n$ identity matrix and the $n \times n$ zero matrix respectively.

In both cases, all the eigenvalues belong to the set $\{-1, 1\}$. □

4. The characteristic polynomial of $\mathfrak{M}_k(1, 0)$

We study the structure of $\mathfrak{M}_k(1, 0)$, which is less trivial than in the previous examples. Denote

$$A_k := \psi^{(k)}(x)\Big|_{(x,y)=(1,1)}, \text{ and } B_k := \psi^{(k)}(y)\Big|_{(x,y)=(1,1)},$$

such that $\mathfrak{M}_k(1,0) = \frac{1}{2}[A_{2^k} + B_{2^k}]$. Note that the pair of matrices (A_{2^k}, B_{2^k}) can be defined equivalently using the following recursion:

$$A_1 = B_1 = (1)_{1\times 1}, \quad A_{2^{k+1}} = \begin{pmatrix} 0_{2^k} & A_{2^k} \\ B_{2^k} & 0_{2^k} \end{pmatrix}_{2^{k+1}\times 2^{k+1}},$$

$$B_{2^{k+1}} = \begin{pmatrix} 0_{2^k} & I_{2^k} \\ I_{2^k} & 0_{2^k} \end{pmatrix}_{2^{k+1}\times 2^{k+1}},$$

where 0_{2^k} and I_{2^k} are respectively the $2^k \times 2^k$ zero and identity matrices.

Lemma 4.1. For each integer $k \geqslant 2$, $\mathfrak{M}_k(1,0)^2 = \begin{pmatrix} P_{2^{k-1}} & 0_{2^{k-1}} \\ 0_{2^{k-1}} & Q_{2^{k-1}} \end{pmatrix}$, where

$$P_{2^{k-1}} = \frac{1}{4}\begin{pmatrix} A_{2^{k-2}} + I_{2^{k-2}} & A_{2^{k-2}} + I_{2^{k-2}} \\ B_{2^{k-2}} + I_{2^{k-2}} & B_{2^{k-2}} + I_{2^{k-2}} \end{pmatrix},$$

$$Q_{2^{k-1}} = \frac{1}{4}\begin{pmatrix} B_{2^{k-2}} + I_{2^{k-2}} & A_{2^{k-2}} + I_{2^{k-2}} \\ B_{2^{k-2}} + I_{2^{k-2}} & A_{2^{k-2}} + I_{2^{k-2}} \end{pmatrix}$$

Proof. It follows from the identity:

$$\begin{aligned}
[\mathfrak{M}_k(1,0)]^2 &= \begin{pmatrix} 0_{2^{k-1}} & \frac{1}{2}\left(A_{2^{k-1}} + I_{2^{k-1}}\right) \\ \frac{1}{2}\left(B_{2^{k-1}} + I_{2^{k-1}}\right) & 0_{2^{k-1}} \end{pmatrix}^2 \\
&= \frac{1}{4}\begin{pmatrix} \left(A_{2^{k-1}} + I_{2^{k-1}}\right)\left(B_{2^{k-1}} + I_{2^{k-1}}\right) & 0_{2^{k-1}} \\ 0_{2^{k-1}} & \left(B_{2^{k-1}} + I_{2^{k-1}}\right)\left(A_{2^{k-1}} + I_{2^{k-1}}\right) \end{pmatrix} \\
&= \frac{1}{4}\left(\begin{array}{cc|cc} A_{2^{k-2}} + I_{2^{k-2}} & A_{2^{k-2}} + I_{2^{k-2}} & 0_{2^{k-2}} & 0_{2^{k-2}} \\ B_{2^{k-2}} + I_{2^{k-2}} & B_{2^{k-2}} + I_{2^{k-2}} & 0_{2^{k-2}} & 0_{2^{k-2}} \\ \hline 0_{2^{k-2}} & 0_{2^{k-2}} & B_{2^{k-2}} + I_{2^{k-2}} & A_{2^{k-2}} + I_{2^{k-2}} \\ 0_{2^{k-2}} & 0_{2^{k-2}} & B_{2^{k-2}} + I_{2^{k-2}} & A_{2^{k-2}} + I_{2^{k-2}} \end{array}\right). \square
\end{aligned}$$

Lemma 4.2. Given positive integer $k \geqslant 2$, let $p(x)$ and $q(x)$ be the characteristics polynomials of

$$\left[\frac{1}{2}\left(A_{2^k} + B_{2^k}\right)\right]^2 \text{ and } \frac{1}{4}\left(A_{2^{k-2}} + B_{2^{k-2}} + 2I_{2^{k-2}}\right),$$

respectively. Then $p(x) = \left(x^{2^{k-2}} q(x)\right)^2$.

Proof. By Lemma 4.1, the characteristics polynomial of $[\mathfrak{M}_k(1,0)]^2$ is the product of the characteristic polynomials of $P_{2^{k-1}}$ and $Q_{2^{k-1}}$. By Lemma 2.2, $P_{2^{k-1}}$ and $Q_{2^{k-1}}$ have the same characteristic polynomials. By Lemma 2.3, the characteristic polynomial of $P_{2^{k-1}}$ is $x^{2^{k-2}}$ times the characteristic polynomial of $\frac{1}{4}\left(A_{2^{k-2}} + B_{2^{k-2}} + 2I_{2^{k-2}}\right)$. Hence, $p(x) = \left(x^{2^{k-2}}q(x)\right)^2$. $\square$

Let the characteristic polynomial of $\mathfrak{M}_k(1,0)$ be denoted by $\mathcal{C}_k(x)$ such that

$$\mathcal{C}_k(x) := |xI_{2^k} - \mathfrak{M}_k(1,0)|$$

Lemma 4.3. Given positive integer $k \geqslant 2$, the polynomial $\mathcal{C}_k(x)$ satisfies the following recurrence relation

$$\mathcal{C}_k(x) = \frac{x^{2^{k-1}}}{2^{2^{k-2}}}\mathcal{C}_{k-2}\left(2x^2 - 1\right).$$

Proof. We have

$$\begin{aligned}
[\mathcal{C}_k(x)]^2 &= |xI_{2^k} - [\mathfrak{M}_k(1,0)]|^2 \\
&= \left|x^2 I_{2^k} - [\mathfrak{M}_k(1,0)]^2\right|, \text{ (by Lemma 2.5)} \\
&= \left((x^2)^{2^{k-2}}\left|x^2 I_{2^{k-2}} - \frac{1}{4}\left(A_{2^{k-2}} + B_{2^{k-2}} + 2I_{2^{k-2}}\right)\right|\right)^2, \\
&\qquad \text{(by Lemma 4.2)} \\
&= \left(x^{2^{k-1}}\left|\frac{1}{2}\left(2x^2 I_{2^{k-2}} - \frac{1}{2}\left(A_{2^{k-2}} + B_{2^{k-2}} + 2I_{2^{k-2}}\right)\right)\right|\right)^2, \\
&= \left(\frac{x^{2^{k-1}}}{2^{2^{k-2}}}\left|\left(2x^2 - 1\right) I_{2^{k-2}} - \frac{1}{2}\left(A_{2^{k-2}} + B_{2^{k-2}}\right)\right|\right)^2,
\end{aligned}$$

Using the fact that the characteristic polynomial has leading coefficient 1 in our definition, we conclude that

$$|xI_{2^k} - [\mathfrak{M}_k(1,0)]| = \frac{x^{2^{k-1}}}{2^{2^{k-2}}}\left|\left(2x^2 - 1\right) I_{2^{k-2}} - \frac{1}{2}\left(A_{2^{k-2}} + B_{2^{k-2}}\right)\right|.$$

Therefore, $\mathcal{C}_k(x) = \frac{x^{2^{k-1}}}{2^{2^{k-2}}}\mathcal{C}_{k-2}\left(2x^2 - 1\right)$. $\square$

Example 4.1. The matrices $\mathfrak{M}_1(1,0)$, $\mathfrak{M}_2(1,0)$, and $\mathfrak{M}_3(1,0)$ are

$$\mathfrak{M}_1(1,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathfrak{M}_2(1,0) = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix},$$

$$\mathfrak{M}_3(1,0) = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Example 4.2. The first values of $\mathcal{C}_k(\lambda)$ are

$$\begin{aligned}
\mathcal{C}_2(\lambda) &= \left(\lambda^2 - 1\right) \cdot \lambda^2, \\
\mathcal{C}_3(\lambda) &= \left(\lambda^2 - 1\right) \cdot \lambda^6, \\
\mathcal{C}_4(\lambda) &= \left(\lambda^2 - 1\right) \cdot \lambda^{10} \cdot \left(\lambda^2 - \frac{1}{2}\right)^2, \\
\mathcal{C}_5(\lambda) &= \left(\lambda^2 - 1\right) \cdot \lambda^{18} \cdot \left(\lambda^2 - \frac{1}{2}\right)^6, \\
\mathcal{C}_6(\lambda) &= \left(\lambda^2 - 1\right) \cdot \lambda^{34} \cdot \left(\lambda^2 - \frac{1}{2}\right)^{10} \cdot \left(\lambda^4 - x^2 + \frac{1}{8}\right)^2, \\
\mathcal{C}_7(\lambda) &= \left(\lambda^2 - 1\right) \cdot \lambda^{66} \cdot \left(\lambda^2 - \frac{1}{2}\right)^{18} \cdot \left(\lambda^4 - \lambda^2 + \frac{1}{8}\right)^6, \\
\mathcal{C}_8(\lambda) &= \left(\lambda^2 - 1\right) \cdot \lambda^{130} \cdot \left(\lambda^2 - \frac{1}{2}\right)^{34} \cdot \left(\lambda^4 - \lambda^2 + \frac{1}{8}\right)^{10} \\
&\quad \cdot \left(\lambda^8 - 2\lambda^6 + \frac{5}{4}\lambda^4 - \frac{1}{4}\lambda^2 + \frac{1}{128}\right)^2.
\end{aligned}$$

5. A monoid generated by the irreducible Chebyshev polynomials of the first kind

The commutative monoid, generated by $T_2(x) = 2x^2 - 1$ with composition (superposition) of polynomials as the binary operation, will be denoted by $\mathfrak{T}_\circ \subseteq \mathbb{Q}[x]$. We will use the notation $T_1(x) = x$ and $T_{2^{r+1}}(x) = T_{2^r}(T_2(x))$ for the elements of $\mathfrak{T}_\circ$. Let $\mathfrak{T} \subseteq \mathbb{Q}[x]$ be the commutative monoid generated by the rational numbers and the elements from $\mathfrak{T}_\circ$, with the ordinary polynomial product as binary operation. For each $T_{2^r}(x) \in \mathfrak{T}_\circ$, the application $\mathfrak{T} \to \mathfrak{T}$ given by $p(x) \mapsto p(T_{2^r}(x))$ is a well-defined commutative monoid morphism.

Theorem 5.1. Given positive integer k, the characteristic polynomial of $\mathfrak{M}_k(1,0)$ is equal to x^2-1 times an element from $\mathfrak{T}$.

Proof. We proceed by induction on $k \geqslant 1$. The result is true for $k=1$ and $k=2$ by direct computation,

$$\mathcal{C}_1(x) = x^2-1, \qquad \mathcal{C}_2(x) = (x^2-1)\,(T_1(x))^2.$$

Let $k=m$ for some $m \geqslant 3$. Suppose that $\mathcal{C}_{m-2}(x) = (x^2-1)p(x)$ for some $p(x) \in \mathfrak{T}$. By Lemma 4.3,

$$\mathcal{C}_m(x) = \frac{x^{2^{m-1}}}{2^{2^{m-2}}}\mathcal{C}_{m-2}(2x^2-1),$$

which implies

$$\mathcal{C}_m(x) = \frac{(T_1(x))^{2^{m-1}}}{2^{2^{m-2}}}\left(\left(2x^2-1\right)^2-1\right)p(2x^2-1).$$

Using $p(2x^2-1) = p(T_2(x)) \in \mathfrak{T}$ and $\left(\left(2x^2-1\right)^2-1\right) = 4\,(T_1(x))^2\,(x^2-1)$, we conclude $\mathcal{C}_m(x)/(x^2-1) \in \mathfrak{T}$. Therefore, the result is true for all $k \geqslant 1$ by induction. □

Corollary 5.1. For $k \geqslant 0$, the matrix $2\,\mathfrak{M}_k(1,0)$ is nilpotent mod 2, i.e. for some integer $N \geqslant 0$, all the entries of $(2\,\mathfrak{M}_k(1,0))^N$ are even integers.

Proof. By Theorem 5.1, we have

$$|xI_{2^k} - \mathfrak{M}_k(1,0)| = \rho\cdot(x^2-1)T_{2^{r_1}}(x)T_{2^{r_2}}(x)T_{2^{r_3}}(x)\cdots T_{2^{r_h}}(x),$$

where $r_1, r_2, \ldots, r_h$ are some nonnegative integer and ρ is a positive rational number. Substituting x by $x/2$ in the above equation, we obtain $|xI_{2^k} - 2\mathfrak{M}_k(1,0)| =$

$$2^{2^k-h-2}\rho\cdot(x^2-4)T_{2^{r_1}}\left(\frac{x}{2}\right)T_{2^{r_2}}\left(\frac{x}{2}\right)T_{2^{r_3}}\left(\frac{x}{2}\right)\cdots T_{2^{r_h}}\left(\frac{x}{2}\right).$$

Each polynomial $2T_{2^{r_j}}$ for $j=1,2,\ldots,h$ has leading coefficient 1 and the characteristic polynomial of $2\mathfrak{M}_k(1,0)$ has leading coefficient 1 too. Hence, $2^{2^k-h-2}\rho = 1$.

We claim that the non-leading coefficients in $T_{2^{r_j}}$ for $j=1,2,\ldots,h$ are even integers. Indeed, $2T_{2^0}\left(\frac{x}{2}\right) = x$ satisfies this claim. If $2T_{2^r}\left(\frac{x}{2}\right)$ satisfies the claim, then

$$2T_{2^{r+1}}\left(\frac{x}{2}\right) = 2T_{2^r}\left(T_2\left(\frac{x}{2}\right)\right) = 2T_{2^r}\left(2\left(\frac{x}{2}\right)^2-1\right) = 2T_{2^r}\left(\frac{x^2-2}{2}\right)$$

also satisfies the claim. The claim follows by induction.

After reducing the entries of $2\,\mathfrak{M}_k(1,0)$ to $\mathbb{Z}/2\mathbb{Z}$, we obtain that its characteristic polynomial is x^{2^k}. Therefore, the matrix $2\,\mathfrak{M}_k(1,0)$ is nilpotent in $\mathbb{F}_2$. $\square$

Example 5.1. The 15th power of $2\,\mathfrak{M}_{10}(a,b)$ for $(a,b) = (1,0)$ and $(a,b) = (1,2)$ are represented[1] in Figure 1 and Figure 2, respectively. The odd entries correspond to the black points and the even entries, to the white points.

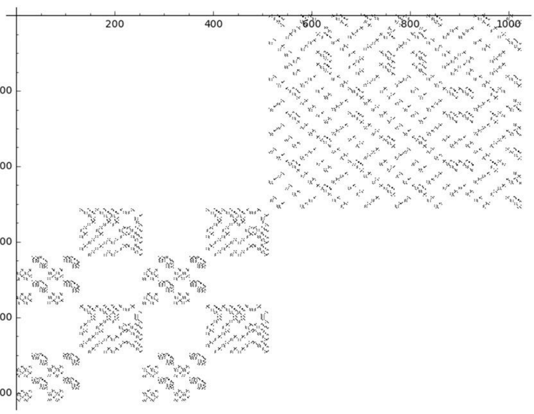

Figure 1. Representation of $(2\,\mathfrak{M}_{10}(1,0))^{15}$.

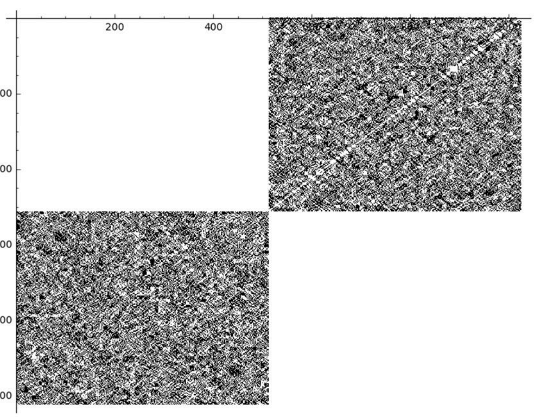

Figure 2. Representation of $(2\,\mathfrak{M}_{10}(1,2))^{15}$.

[1] These pictures were obtained in SageMath using a program created by the authors.

The following common property seem to be true because of the empirical evidences.

Conjecture 5.1. For $a \geqslant 0$, $b \geqslant 0$, and $k \geqslant 0$, the matrix $2\,\mathfrak{M}_k(a,b)$ is nilpotent mod 2, i.e. for some integer $N \geqslant 0$, all the entries of $(2\,\mathfrak{M}_k(a,b))^N$ are even integers.

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