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RESEARCH ARTICLE

Algebraic Morse theory and homological perturbation theory

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Abstract. We show that the main result of algebraic Morse theory can be obtained as a consequence of the perturbation lemma of Brown and Gugenheim.

1. Introduction

Robin Forman introduced discrete Morse theory in [For98] as a combinatorial adaptation of the classical Morse theory suited for studying the topology of CW-complexes. Its fundamental idea is also applicable in purely algebraical situations (see e.g. [Jon03], [Koz05], [JW09], [Skö06]).

Homological perturbation theory on the other hand builds on the perturbation lemma [Bro65], [Gug72]. In addition to its applications in algebraic topology, it has also found uses in e.g. the study of group cohomology [Lam92], [Hue89], resolutions in commutative algebra [JLS02], as well as in operadic settings, [Ber14].

In this note we show how to derive the main result of algebraic Morse theory from the perturbation lemma. In related work, Berglund [Ber], has also treated connections between algebraic Morse theory and homological perturbation theory.

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2. Definitions

We will briefly review the definitions of the main objects of study.

A *contraction* is a diagram of chain complexes of (left or right) modules over a ring R

$$\mathbf{D} \underset{f}{\overset{g}{\rightleftarrows}} \mathbf{C} \circlearrowright h$$

where f and g are chain maps and h is a degree 1 map satisfying the identities

$$fg = 1, \quad gf = 1 + dh + hd$$

and

$$fh = 0, \quad hg = 0, \quad h^2 = 0.$$

A contraction is *filtered* if there is a bounded below exhaustive filtration on the complexes which is preserved by the maps f, g and h. A *perturbation* of a chain complex $\mathbf{C}$ is a map $t : \mathbf{C} \to \mathbf{C}$ of degree -1 such that $(d+t)^2 = 0$. Given a perturbation t on $\mathbf{C}$, we let $\mathbf{C}^t$ be the complex obtained by equipping $\mathbf{C}$ with the new differential $d + t$.

We can now state the perturbation lemma.

Theorem 1 (Brown, Gugenheim)**.**

$$\mathbf{D} \underset{f}{\overset{g}{\rightleftarrows}} \mathbf{C} \circlearrowright h$$

and a filtration lowering perturbation t of $\mathbf{C}$, the diagram

$$\mathbf{D}^{t'} \underset{f'}{\overset{g'}{\rightleftarrows}} \mathbf{C}^t \circlearrowright h'$$

where

$$f' = f + f\mathcal{S}h, \quad g' = g + h\mathcal{S}g, \quad h' = h + h\mathcal{S}h, \quad t' = f\mathcal{S}g$$

and

$$\mathcal{S} = \sum_{n=0}^{\infty} t(ht)^n$$

defines a contraction.

Let us next review some terminology of algebraic Morse theory. By a *based complex* of R-modules we mean a chain complex $\mathbf{C}$ of R-modules together with direct sum decompositions $C_n = \bigoplus_{\alpha \in I_n} C_\alpha$ where $\{I_n\}$ is a family of mutually disjoint index sets. For $f : \bigoplus_n C_n \to \bigoplus_n C_n$ a graded map, we write $f_{\beta,\alpha}$ for the component of f going from C_α to C_β, and given a based complex $\mathbf{C}$ we construct a digraph $\mathcal{G}(\mathbf{C})$ with vertex set $V = \bigcup_n I_n$ and with a directed edge $\alpha \to \beta$ whenever the component $d_{\beta,\alpha}$ is non-zero.

A subset M of the edges of $\mathcal{G}(\mathbf{C})$ such that no vertex is incident to more than one edge of M is called a *Morse matching* if, for each edge $\alpha \to \beta$ in M, the corresponding component $d_{\beta,\alpha}$ is an isomorphism, and furthermore there is a well founded partial order $\prec$ on each I_n such that $\gamma \prec \alpha$ whenever there is a path $\alpha^{(n)} \to \beta \to \gamma^{(n)}$ in the graph $\mathcal{G}(\mathbf{C})^M$, which is the graph obtained from $\mathcal{G}(\mathbf{C})$ by reversing the edges from M.

Given the matching M, we define the set M^0 to be the vertices that are not incident to an arrow from M.

For α and β vertices in $\mathcal{G}(\mathbf{C})^M$ we can now consider all directed paths from α to β. For each such path γ, we get a map from C_α to C_β by, for each edge $\sigma \to \tau$ in γ which is not in M take the map $d_{\tau,\sigma}$, and for each edge $\sigma \to \tau$ in γ which is the reverse of an edge in M take the map $-d_{\sigma,\tau}^{-1}$ and composing them. Summing these maps over all paths from α to β defines the map $\Gamma_{\beta,\alpha} : C_\alpha \to C_\beta$.

3. The main result

From the based complex $\mathbf{C}$ with $C_n = \bigoplus_{\alpha \in I_n} C_\alpha$ furnished with a Morse matching M, we define another based complex $\tilde{\mathbf{C}}$ by letting it be isomorphic to $\mathbf{C}$ as a graded module, and defining the differential $\tilde{d}$ in $\tilde{\mathbf{C}}$ as

$$\tilde{d}(x) = \begin{cases} d_{\beta,\alpha}(x), & \text{if } \alpha \to \beta \in M, \\ 0, & \text{otherwise;} \end{cases} \qquad \text{for } x \in C_\alpha.$$

We also need a based complex coming from the vertices in M^0, so we define $\tilde{\mathbf{C}}^M$ by

$$\tilde{C}_n^M = \bigoplus_{\alpha \in I_n \cap M^0} C_\alpha, \quad d_{\tilde{\mathbf{C}}^M} = 0,$$

and maps $\tilde{f} : \tilde{\mathbf{C}} \to \tilde{\mathbf{C}}^M$, $\tilde{g} : \tilde{\mathbf{C}}^M \to \tilde{\mathbf{C}}$ and $\tilde{h} : \tilde{\mathbf{C}} \to \tilde{\mathbf{C}}[1]$ given by

$$\begin{aligned}
\tilde{f}(x) &= \begin{cases} x, & \text{if } \alpha \in M^0, \\ 0, & otherwise, \end{cases} \\
\tilde{g}(x) &= x, && x \in C_\alpha. \\
\tilde{h}(x) &= \begin{cases} -d^{-1}_{\alpha,\beta}(x), & \text{if } \beta \to \alpha \in M, \\ 0, & \text{otherwise;} \end{cases}
\end{aligned}$$

With this notation we can now formulate the following lemma.

Lemma 1. The diagram

$$\tilde{\mathbf{C}}^M \underset{\tilde{f}}{\overset{\tilde{g}}{\rightleftarrows}} \tilde{\mathbf{C}} \circlearrowright \tilde{h}$$

is a contraction.

Proof. We first need to verify that $\tilde{f}$ and $\tilde{g}$ are chain maps, which is readily seen. Next we check the identities

$$\tilde{f}\tilde{g} = 1, \quad \tilde{g}\tilde{f} = 1 + \tilde{d}\tilde{h} + \tilde{h}\tilde{d}.$$

The first one is obvious, and the second follows from the fact that for a basis element $x \in C_\alpha$, $\tilde{d}\tilde{h}(x) = -x$ if there is an edge $\beta \to \alpha$ in M, and 0 otherwise; and similarly $\tilde{h}\tilde{d}(x) = -x$ if there is an edge $\alpha \to \beta$ in M, and 0 otherwise. The identities

$$\tilde{h}\tilde{g} = 0, \quad \tilde{f}\tilde{h} = 0, \quad \tilde{h}^2 = 0$$

follow from that vertices in M^0 are not incident to any edge in M (the first two) and that no vertex is incident to more than one edge in M (the third). □

Let us now define the perturbation t on $\tilde{\mathbf{C}}$ as $t = d - \tilde{d}$, where d is the differential on $\mathbf{C}$, so

$$t(x) = \sum_{\alpha \to \beta \notin M} d_{\beta,\alpha}(x)$$

for $x \in C_\alpha$. This makes $\tilde{\mathbf{C}}^t$ and $\mathbf{C}$ isomorphic as based complexes.

Lemma 2. The diagram

$$\mathbf{C}^M \underset{f}{\overset{g}{\rightleftarrows}} \mathbf{C} \circlearrowright h$$

where, for $x \in C_\alpha$ with $\alpha \in I_n$,

$$d_{\mathbf{C}^M}(x) = \sum_{\beta \in M^0 \cap I_{n-1}} \Gamma_{\beta,\alpha}(x), \qquad f(x) = \sum_{\beta \in M^0 \cap I_n} \Gamma_{\beta,\alpha}(x),$$

$$g(x) = \sum_{\beta \in I_n} \Gamma_{\beta,\alpha}(x), \qquad h(x) = \sum_{\beta \in I_{n+1}} \Gamma_{\beta,\alpha}(x),$$

is a filtered contraction.

Proof. From Lemma 1 together with the fact that there are no infinite paths in $\mathcal{G}(\mathbf{C})^M$, the Morse graph of $\mathbf{C}$, we can deduce that ht is locally nilpotent, and we can thus invoke the perturbation lemma. It is not so hard to see that the perturbed differential on $\tilde{\mathbf{C}}^M$ is given by

$$d(x) = \sum_{i=0}^{\infty} t(ht)^i(x) = \sum_{\beta \in M^0 \cap I_{n-1}} \Gamma_{\beta,\alpha}(x)$$

and the maps f, g and h by

$$f(x) = \sum_{i=0}^{\infty} f(ht)^i(x) = \sum_{\beta \in M^0 \cap I_n} \Gamma_{\beta,\alpha}(x)$$

$$g(x) = \sum_{i=0}^{\infty} g(ht)^i(x) = \sum_{\beta \in I_n} \Gamma_{\beta,\alpha}(x)$$

$$h(x) = \sum_{i=0}^{\infty} (ht)^i h(x) = \sum_{\beta \in I_{n+1}} \Gamma_{\beta,\alpha}(x)$$

where $x \in C_\alpha$. □

The above result is also shown (without the use of the perturbation lemma) in [Ber] using a result from [JW09].

From the preceding lemma, the main result of algebraic Morse theory now follows.

Theorem 2. Let $\mathbf{C}$ be a based complex with a Morse matching M, then there is a differential on the graded module $\bigoplus_{\alpha \in M^0} C_\alpha$ such that the resulting complex is homotopy equivalent to $\mathbf{C}$.

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