Algebra and Discrete Mathematics
Volume **27** (2019). Number 1, pp. 99–116

RESEARCH ARTICLE

On certain families of sparse numerical semigroups with Frobenius number even

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Communicated by V. Mazorchuk

Abstract. This paper is about sparse numerical semigroups and applications in the Weierstrass semigroups theory. We describe and find the genus of certain families of sparse numerical semigroups with Frobenius number even and we also study the realization of the elements on these families as Weierstrass semigroups.

Introduction

Let $\mathbb{Z}$ be the set of integers numbers and $\mathbb{N}_0$ be the set of non-negative integers. A subset $H = \{0 = n_0(H) < n_1(H) < \cdots\}$ of $\mathbb{N}_0$ is a *numerical semigroup* if its is closed respect to addition and its complement $\mathbb{N}_0 \setminus H$ is finite. The cardinality of the set $\mathrm{Gaps}(H) := \mathbb{N}_0 \setminus H$ is called *genus* of the numerical semigroup H and is denoted by $g = g(H)$. Note that $g(H) = 0$ if and only if $H = \mathbb{N}_0$. If $g(H) > 0$ the elements of $\mathrm{Gaps}(H)$ are called *gaps*. The smallest integer $c = c(H)$ such that $c + h \in H$, for all $h \in \mathbb{N}_0$ is called the *conductor* of H. The least positive integer $n_1 = n_1(H) \in H$ is called the *multiplicity* of H. As $\mathbb{N}_0 \setminus H$ is finite, the set $\mathbb{Z} \setminus H$ has a maximum, which is called *Frobenius number* and will be denoted by $\ell_g = \ell_g(H)$. A property known of this number is that $\ell_g(H) \leqslant 2g - 1$, see

[1]Partially supported by CNPq 446913/2014-6 and FAPEMIG APQ-01607-14.
[2]Partially supported by PNPD/CAPES, by Programa de Pós-doutorado da FAMAT, Câmpus Santa Mônica, Universidade Federal de Uberlândia.
2010 MSC: 20M10, 20M14, 14H55.
Key words and phrases: Arf numerical semigroup, numerical semigroup, sparse numerical semigroup, Weierstrass semigroup, weight of numerical semigroup.

[15]. In particular, $H = \mathbb{N}_0$ if and only if -1 is the Frobenius number of H. Aa a consequence of this fact, from now on we use the notation $\ell_0 = \ell_0(H) := -1$, for all numerical semigroup H. When $g > 0$, we denote $\mathrm{Gaps}(H) = \big\{1 = \ell_1(H) < \cdots < \ell_g(H)\big\}$. So, $c = \ell_g(H) + 1$ and is clear that $c = n_{c-g}(H)$. For simplicity of notation we shall write ℓ_i for $\ell_i(H)$ and n_k for $n_k(H)$, for all integers i, k such that $0 \leqslant i \leqslant g$ and $k \geqslant 0$, when there is no danger of confusion. More details about numerical semigroups theory, see e.g. [17].

Currently, there are several families that have been of interest in the literature due to their properties and applications. Examples of such families are the sparse semigroups, which were introduced in [14]. A numerical semigroup $H = \{0 = n_0 < n_1 < \cdots\}$ of genus $g > 0$ with $\mathrm{Gaps}(H) = \{\ell_1 < \cdots < \ell_g\}$ is called *sparse numerical semigroup* if $\ell_i - \ell_{i-1} \leqslant 2$, for all integer i such that $1 \leqslant i \leqslant g$, or equivalently $n_i - n_{i-1} \geqslant 2$, for all integer i such that $1 \leqslant i \leqslant c - g$, where c is the conductor of H. For convenience, we considerer the numerical semigroup $\mathbb{N}_0$ as sparse. Thus, the concept of sparse numerical semigroups it's in a way a generalization of the concept of *Arf numerical semigroups*, which was introduced in [1].

Among other applications, the study of numerical semigroup is related to Algebraic Geometry in the treatment of algebraic curves and their Weierstrass semigroups. More explicitly, given a numerical semigroup H, does it exist a curve $\mathcal{X}$ such that for some point $P \in \mathcal{X}$ has $H = H(P)$?, where $H(P)$ is the Weierstrass semigroup of $\mathcal{X}$ at P. If the answer is yes, we say that the numerical semigroup H is Weierstrass. Studies to answer this question have been done for decades, see e.g. [4], [12], [15] and [19]. From a geometrical point of view, sparse numerical semigroups are closely related to Weierstrass semigroups arising in double covering of curves, cf. [19]. Its arithmetical structure is strongly influenced by the parity of ℓ_g. In this work, we study certain families of sparse numerical semigroups which are examples of Weierstrass semigroups. In addition, we manage to describe and find the genus of the semigroups on these families. These aspects are very important in the study of numerical semigroups theory. It is important to note that in [6] the authors find an upper bound for the genus of sparse numerical semigroups with Frobenius number even. Here, in this paper, we get a better bound for the genus of the semigroups on the families of sparse numerical semigroups studied.

This paper is organized as follows. Section 1, contains basic concepts about numerical semigroups and backgrounds for the next sections. In Section 2, we study certain families of sparse numerical semigroups with

Frobenius number even. As the main results, we describe and find the genus of the semigroups on these families of sparse numerical semigroups. Finally, in Section 3 we study the realization of the sparse numerical semigroups determined in the previous section as Weierstrass semigroups.

1. Preliminaries

1.1. Basic concepts

Let $H = \{0 = n_0 < n_1 < \cdots\}$ be a numerical semigroup of genus $g > 0$ and $\mathrm{Gaps}(H) = \{\ell_1 < \cdots < \ell_g\}$. For each $1 \leqslant i \leqslant g$, the ordered pair (ℓ_{i-1}, ℓ_i) will be called *leap on* H (or simply *leap*). The set of leaps on H will be denoted by

$$\mathcal{V} = \mathcal{V}(H) := \{(\ell_{i-1}, \ell_i) : 1 \leqslant i \leqslant g\}.$$

Note that $|\mathcal{V}| = g$. The ordered pair (ℓ_{i-1}, ℓ_i) will be called *single leap* if $\ell_i - \ell_{i-1} = 1$ and *double leap* if $\ell_i - \ell_{i-1} = 2$.

Based on this set, for a positive integer m, let us define the subset

$$\mathcal{V}_m = \mathcal{V}_m(H) := \{(\ell_{i-1}, \ell_i) : \ell_i - \ell_{i-1} = m,\ 1 \leqslant i \leqslant g\}$$

and for an interval $[a, b]$, with $-1 \leqslant a < b \leqslant \ell_g$, let us define the subset

$$\mathcal{V}_{[a,b]} = \{(\ell_{i-1}, \ell_i) : \ell_{i-1}, \ell_i \in [a, b],\ 1 \leqslant i \leqslant g\}.$$

For convenience, we define $\mathcal{V}_m(\mathbb{N}_0) := \varnothing$, for all positive integer m. To simplify the notation we will denote the cardinality of the set $\mathcal{V}_m(H)$ by $v_m(H)$, that is, $v_m = v_m(H) := |\mathcal{V}_m(H)|$.

1.2. Arf numerical semigroup

A numerical semigroup $H = \{0 = n_0 < n_1 < \cdots\}$ is called *Arf numerical semigroup* if

$$n_i + n_j - n_k \in H, \tag{1}$$

for all integers i, j, k such that $0 \leqslant k \leqslant j \leqslant i$, or equivalently $n_i + n_j - n_k \in H$, for all integers i, j, k such that $0 \leqslant k \leqslant j \leqslant i \leqslant c - g$, where c is the conductor and g is the genus of H, respectively. The Arf numerical semigroups was introduced in [1]. For more details about this family of numerical semigroups, see e.g. [2], [5], [16] and [18].

If $g > 0$ and $\mathrm{Gaps}(H) = \{\ell_1 < \cdots < \ell_g\}$, the Arf property (1) implies that

$$\ell_i - \ell_{i-1} \leqslant 2, \tag{2}$$

for all integer i such that $1 \leqslant i \leqslant g$, or equivalently $n_i - n_{i-1} \geqslant 2$, for all integer i such that $1 \leqslant i \leqslant c-g$ (see [14, Corollary 1] and [20, Corollary 2.1.4]). Using this property is not difficult to see that if H is an Arf numerical semigroup, then H is a sparse numerical semigroup.

For each non-negative integer g, let

$$\mathbb{N}_g := \{0\} \cup \{n \in \mathbb{N} : n \geqslant g+1\}$$

(in the case $g = 0$, it notation is itself the $\mathbb{N}_0$). It is clear that $\mathbb{N}_g$ is a numerical semigroup of genus g. The semigroup $\mathbb{N}_g$ is called *ordinary numerical semigroup* and is a canonical example of Arf numerical semigroups and, consequently, a example of sparse numerical semigroups.

1.3. Sparse numerical semigroups

Theorem 1.1 ([18], Theorem 2.2)**.** *Let H be a numerical semigroup of genus g.*

(1) *H is an sparse numerical semigroup if and only if $v_1(H)+v_2(H)=g$. In this case, $v_m(H)=0$, for all positive integer $m \geqslant 3$.*
(2) *If H is an sparse numerical semigroup, then the Frobenius number of H is $\ell_g = v_1(H)+2v_2(H)-1$.*
(3) *If $g>0$ and $\ell_g(H)=2g-K$, for some positive integer K, then H is an sparse numerical semigroup if and only if $v_1(H)=K-1$ and $v_2(H)=g-K+1$.*

For a numerical semigroup $H=\{0=n_0<n_1<\cdots\}$, define $M = M(H) := n_1-1$. The parameter M was introduced in [14], where if $\ell_g(H)=2g-K$, for some positive integer K, we have that

$$0 \leqslant M \leqslant K. \tag{3}$$

If $g>1$ and $\mathrm{Gaps}(H)=\{1=\ell_1<\cdots<\ell_g\}$, define

$$S_M = S_M(H) := \left|\left\{(\ell_{i-1},\ell_i) : \ell_i-\ell_{i-1}=1,\ M+1 \leqslant i \leqslant g\right\}\right|. \tag{4}$$

Note that $S_M < v_1$, since $\left\{(\ell_{i-1},\ell_i) : \ell_i-\ell_{i-1}=1,\ M+1 \leqslant i \leqslant g\right\} \subsetneq \mathcal{V}_1$.

Before the next result, let us remember that a semigroup H is called *γ-hyperelliptic* if it has exactly γ even gaps. If $\gamma=0$, H is called simply *hyperelliptic.*

Proposition 1.2. *Let $H=\{0=n_0<n_1<\cdots\}$ be a numerical semigroup of genus g and $M=M(H)=n_1-1$. Then*:

(1) $0 \leqslant M \leqslant g$;

(2) *$M(H) = 0$ if and only if $H = \mathbb{N}_0$;*
(3) *$M(H) = g$ if and only if $H = \mathbb{N}_g$;*
(4) *$M(H) = 1$ if and only if H is hyperelliptic;*
(5) *If $g > 0$ and $\operatorname{Gaps}(H) = \{\ell_1 < \cdots < \ell_g\}$, then M is the largest integer such that $\ell_M = M$.*

Proof. The assertions (1), (2) and (3) are clear. Item (4), follows from the fact that $M = 1$ if and only if $n_1 = 2$. Finally, item (5) follows directly from the fact that all integers belong to interval $[1, n_1 - 1]$ are gaps. □

Corollary 1.3. *Let H be a sparse numerical semigroup of genus $g > 1$ with Frobenius number $\ell_g = 2g - K$, for some positive integer K, and $\operatorname{Gaps}(H) = \{\ell_1 < \cdots < \ell_g\}$. Let $M = M(H)$ and $S_M = S_M(H)$ be defined in (4). If $M < g$, then $S_M = K - M$.*

Proof. By the definition of S_M, since $M < g$, we have that $S_M \neq 0$ and $S_M = v_1(H) - (M - 1)$. Now, by Theorem 1.1 item (2), follows that $S_M = K - 1 - (M - 1) = K - M$. □

In Proposition 1.2, we present the semigroups for the cases $M = 0, 1$ and $M = g$. The case $M = 2$ is treated in [14, Proposition 3]. The following result treat of the case $M = K$.

Theorem 1.4. *Let H be a numerical semigroup of genus $g > 0$ with Frobenius number $\ell_g = 2g - K$, for some positive integer K. The following statements are equivalent:*
(1) *H is an sparse numerical semigroup and $M = K$;*
(2) *$H = \mathbb{N}_K$ or $H = \{K + 2i - 1 : 1 \leqslant i \leqslant g - K\} \cup \mathbb{N}_{2g-K}$;*
(3) *$v_1(H) = M - 1$ and $v_2(H) = g - M + 1$.*
In this case, $\operatorname{Gaps}(H) = \{1, \ldots, \ell_K, \ldots, \ell_g\}$, where $\ell_i = 2i - K$, for all integer i such that $K \leqslant i \leqslant g$.

Proof. We prove that (1) ⇒ (2) ⇒ (3) ⇒ (1). By [14, Theorem 1 (2)] follows that (1) ⇒ (2). Now, suppose (2). If $H = \mathbb{N}_K$, then $M = K$ and by Theorem 1.1 (3) follows that $v_1(H) = M - 1$ and $v_2(H) = g - M + 1$. If $H = \{K + 2i - 1 : 1 \leqslant i \leqslant g - K\} \cup \mathbb{N}_{2g-K}$, then $\operatorname{Gaps}(H) = \{1, \ldots, K, \ell_{K+1}, \ldots, \ell_g\}$, where $\ell_{K+i} = K + 2i$, for all integer i such that $1 \leqslant i \leqslant g - K$. So, $M = K$, $v_1(H) = K - 1 = M - 1$ and $v_2(H) = g - K + 1 = g - M + 1$. Thus, we have (2) ⇒ (3). The implication (3) ⇒ (1), follows directly from the Theorem 1.1 items (1) and (3).

The assertion on the gaps of H follows as in the proof of the implication (2) ⇒ (3). □

2. Sparse numerical semigroups with Frobenius number even

For each positive integer k, consider the family $\mathscr{H}_k^{\mathrm{sfe}}$ of sparse numerical semigroups H with genus $g = g(H)$ and Frobenius number even of the form $2g - 2k$. That is,

$$\mathscr{H}_k^{\mathrm{sfe}} := \{H \colon H \text{ is a sparse numerical semigroup} \\ \text{with genus } g \text{ and } \ell_g = 2g - 2k\}.$$

In this section, we will study the classification of the elements of some proper subsets in $\mathscr{H}_k^{\mathrm{sfe}}$ as well as the cardinality of these subsets. If $H \in \mathscr{H}_k^{\mathrm{sfe}}$ in [14, Theorem 2] is proved that $g(H) \leqslant 6k - 3$ and that the family $\mathscr{H}_k^{\mathrm{sfe}}$ is finite. In [20, Question 2.3.10], was conjectured that $g(H) \leqslant 4k - 1$. This conjecture was proved by Contiero, Moreira and Veloso in [6, Corollary 3.7]. In Theorem 2.8 and Theorem 2.13, we get a better bound for $g(H)$ for certain $H \in \mathscr{H}_k^{\mathrm{sfe}}$.

In the previous section, in particular, we describe all the sparse numerical semigroups belongs to $\mathscr{H}_k^{\mathrm{sfe}}$ with $M = 0, 1$ and $M = 2k$. The next result describe the case $M = 3$.

Lemma 2.1. *Let $k \geqslant 2$ be an integer and H be a numerical semigroup of genus g and Frobenius number $\ell_g = 2g - 2k$. If $M = 3$, the following statements are equivalent*:
(1) *H is be an Arf numerical semigroup*;
(2) *H is be an sparse numerical semigroup*;
(3) *$H = \langle 4, 4k - 1, 4k + 1, 4k + 2\rangle$*;
(4) *H is k-hyperelliptic.*
In this case, $g = 3k - 1$ and

$$\mathrm{Gaps}(H) = \{1, \ldots, 2g - 2k\} \setminus \left\{4i \,:\, i \in \mathbb{N}, 1 \leqslant i \leqslant \tfrac{2g-2k-2}{4}\right\}.$$

Proof. We prove that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (1). Implication (1) $\Rightarrow$ (2) is always true. Before show the next implications, since $M = 3$, we have that the multiplicity of H is $n_1 = 4$. Let $G := \{1, \ldots, 2g - 2k\} \setminus \left\{4i \,:\, i \in \mathbb{N}, 1 \leqslant i \leqslant \frac{2g-2k-2}{4}\right\}$. It is clear that $\mathrm{Gaps}(H)$ is a subset of G. Since $4 \in H$ and $\ell_g = 2g - 2k$ is even, we have that $\ell_g = 4i_0 + 2$, for some $i_0 \in \mathbb{N}$. Note that if $\ell \in G$ is even, then $\ell = 4i + 2$, for some $i \in \mathbb{N}_0$ with $i \leqslant i_0$. So, $\ell \in G$ and ℓ even implies that $\ell \notin H$, because otherwise we will have that $\ell_g \in H$, since $\ell = 4i + 2$, $4 \in H$, $\ell_g = 4i_0 + 2$ and $i \leqslant i_0$.

Now, suppose (2). Let $\ell \in G$. If ℓ is even, we have seen above that $\ell \in \mathrm{Gaps}(H)$. If $\ell \equiv 1 \pmod 4$, then $\ell - 1 \in H$ and we have that $\ell \in \mathrm{Gaps}(H)$, since H is sparse. If $\ell \equiv 3 \pmod 4$, then $\ell + 1 \in H$ and we also have that $\ell \in H$, since H is sparse. Thus, $\mathrm{Gaps}(H) = G$, and so $H = \langle 4, 4k-1, 4k+1, 4k+2\rangle$. Thus, we have (2) $\Rightarrow$ (3).

Now, we prove the implication (3) $\Rightarrow$ (4). If $H = \langle 4, 4k-1, 4k+1, 4k+2\rangle$, then $2 \in \mathrm{Gaps}(H)$ and the number of even gaps on the interval $[4, 4k]$ is equal to $\frac{4k-4}{4} = k - 1$. So, H is k-hyperelliptic.

To prove the implication (4) $\Rightarrow$ (3), first note that $G \cap 2\mathbb{N} = \mathrm{Gaps}(H) \cap 2\mathbb{N}$. Suppose that H is k-hyperelliptic. Then, $G \cap 2\mathbb{N} = \mathrm{Gaps}(H) \cap 2\mathbb{N}$ and $|\, G \cap 2\mathbb{N}\,| = g - k - \frac{2g-2k-2}{4}$ implies that $g - k - \frac{2g-2k-2}{4} = k$, that is, $g = 3k - 1$. Since $M = 3$, $g = 3k - 1$ and $\ell_g = 2g - 2k = 4k - 2$, we conclude that $\mathrm{Gaps}(H) = [1, 4k-2] \setminus \{4i : i \in \mathbb{N}, 1 \leqslant i \leqslant k-1\}$, and then $H = \langle 4, 4k-1, 4k+1, 4k+2\rangle$.

Finally, we prove the implication (3) $\Rightarrow$ (1). If $H = \langle 4, 4k-1, 4k+1, 4k+2\rangle$, then $H = 4\mathbb{N}_0 \cup \{n \in \mathbb{N} : n \geqslant 4k-1\}$. Suppose that $H = \{0 = n_0 < 4 = n_1 < n_2 < \cdots\}$ and let i, j, s integers such that $0 \leqslant s \leqslant j \leqslant i$. If n_i, n_j or n_s belongs to $\{n \in \mathbb{N} : n \geqslant 4k-1\}$, then it is clear that $n_i + n_j - n_s \in \{n \in \mathbb{N} : n \geqslant 4k-1\}$ and so $n_i + n_j - n_s \in H$. If $n_i, n_j, n_s \in 4\mathbb{N}_0$ then it is also clear that $n_i + n_j - n_s \in 4\mathbb{N}_0$ and so $n_i + n_j - n_s \in H$. Therefore, H is Arf. $\square$

For $H \in \mathscr{H}_k^{\mathrm{sfe}}$, by Equation (3) and Proposition 1.2, follows that

$$2 \leqslant M(H) \leqslant 2k. \tag{5}$$

For each integer J such that $0 \leqslant J \leqslant 2k-2$, consider the set $\mathscr{H}_{2+J}^k$ of sparse numerical semigroups $H \in \mathscr{H}_k^{\mathrm{sfe}}$ such that $M(H) = 2 + J$. That is,

$$\mathscr{H}_{2+J}^k = \Big\{H \in \mathscr{H}_k^{\mathrm{sfe}} : M(H) = 2 + J\Big\}. \tag{6}$$

Since

$$\mathscr{H}_k^{\mathrm{sfe}} = \dot{\bigcup_{J=0}^{2k-2}} \mathscr{H}_{2+J}^k,$$

in order to study the cardinality of $\mathscr{H}_k^{\mathrm{sfe}}$ it is enough to study the cardinality of $\mathscr{H}_{2+J}^k$, for all integer J such that $0 \leqslant J \leqslant 2k-2$. Next we will study the cases: $J = 2k-2$, $J = 2k-3$ and $J = 2k-4$.

The following result shows that all numerical semigroups H belongs to $\mathscr{H}_{2k}^k$ are Arf and $(g(H) - k)$-hyperelliptic.

Proposition 2.2. *Let H be a numerical semigroup of genus $g>0$ with Frobenius number $\ell_g = 2g-2k$, for some positive integer k. If $M=2k$, then the following statements are equivalent*:
(1) *H is an Arf numerical semigroup*;
(2) *H is an sparse numerical semigroup*;
(3) *$H = \mathbb{N}_{2k}$ or $H=\{2k+2i-1 : 1\leqslant i\leqslant g-2k\}\cup\mathbb{N}_{2g-2k}$*;
(4) *$v_1(H)=2k-1$ and $v_2(H)=g-2k+1$*;
(5) *H is $(g-k)$-hyperelliptic.*
In this case, $2k\leqslant g\leqslant 3k$ and $\mathrm{Gaps}(H)=\{1,\ldots,2k-1,\ell_{2k},\ldots,\ell_g\}$, *where $\ell_i = 2(i-k)$, for all integer i such that $2k\leqslant i\leqslant g$.*

Proof. We prove that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4), (3) $\Leftrightarrow$ (5) and (3) $\Rightarrow$ (1). The implication (1) $\Rightarrow$ (2) is always true. The implication (2) $\Rightarrow$ (3), the equivalence (3) $\Leftrightarrow$ (4) and the assertion on the gaps of H is a particular case from Theorem 1.4, by taking $M=K=2k$.

Now, suppose (3). Then, $n_i = 2k+2i-1$, for all integer i such that $1\leqslant i\leqslant g-2k+1$, Let i,j integers such that $0\leqslant j\leqslant i\leqslant g-2k+1$. If $j=0$ is clear that $2n_i-n_j\in H$; other case $2n_i-n_j = 2k+4i-2j-1 \geqslant 2k+2i-1\geqslant 2k+1>\ell_{2k}$. Since in the interval $[\ell_{2k},\ell_g]$ the gaps ℓ_i's are even numbers and $2n_i-n_j$ is odd, follow that $2n_i-n_j\in H$. This shows that (3) $\Rightarrow$ (1). Also, by Theorem 1.4, (3) implies that $\mathrm{Gaps}(H) = \{1,\ldots,2k-1,\ell_{2k},\ldots,\ell_g\}$, where $\ell_i=2(i-k)$, for all integer i such that $2k\leqslant i\leqslant g$, and thus (3) $\Rightarrow$ (5). Finally, suppose (5). Since in the interval $[2,2g-2k]$ there are $g-k$ elements even, these elements should be gaps and so (5) $\Rightarrow$ (3).

Note that, since $n_1 = 2k+1$, from item (3) follows that $2(2k+1)\geqslant \ell_g+2 = (2g-2k)+2$. Therefore, we have that $g\leqslant 3k$. $\square$

Theorem 2.3. *For all integer $k\geqslant 1$,*

$$\mathscr{H}^k_{2k} = \{\mathbb{N}_{2k}\}\cup\left\{H_{(k,r)} : r\in\mathbb{N},\ 1\leqslant r\leqslant k\right\},$$

where

$$H_{(k,r)} = \{2k+2i-1 : i\in\mathbb{N},\ 1\leqslant i\leqslant r\}\cup\mathbb{N}_{2k+2r},$$

for $1\leqslant r\leqslant k$. In addition, $g\left(H_{(k,r)}\right) = 2k+r$. In particular, the cardinality of $\mathscr{H}^k_{2k}$ is $\left|\mathscr{H}^k_{2k}\right| = k+1$.

Proof. For each integer r such that $1\leqslant r\leqslant k$, let $H_{(k,r)} = \{2k+2i-1 : i\in\mathbb{N},\ 1\leqslant i\leqslant r\}\cup\mathbb{N}_{2k+2r}$. By the definition, is clear that $H_{(k,r)}$ is a numerical semigroup of genus $g\left(H_{(k,r)}\right) = 2k+r$ and $\ell_g\left(H_{(k,r)}\right) = 2g\left(H_{(k,r)}\right)-2k$.

So, by Proposition 2.2, we have that $H_{(k,r)}$ belongs to $\mathscr{H}_{2k}^{k}$, for all integer r such that $1 \leqslant r \leqslant k$. Also, by Proposition 2.2, $\mathbb{N}_{2k}$ is contained in $\mathscr{H}_{2k}^{k}$.

Now, let $H \in \mathscr{H}_{2k}^{k}$ of genus g and $H \neq \mathbb{N}_{2k}$. By Proposition 2.2, we have $2k+1 \leqslant g \leqslant 3k$ and $H = \{2k+2i-1 : i \in \mathbb{N}, 1 \leqslant i \leqslant g-2k\} \cup \mathbb{N}_{2g-2k}$. Let $r := g-2k$. Then, $1 \leqslant r \leqslant k$ and so $H = H_{(k,r)}$. This completes the proof. $\square$

A numerical semigroup H of genus $g > 0$ is called *quasi-symmetric* if the Frobenius number of H is $\ell_g = 2g-2$. In [2] and [3], has been proved that there are only two Arf numerical semigroups quasi-symmetric: $\langle 3,4,5\rangle$ and $\langle 3,5,7\rangle$. In [14, Example 4] and [20, Example 2.3.13], was proved that Arf numerical semigroups quasi-symmetric is equivalent to sparse numerical semigroups quasi-symmetric. This result can be obtained directly from the previous theorem by taking $k=1$. That is, the family of sparse numerical semigroups quasi-symmetric is

$$\mathscr{H}_1^{\mathrm{sfe}} = \big\{\mathbb{N}_2, \{3\} \cup \mathbb{N}_4\big\} = \mathscr{H}_2^1.$$

The following is a further demonstration of the [14, Example 5] about the classification of sparse numerical semigroups of genus g with Frobenius number $\ell_g = 2g-4$.

Corollary 2.4. *The family of sparse numerical semigroups of genus g and Frobenius number $2g-4$ is*

$$\mathscr{H}_2^{\mathrm{sfe}} = \big\{\{3,6\} \cup \mathbb{N}_8, \{3,6,9\} \cup \mathbb{N}_{10}, \{4\} \cup \mathbb{N}_6, \mathbb{N}_4, \{5\} \cup \mathbb{N}_6, \{5,7\} \cup \mathbb{N}_8\big\}.$$

In this case,

$$\begin{aligned}
\mathscr{H}_2^2 &= \big\{\{3,6\} \cup \mathbb{N}_8, \{3,6,9\} \cup \mathbb{N}_{10}\big\},\\
\mathscr{H}_3^2 &= \big\{\{4\} \cup \mathbb{N}_6\big\},\\
\mathscr{H}_4^2 &= \big\{\mathbb{N}_4, \{5\} \cup \mathbb{N}_6, \{5,7\} \cup \mathbb{N}_8\big\}.
\end{aligned}$$

Proof. First, note that $\mathscr{H}_2^{\mathrm{sfe}} = \mathscr{H}_2^2 \cup \mathscr{H}_3^2 \cup \mathscr{H}_4^2$. From [14, Proposition 3], we have that $\mathscr{H}_2^2 = \big\{\{3,6\} \cup \mathbb{N}_8, \{3,6,9\} \cup \mathbb{N}_{10}\big\}$. From the item (3) of Lemma 2.1 follows that $\mathscr{H}_3^2 = \big\{\{4\} \cup \mathbb{N}_6\big\}$ and taking $k=2$ in the previous theorem follows that $\mathscr{H}_4^2 = \big\{\mathbb{N}_4, \{5\} \cup \mathbb{N}_6, \{5,7\} \cup \mathbb{N}_8\big\}$. $\square$

Lemma 2.5. *Let $k \geqslant 2$ be an integer. For each pair of integers r and α_r such that $1 \leqslant r \leqslant k-1$ and $1 \leqslant \alpha_r \leqslant r$, let*

$$\begin{aligned}
H_{(k,r)}^{\alpha_r} := {} & \{2(k+\mu-1) : 1 \leqslant \mu \leqslant \alpha_r\} \cup \\
& \{2k+2\nu+1 : \alpha_r \leqslant \nu \leqslant r-1\} \cup \mathbb{N}_{2k+2r}.
\end{aligned}$$

Then, $H^{\alpha_r}_{(k,r)} \in \mathscr{H}^{\,k}_{2k-1}$ *and* $g\left(H^{\alpha_r}_{(k,r)}\right) = 2k + r$*, for all pairs* (r, α_r)*.*

Proof. Let r, α_r and $H^{\alpha_r}_{(k,r)}$ be as above. By the definition, it is clear that $H^{\alpha_r}_{(k,r)}$ is a numerical semigroup with set of gaps given by

$$\begin{aligned}\mathrm{Gaps}\left(H^{\alpha_r}_{(k,r)}\right) = \{i : 1 \leqslant i \leqslant 2k-1\} \cup \{2k + 2\mu - 1 : 1 \leqslant \mu \leqslant \alpha_r\} \cup \\ \{2k + 2\nu : \alpha_r \leqslant \nu \leqslant r\}.\end{aligned}$$

In particular, $g\left(H^{\alpha_r}_{(k,r)}\right) = (2k-1) + \alpha_r + (r - \alpha_r + 1) = 2k + r$. Also, $M\left(H^{\alpha_r}_{(k,r)}\right) = 2k - 1$ and $\ell_g\left(H^{\alpha_r}_{(k,r)}\right) = 2k + 2r = 2g - 2k$. Therefore, $H^{\alpha_r}_{(k,r)} \in \mathscr{H}^{\,k}_{2k-1}$, for all pairs (r, α_r) as above. $\square$

Henceforth $H^{\alpha_r}_{(k,r)}$ is the sparse numerical semigroup defined in the Lemma 2.5.

Remark 2.6. Let $k \geqslant 2$ be an integer and (r, α_r) be as defined in the previous lemma. Then,

$$\mathrm{Gaps}\left(H^{\alpha_r}_{(k,r)}\right) = \left\{\ell_1\left(H^{\alpha_r}_{(k,r)}\right), \ldots, \ell_{2k+r}\left(H^{\alpha_r}_{(k,r)}\right)\right\},$$

where

$$\ell_i\left((H^{\alpha_r}_{(k,r)}\right) = \begin{cases} i, & \text{if } 1 \leqslant i \leqslant 2k-1; \\ 2(i-k)+1, & \text{if } 2k \leqslant i \leqslant 2k-1+\alpha_r; \\ 2(i-k), & \text{if } 2k+\alpha_r \leqslant i \leqslant 2k+r. \end{cases}$$

The following theorem will give us a new bound for the genus of semigroups in $\mathscr{H}^k_{2k-1}$. Before, we will make an observation that will be very useful in the following.

Remark 2.7. Let H be a sparse numerical semigroup of genus $g > 1$ with Frobenius number $\ell_g = 2g - K$, for some positive integer K, and $\mathrm{Gaps}(H) = \{\ell_1 < \cdots < \ell_g\}$. Let r and s integers such that $1 \leqslant r < s \leqslant g$. If $\mathcal{V}_{[\ell_r, \ell_s]} \cap \mathcal{V}_1 = \left\{(\ell_{j-1}, \ell_j)\right\}$, for some $j \in \{r+1, \ldots, s\}$, then

$$\ell_v = \ell_{j-1} - 2(j - v - 1),$$

for all integer v such that $r \leqslant v \leqslant j - 1$, and

$$\ell_w = \ell_j + 2(w - j),$$

for all integer w such that $j \leqslant w \leqslant s$. In particular, $\ell_{j-1} \equiv \ell_v \pmod 2$, for all integer v such that $r \leqslant v \leqslant j-1$, and $\ell_j \equiv \ell_w \pmod 2$, for all integer w such that $j \leqslant w \leqslant s$.

Theorem 2.8. *Let $k \geqslant 2$ be an integer. If $H \in \mathscr{H}^k_{2k-1}$, then*

$$2k+1 \leqslant g(H) \leqslant 3k-1.$$

Proof. Let $k \geqslant 2$ be an integer and let $H \in \mathscr{H}^k_{2k-1}$ with $g(H) = g$ and $\mathrm{Gaps}(H) = \{1 = \ell_1 < \cdots < \ell_g\}$. Since H is a sparse semigroup and $M = 2k-1$, we have that $\ell_{M+1} = 2k+1$. If $g \leqslant 2k$, then $\ell_g \leqslant 2k$, a contradiction with the value of ℓ_{M+1}. So, $g \geqslant 2k+1$. Now, suppose that $g = 3k+s$, for some integer $s \geqslant 0$. So, $\ell_g = 4k+2s$. By the Corollary 1.3, $S_M = 1$, that is, there exist a unique single leap (ℓ_{j-1}, ℓ_j), for some integer j such that $2k+1 = M+2 \leqslant j \leqslant g$. Then, by the Remark 2.7, all the even numbers greater than $2k+1$ and smaller than ℓ_{j-1} are non-gaps. Moreover all the even numbers from ℓ_j to ℓ_g are gaps. More precisely, we have $\ell_g = \ell_j + 2(g-j)$. So, $\ell_j = (4k+2s) - 2(3k+s-j) = 2j-2k$. We conclude the proof by studying the following cases separately: $j < 3k$ and $j \geqslant 3k$. Firstly, assume that $j < 3k$. Then, $\ell_j = 2j-2k < 4k \leqslant 4k+2s$. So, $4k$ is a gap, a contradiction since $2k$ is a non-gap. Now, suppose that $j \geqslant 3k$. Then, $2k \leqslant 2j-4k < 2j-2k-1 = \ell_{j-1}$. Thus, $2j-4k \in H$. Therefore, $\ell_j = 2j-2k = 2k+(2j-4k) \in H$, a contradiction, and the proof is complete. $\square$

Theorem 2.9. *For all integer $k \geqslant 2$,*

$$\mathscr{H}^k_{2k-1} = \Big\{H^{\alpha_r}_{(k,r)} : (r, \alpha_r) \in \mathbb{N}^2,\ 1 \leqslant r \leqslant k-1,\ 1 \leqslant \alpha_r \leqslant r\Big\}.$$

Proof. Firstly, by the Lemma 2.5, we have that

$$\Big\{H^{\alpha_r}_{(k,r)} : (r, \alpha_r) \in \mathbb{N}^2,\ 1 \leqslant r \leqslant k-1,\ 1 \leqslant \alpha_r \leqslant r\Big\} \subset \mathscr{H}^k_{2k-1}.$$

Now, let $H \in \mathscr{H}^k_{2k-1}$ with $g(H) = g$. By Theorem 2.8, we have $2k+1 \leqslant g \leqslant 3k-1$. Let $r := g-2k$. So, $1 \leqslant r \leqslant k-1$. Since $\ell_g = 2g-2k$, we have that $\ell_g = \ell_{2k+r} = 2k+2r$. Let $\mathrm{Gaps}(H) = \{1 = \ell_1 < \cdots < \ell_g\}$. By Corollary 1.3, $S_M = 1$, that is, there exist a unique single leap (ℓ_{j-1}, ℓ_j), for some integer j such that $2k+1 = M+2 \leqslant j \leqslant g = 2k+r$. Let $\alpha_r := j-2k$. By Remark 2.7, $\ell_v = \ell_{2k} + 2(v-2k)$, for all integer v such that $2k \leqslant v \leqslant j-1$ and $\ell_w = \ell_{2k+r} + 2\big[w-(2k+r)\big]$, for all integer w such that $j \leqslant w \leqslant 2k+r$. Since $M = 2k-1$, we have $\ell_{2k-1} = 2k-1$ and $\ell_{2k} = 2k+1$. Thus, we concluded that $\ell_v = 2(v-k)+1$, for all integer v such that $2k \leqslant v \leqslant j-1 = 2k-1+\alpha_r$, and $\ell_w = 2(w-k)$, for all integer w such that $2k+\alpha_r = j \leqslant w \leqslant 2k+r = g$. Therefore, by Remark 2.6, we have that $H = H^{\alpha_r}_{(k,r)}$ and follows the result. $\square$

Corollary 2.10. *For all integer* $k \geqslant 2$, $\left|\mathscr{H}^k_{2k-1}\right| = \binom{k}{2}$.

Proof. By the Theorem 2.9, we have that

$$\mathscr{H}^k_{2k-1} = \dot{\bigcup_{r=1}^{k-1}} \left\{ H^{\alpha_r}_{(k,r)} : \alpha_r \in \mathbb{N},\ 1 \leqslant \alpha_r \leqslant r \right\}.$$

Therefore,

$$\left|\mathscr{H}^k_{2k-1}\right| = \sum_{r=1}^{k-1} r = \binom{k}{2}. \qquad \square$$

Lemma 2.11. *Let* $k \geqslant 3$ *be an integer. Let* r, s *and* α_s *be a triple of integers such that* $1 \leqslant r \leqslant k-2$, $1 \leqslant s \leqslant r$ *and* $1 \leqslant \alpha_s \leqslant r-s+1$. *For* $r \in \{k, k+1\}$, *let* $s = 1$ *and* $\alpha_s = k$. *For each triple* (r, s, α_s), *consider*

$$\begin{aligned} H^{(s,\alpha_s)}_{(k,r)} :=& \{2k + 2\lambda - 3 : 1 \leqslant \lambda \leqslant s\} \cup \\ & \{2(k + s + \mu - 1) : 1 \leqslant \mu \leqslant \alpha_s - 1\} \cup \\ & \{2k + 2\nu + 2s - 1 : \alpha_s \leqslant \nu \leqslant r - s\} \cup \mathbb{N}_{2k+2r}. \end{aligned}$$

Then, $H^{(s,\alpha_s)}_{(k,r)} \in \mathscr{H}^k_{2k-2}$ *and* $g\left(H^{(s,\alpha_s)}_{(k,r)}\right) = 2k + r$, *for all triples* (r, s, α_s).

Proof. Let r, s, α_s and $H^{(s,\alpha_s)}_{(k,r)}$ be as above. By the definition, it is clear that $H^{(s,\alpha_s)}_{(k,r)}$ is a numerical semigroup with set of gaps given by

$$\begin{aligned} \mathrm{Gaps}\left(H^{(s,\alpha_s)}_{(k,r)}\right) =& \{i : 1 \leqslant i \leqslant 2k-2\} \,\dot\cup\, \{2(k + \lambda - 1) : 1 \leqslant \lambda \leqslant s\} \,\dot\cup \\ & \{2k + 2\mu + 2s - 3 : 1 \leqslant \mu \leqslant \alpha_s\} \,\dot\cup \\ & \{2(k + \nu + s) : \alpha_s - 1 \leqslant \nu \leqslant r - s\}. \end{aligned}$$

In particular, $g\left(H^{(s,\alpha_s)}_{(k,r)}\right) = (2k-2) + s + \alpha_s + (r - s - \alpha_s + 2) = 2k + r$. Also, $M\left(H^{(s,\alpha_s)}_{(k,r)}\right) = 2k - 2$ and $\ell_g\left(H^{(s,\alpha_s)}_{(k,r)}\right) = 2k + 2r = 2g - 2k$. Therefore, $H^{(s,\alpha_s)}_{(k,r)} \in \mathscr{H}^k_{2k-2}$, for all triple (r, s, α_s) as above. $\square$

Henceforth $H^{(s,\alpha_s)}_{(k,r)}$ is the sparse numerical semigroup defined in the Lemma 2.11.

Remark 2.12. Let $k \geqslant 3$ be an integer and (r, s, α_s) be as defined in the previous lemma. Then,

$$\mathrm{Gaps}\left(H^{(s,\alpha_s)}_{(k,r)}\right) = \left\{ \ell_1\left(H^{(s,\alpha_s)}_{(k,r)}\right), \ldots, \ell_{2k+r}\left(H^{(s,\alpha_s)}_{(k,r)}\right) \right\},$$

where

$$\ell_i\left(H_{(k,r)}^{(s,\alpha_s)}\right) = \begin{cases} i, & \text{if } 1 \leqslant i \leqslant 2k-2; \\ 2(i-k+1), & \text{if } 2k-1 \leqslant i \leqslant 2k+s-2; \\ 2(i-k)+1, & \text{if } 2k+s-1 \leqslant i \leqslant 2k+s+\alpha_s-2; \\ 2(i-k), & \text{if } 2k+s+\alpha_s-1 \leqslant i \leqslant 2k+r. \end{cases}$$

Theorem 2.13. *Let $k \geqslant 3$ be an integer. If $H \in \mathscr{H}_{2k-2}^k$, then $2k+1 \leqslant g(H) \leqslant 3k+1$ and $g(H) \neq 3k-1$.*

Proof. Let $k \geqslant 3$ be an integer and let $H \in \mathscr{H}_{2k-2}^k$ with $g(H) = g$ and $\mathrm{Gaps}(H) = \{1 = \ell_1 < \cdots < \ell_g\}$. Since $M = M(H) = 2k-2$ and H is a sparse semigroup, we have that $\ell_{2k-1} = \ell_{M+1} = 2k$ and, by the Corollary 1.3, $S_M = 2$. If $g < 2k$, then $\ell_g < 2k$, a contradiction with the value of ℓ_{M+1}. If $g = 2k$, then $\ell_g = 2k$. So, $g = M+1 = 2k-1 = g-1$, a contradiction. Therefore, $g \geqslant 2k+1$. It is clear that $g \neq 3k-1$, because if $g = 3k-1$, then $\ell_g = 4k-2 \in \mathrm{Gaps}(H)$, a contradiction, since $2k-1 \in H$. Now, suppose that H has genus $g = 3k+s$, for some integer $s \geqslant 2$. So, $\ell_g = 4k+2s$ and follows that $\ell_g/2 = 2k+s \in \mathrm{Gaps}(H)$. Since $2k-1 \in H$, we have that $\ell := 2k+2s+1 \in \mathrm{Gaps}(H)$. We prove that $2k+2s \in H$. In fact, suppose that $2k+2s \in \mathrm{Gaps}(H)$. Thus, $(\ell-1, \ell)$ is a single leap. If $\ell_{2k} = 2k+1$, then $(2k, 2k+1)$ is a single leap. Note that, if $\ell_{2k} = 2k+2$, then $2k+1 \in H$ and so $2k+2s-1 \in \mathrm{Gaps}(H)$ (since $s \geqslant 2$), or equivalently, $(\ell-2, \ell-1)$ is a single leap. On the other hand, since $S_M = 2$, by Remark 2.7, follows that $(2k+2s+1) \equiv (4k+2s) \pmod 2$, a contradiction. Therefore, $2k+2s \in H$. Then, $2k+2s-1 \in \mathrm{Gaps}(H)$ and $4k+2s-1 = (2k-1)+(2k+2s) \in H$. Thus, $\ell_{g-1} = 4k+2s-2$ and follows that $2k+s-1 \in \mathrm{Gaps}(H)$. So, $(2k+s-1, 2k+s)$ is a single leap. Therefore, since $2k+2s+1$ is odd and $4k+2s-2$ is even, we have that $\left|\mathcal{V}_{[2k+2s+1, 4k+2s-2]} \cap \mathcal{V}_1(H)\right| = 1$. We conclude the proof by studying the following cases separately: $s = 2$ and $s \geqslant 3$. Firstly, assume that $s = 2$. In this case, $(2k+1, 2k+2)$ and $(2k+2, 2k+3)$ are both single leaps. So, $S_M = 3$, a contradiction. Now, suppose that $s \geqslant 3$. Then, $2k+s-2 \in H$ and consequently $4k+2s-4 \in H$. So, $\ell_{g-2} = 4k+2s-3$, a contradiction, since $4k+2s-3 = (2k-1)+(2k+2s-2) \in H$. $\square$

Theorem 2.14. *For all integer $k \geqslant 3$,*

$$\mathscr{H}_{2k-2}^k = \left\{H_{(k,r)}^{(s,\alpha_s)} : (r, s, \alpha_s) \in \mathbb{N}^3,\ 1 \leqslant r \leqslant k-2,\ 1 \leqslant s \leqslant r,\ 1 \leqslant \alpha_s \leqslant r-s+1\right\} \cup \left\{H_{(k,r)}^{(1,k)} : r \in \{k, k+1\}\right\}.$$

Proof. Firstly, by the Lemma 2.11, we have that

$$\left\{H_{(k,r)}^{(s,\alpha_s)} : (r,s,\alpha_s) \in \mathbb{N}^3,\, 1 \leqslant r \leqslant k-2,\, 1 \leqslant s \leqslant r, 1 \leqslant \alpha_s \leqslant r-s+1\right\} \cup \left\{H_{(k,r)}^{(1,k)} : r \in \{k,k+1\}\right\} \subset \mathscr{H}_{2k-2}^{k}.$$

Now, let $H \in \mathscr{H}_{2k-2}^{k}$ with $g(H) = g$. By Theorem 2.13, we have $2k+1 \leqslant g \leqslant 3k+1$ and $g \neq 3k-1$. Let $r := g-2k$, then $1 \leqslant r \leqslant k-2$ or $r \in \{k,k+1\}$. Since $\ell_g = 2g-2k$, we have that $\ell_g = \ell_{2k+r} = 2k+2r$. Let $\mathrm{Gaps}(H) = \{1 = \ell_1 < \cdots < \ell_g\}$ be the gaps set of H. By Corollary 1.3, $S_M = 2$. That is, there exists exactly two single leaps (ℓ_{i-1}, ℓ_i) and (ℓ_{j-1}, ℓ_j), for some integers i and j such that $2k \leqslant i < j \leqslant g = 2k+r$.

We affirm that if $r \in \{k,k+1\}$, then $i = 2k$ and $j = 3k$. In fact, first we will prove that $(2k, 2k+1)$ is a single leap. Indeed, since $M = 2k-2$, we have that $\ell_{2k-2} = 2k-2$ and $\ell_{2k-1} = 2k$. In particular, $2k-1 \in H$. Suppose that $2k+1 \in H$. Thus, we have the following: if $r = k$, then $\ell_g = 4k = (2k-1)+(2k+1) \in H$; and if $r = k+1$, then $\ell_g = 4k+2 = 2(2k+1) \in H$. Therefore, in both cases, we get a contradiction. Thus, $(2k, 2k+1)$ is a single leap and so $i = 2k$. Now, we will prove that $j = 3k$. Since, $4k-2 = 2(2k-1) \in H$, follows that $4k-1$ is a gap. Then, by Remark 2.7, $2k+2 \in H$, since $(2k+2) \not\equiv (4k-1) \pmod 2$. Thus, $4k+1 = (2k-1)+(2k+2) \in H$ and so $4k$ is a gap. Consequently, $(4k-1, 4k)$ is a single leap. Note that, if $r = k$, then $\ell_{3k} = \ell_g = 4k$, and if $r = k+1$, then $\ell_{3k} = \ell_{g-1} = 4k$. Therefore, in both cases, $j = 3k$.

For $r \in \{1,\ldots,k-2\} \cup \{k,k+1\}$, let $s := i-2k+1$ and $\alpha_s := j-i$. Note that if $r \in \{1,\ldots,k-2\}$, then $1 \leqslant s \leqslant r$ and $1 \leqslant \alpha_s = (j-2k)-s+1 \leqslant r-s+1$, and if $r \in \{k,k+1\}$, then $s = 1$ and $\alpha_s = k$.

We will prove that $H = H_{(k,r)}^{(s,\alpha_s)}$, with s and α_s as above. In fact, by Remark 2.7, we have that $\ell_u = \ell_{i-1} - 2(i-u-1)$, for all integer u such that $2k-1 \leqslant u \leqslant i-1$, $\ell_v = \ell_i + 2(v-i)$, for all integer v such that $i \leqslant v \leqslant j-1$, and $\ell_w = \ell_j + 2(w-j)$, for all integer w such that $j \leqslant w \leqslant g$. Since $\ell_g = 2g-2k$, it is not hard conclude that $\ell_j = 2(j-k)$ and, therefore, $\ell_{j-1} = 2(j-k)-1$. In the same way we can conclude that $\ell_i = 2(i-k)+1$ and $\ell_{i-1} = 2(i-k)$, and that $\ell_u = 2(u-k+1)$, for all integer u such that $2k-1 \leqslant u \leqslant i-1 = 2k+s-2$, $\ell_v = 2(v-k)+1$, for all integer v such that $2k+s-1 = i \leqslant v \leqslant j-1 = 2k+s+\alpha_s-2$ and $\ell_w = 2(w-k)$, for all integer w such that $2k+s+\alpha_s-1 = j \leqslant w \leqslant g = 2k+r$. Therefore, by Remark 2.12, we have that $H = H_{(k,r)}^{(s,\alpha_s)}$. $\square$

Corollary 2.15. *For all integer $k \geqslant 3$, $\left|\mathscr{H}_{2k-2}^{k}\right| = \binom{k}{3} + 2$.*

Proof. By the Theorem 2.14, we have that

$$\begin{aligned}\mathscr{H}^k_{2k-2} = \dot{\bigcup_{r=1}^{k-2}} \dot{\bigcup_{s=1}^{r}} \left\{ H^{(s,\alpha_s)}_{(k,r)} : \alpha_s \in \mathbb{N},\, 1 \leqslant \alpha_s \leqslant r-s+1 \right\} \dot\cup \\ \left\{ H^{(1,k)}_{(k,r)} : r \in \{k, k+1\} \right\}.\end{aligned}$$

Therefore,

$$\left| \mathscr{H}^k_{2k-2} \right| = \sum_{r=1}^{k-2} \sum_{s=1}^{r} (r-s+1) + 2 = \sum_{r=1}^{k-2} \binom{r+1}{2} + 2 = \binom{k}{3} + 2. \quad \square$$

3. On Weierstrass semigroup

Let $\mathcal{X}$ be a non-singular, projective, irreducible, algebraic curve of genus $g \geqslant 1$ over a field $\mathbf{K}$. Let $\mathbf{K}(\mathcal{X})$ be the field of rational functions on $\mathcal{X}$ and for $f \in \mathbf{K}(\mathcal{X})$, $(f)_\infty$ will denote the divisor of poles of f. Let P be a point on $\mathcal{X}$. The set

$$H(P) := \{n \in \mathbb{N}_0 : \text{there exist } f \in \mathbf{K}(\mathcal{X}) \text{ with } (f)_\infty = nP\},$$

is a numerical semigroup called *Weierstrass semigroup* of $\mathcal{X}$ at P.

Given a numerical semigroup H, does it exist a curve $\mathcal{X}$ such that for some point $P \in \mathcal{X}$ has $H = H(P)$? If the answer is yes, we say that the numerical semigroup H is *Weierstrass*. Studies to answer this question have been done for decades, see e.g. [4], [8], [10], [11], [12], [13], [15] and [19]. In addition to the genus $g(H)$, Frobenius number $\ell_g(H)$ and multiplicity $n_1(H)$, an important concept in this study is the *weight* of a numerical semigroup H. If $\mathrm{Gaps}(H) = \{1 = \ell_1 < \cdots < \ell_g\}$ be the gaps set of H, the *weight* of H is

$$w(H) = \sum_{i=1}^{g} (\ell_i - i).$$

As a particular result, a numerical semigroup H is Weierstrass if the following condition hold:

$$\text{either } w(H) \leqslant g(H)/2, \text{ or } g(H)/2 < w(H) \leqslant g(H) - 1 \text{ and } 2n_1(H) > \ell_g(H) \qquad (*)$$

(see Eisenbud-Harris [7], Komeda [9]).

Next, we will see which of the semigroups in the families studied in the previous section are Weierstrass.

Lemma 3.1. *Let $k \geqslant 2$ be an integer. If $H = H^{\alpha_r}_{(k,r)} \in \mathscr{H}^k_{2k-1}$, then $w(H) = \alpha_r + \frac{r(r+1)}{2}$ and $2n_1(H) > \ell_g(H)$.*

Proof. Since $H = H^{\alpha_r}_{(k,r)} \in \mathscr{H}^k_{2k-1}$, by Remark 2.6, we have that

$$w(H) = \sum_{i=2k}^{2k-1+\alpha_r}(i-2k+1) + \sum_{i=2k+\alpha_r}^{2k+r}(i-2k) = \sum_{i=1}^{r} i + \alpha_r = \frac{r(r+1)}{2} + \alpha_r.$$

On the other hand, by Lemma 2.5, $\ell_g(H) = 2k+2r \leqslant 4k-2 < 4k = 2(2k) = 2n_1(H)$, since $r \leqslant k-1$. □

Proposition 3.2. *Let $k \geqslant 2$ be an integer and $H = H^{\alpha_r}_{(k,r)} \in \mathscr{H}^k_{2k-1}$. If $\alpha_r + \frac{r(r-1)}{2} \leqslant 2k-1$, then H is Weierstrass. In particular, if $k \geqslant 3$ and $r \in \left\{1, \ldots, \left\lfloor \frac{-1+\sqrt{16k-7}}{2} \right\rfloor \right\}$, then H is Weierstrass.*

Proof. By previous lemma, $2n_1(H) > \ell_g(H)$. So, from the condition $(*)$ above, follows that H is Weierstrass if $w(H) \leqslant g(H)-1$. On the other hand, by Lemma 2.5, $g(H) = 2k+r$, and, by Lemma 3.1, $w(H) = \alpha_r + \frac{r(r+1)}{2}$. Thus, H is Weierstrass if $\alpha_r + \frac{r(r+1)}{2} \leqslant 2k+r-1$, that is, if $\alpha_r + \frac{r(r-1)}{2} \leqslant 2k-1$.

Now, if $r \in \left\{1, \ldots, \left\lfloor \frac{-1+\sqrt{16k-7}}{2} \right\rfloor \right\}$, then $r + \frac{r(r-1)}{2} \leqslant 2k-1$. Also, since $k \geqslant 3$, we have that $r \leqslant k-1$. So, since $\alpha_r \leqslant r$, the required result follows. □

Lemma 3.3. *Let $k \geqslant 3$ be an integer. If $H = H^{(s,\alpha_s)}_{(k,r)} \in \mathscr{H}^k_{2k-2}$, then $w(H) = 2s-1+\alpha_s+\frac{r(r+1)}{2}$. In addition, if $r \notin \{k, k+1\}$, then $2n_1(H) > \ell_g(H)$.*

Proof. Since $H = H^{(s,\alpha_s)}_{(k,r)} \in \mathscr{H}^k_{2k-2}$, by Remark 2.12, we have that

$$\begin{aligned} w(H) &= \sum_{i=2k-1}^{2k+s-2}(i-2k+2) + \sum_{i=2k+s-1}^{2k+s+\alpha_s-2}(i-2k+1) + \sum_{i=2k+s+\alpha_s-1}^{2k+r}(i-2k) \\ &= \sum_{i=-1}^{r} i + 2s + \alpha_s \\ &= -1 + \frac{r(r+1)}{2} + 2s + \alpha_s. \end{aligned}$$

On the other hand, by Lemma 2.11, $\ell_g(H) = 2k+2r \leqslant 4k-2 = 2(2k-1) = 2n_1(H)$, since $r \leqslant k-2$. □

Proposition 3.4. *Let $k \geqslant 3$ be an integer and $H = H_{(k,r)}^{(s,\alpha_s)} \in \mathscr{H}_{2k-2}^{k}$. If $r \notin \{k, k+1\}$ and $2s + \alpha_s + \frac{r(r-1)}{2} \leqslant 2k$, then H is Weierstrass. In particular, if $k \geqslant 4$ and $r \in \left\{1, \dots, \left\lfloor \frac{-3+\sqrt{16k+1}}{2} \right\rfloor \right\}$, then H is Weierstrass.*

Proof. Using the lemmas 2.11 and 3.3, the proof of the first statement is analogous to the proof of the Proposition 3.2.

Now, if $r \in \left\{1, \dots, \left\lfloor \frac{-3+\sqrt{16k+1}}{2} \right\rfloor \right\}$, then $2r + \frac{r(r-1)}{2} \leqslant 2k - 1$. Also, since $k \geqslant 4$, we have that $r \leqslant k - 2$. So, since $s \leqslant r$ and $\alpha_s \leqslant r - s + 1$, the required result follows. $\square$

We observe that, for $r \in \{k, k+1\}$, the condition $(*)$ above cannot be used to conclude that the semigroups $H_{(k,r)}^{(1,k)}$'s are Weierstrass.

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Received by the editors: 20.07.2017
and in final form 22.11.2017.