Algebra and Discrete Mathematics
Volume **25** (2018). Number 2, pp. 269–285

RESEARCH ARTICLE

On functional equations and distributive second order formulae with specialized quantifiers*

Yuri Movsisyan

Communicated by A. Zhuchok

Abstract. The structure of invertible algebras with distributive second order formulae with specialized quantifiers is given. As a consequence, the applications for solutions of the some functional equations of distributivity on quasigroups are provided.

Introduction

Let A be a binary operation on the set Q and A' be a binary operation on the set Q'. Operations A and A' are called isotopic if there exist bijective mappings $\alpha, \beta, \gamma : Q \to Q'$ such, that:

$$\gamma A(x, y) = A'(\alpha x, \beta y)$$

or

$$A(x, y) = \gamma^{-1} A'(\alpha x, \beta y)$$

for every $x, y \in Q$. The groupoids $Q(A)$ and $Q'(A')$ are called isotopic if the operation A and A' are isotopic [5, 6].

A groupoid $Q(\cdot)$ is called a quasigroup if the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions $x, y \in Q$ for every $a, b \in Q$ [5, 6, 11, 16,

*This research is supported by the State Committee of Science of the Republic of Armenia, grant: 10-3/1-41.

2010 MSC: 20N05, 20N02, 03C85, 03C05.

Key words and phrases: functional equation, hyperidentity, second order formula, quasigroup, moufang loop, invertible algebra, isotopy.

17, 31, 35, 57, 58, 66, 68]. The algebra $(Q; \Sigma)$ with quasigroup operations is called an invertible algebra [10, 43, 45, 46].

A quasigroup $Q(\cdot)$ with a unit (identity element) is called a loop. A loop $Q(\cdot)$ is called a Moufang loop [11, 41, 57] if it satisfies the identity:

$$(x \cdot y) \cdot (z \cdot x) = (x \cdot (y \cdot z)) \cdot x.$$

A commutative Moufang loop is defined by the following identity [11, 57]:

$$(x \cdot x) \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z).$$

A quasigroup $Q(\cdot)$ is called distributive [11, 57] if it satisfies the following identities of distributivity:

$$x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z),$$

$$(x \cdot y) \cdot z = (x \cdot z) \cdot (y \cdot z).$$

For functional equations in algebra, logics, real analysis and topology see [1–3, 10, 26–30, 37–39].

The problems of solution of the following two general functional equations of distributivity

$$A(x, B(y, z)) = H(G(x, y), K(x, y)),$$
$$A(B(y, z), x) = H(G(y, x), K(z, x))$$

on quasigroups are the unsolved problems of quasigroup theory [1–3, 10, 26]. Moreover, the following functional equations still remain unsolved on quasigroups, as well:

$$A(x, A(y, z)) = H(G(x, y), K(x, y)),$$
$$A(A(y, z), x) = H(G(y, x), K(z, x)),$$
$$A(x, A(y, z)) = H(K(x, y), K(x, y)),$$
$$A(A(y, z), x) = H(K(y, x), K(z, x)),$$
$$A(x, B(y, z)) = B(A(x, y), A(x, z)),$$
$$A(B(y, z), x) = B(A(y, x), A(z, x)).$$

Here, the solution is unknown even in the case $A = B$.

Note that the solution of the following systems of functional equations are also unknown on quasigroups:

$$\begin{cases} A(x, B(y, z)) = B(A(x, y), A(x, z)), \\ A(B(y, z), x) = B(A(y, x), A(z, x)), \end{cases}$$

and

$$\begin{cases} A(x, B(y, z)) = B(A(x, y), A(x, z)), \\ B(A(y, z), x) = A(B(y, x), B(z, x)). \end{cases}$$

Here, in the case of $A = B$, the solution follows from [9]. Namely, it is proved in [9] that every distributive quasigroup is isotopic to a certain commutative Moufang loop.

The following propositions concerning to the solution of the related functional equations are consequences from the main result of current paper.

1) If quasigroups $Q(A), Q(K)$ and groupoid $Q(H)$ satisfies the following identities:

$$A(x, A(y, z)) = A(A(x, y), A(x, y)),$$
$$A(A(y, z), x) = H(K(y, x), K(z, x)),$$

then $Q(A)$ and $Q(K)$ are isotopic to a commutative Moufang loop.

2) If quasigroups $Q(A), Q(K)$ and groupoid $Q(H)$ satisfies the following identities:

$$A(A(y, z), x) = A(A(y, x), A(z, x)),$$
$$A(x, A(y, z)) = H(K(x, y), K(x, z)),$$

then $Q(A)$ and $Q(K)$ are isotopic to a commutative Moufang loop.

3) If quasigroups $Q(A)$, $Q(K)$, $Q(K')$ and groupoids $Q(H)$ and $Q(H')$ satisfies the following identities:

$$A(x, x) = x,$$
$$A(x, A(y, z)) = H(K(x, y), K(x, z)),$$
$$A(A(y, z), x) = H'(K'(y, x), K'(z, x)),$$

then $Q(A)$, $Q(K)$ and $Q(K')$ are isotopic to a commutative Moufang loop.

1. Auxiliary results and concepts

Lemma 1. *If a binary algebra $Q(A, B, H, K)$ satisfies the following identity:*

$$A(x, B(y, z)) = H(K(x, y), K(x, z)), \tag{1}$$

where $Q(A)$ and $Q(K)$ are quasigroups, $Q(B)$ and $Q(H)$ are groupoids, then A and K are isotopic to a quasigroup operation A_0, and the operations B and H are isotopic to an idempotent operation B_0 such that:

$$A_0(x, B_0(y, z)) = B_0(A_0(x, y), A_0(x, z)). \tag{2}$$

Besides:

$$A(x, B_0(y,z)) = B_0^{L_A^{-1}}(A(x,y), A(x,z)), \tag{3}$$

where $L_A(x) = A(0,x)$, *while the element* 0 *is an arbitrary fixed element of* Q, *and*

$$B_0^{\varphi}(x,y) = \varphi^{-1}B_0(\varphi x, \varphi y).$$

In particular, if the operation B *is idempotent, then*

$$A(x, B(y,z)) = B^{L_A^{-1}}(A(x,y), A(x,z)). \tag{4}$$

In addition, if $Q(B)$ *is a quasigroup, then* $Q(H)$ *also is a quasigroup.*

Proof. If making the substitution $y = z$ in the equality (1), we obtain:

$$A(x, \Theta_B(y)) = \Theta_H K(x,y), \tag{5}$$

where $\Theta_B(y) = B(y,y)$. Let $R_A(x) = A(x,0)$.

If in equality (5) $y = 0$, then

$$A(x, \Theta_B(0)) = \Theta_H K(x,0),$$

i.e.

$$S(x) = \Theta_H R_K(x),$$

where $S(x) = A(x, \Theta_B(0))$ is a bijection. Hence, $\Theta_H = SR_K^{-1}$ also is a bijection. If in equality (5) $x = 0$, then

$$L_A\Theta_B(y) = \Theta_H L_K(y),$$
$$\Theta_B = L_A^{-1}\Theta_H L_K,$$

i.e. Θ_B is a bijection. According to (5), the operations A and K are isotopic and

$$K(x,y) = \Theta_H^{-1}A(x, \Theta_B y).$$

If in the equality (1) $x = 0$, then

$$L_A B(y,z) = H(L_k y, L_K z),$$
$$H(y,z) = L_A B\left(L_K^{-1}y, L_K^{-1}z\right).$$

Let us consider the following new operations:

$$B_0(x,y) = B\left(\Theta_B^{-1}x, \Theta_B^{-1}y\right),$$

and

$$A_0(x, y) = L_A^{-1} A(x, y).$$

Since $x = B\left(\Theta_B^{-1}x, \Theta_B^{-1}x\right)$, the operation B_0 is idempotent (and isotopic to B).

Substituting the values of operations K and H in identity (1), we obtain the equalities (2) and (3). If, in addition, the operation B is idempotent, then Θ_B is the identical mapping and $B_0 = B$. Hence, from (3), (4) follows . □

Lemma 2. *If a binary algebra $Q(A, B, H, K)$ satisfies the following identity:*

$$A(B(y, z), x) = H(K(y, x), K(z, x)), \tag{6}$$

where $Q(A)$ and $Q(K)$ are quasigroups, $Q(B)$ and $Q(H)$ are groupoids, then A and K are isotopic to a quasigroup operation A_0, and operations B and H are isotopic to an idempotent operation B_0 such that:

$$A_0(B_0(y, z), x) = B_0(A_0(y, x), A_0(z, x)). \tag{7}$$

Besides:

$$A(B_0(y, z), x) = B_0^{R_A^{-1}}(A(y, x), A(z, x)), \tag{8}$$

where $R_A(x) = A(x, 0)$, while the element 0 is an arbitrary fixed element of Q. In particular, if the operation B is idempotent, then

$$A(B(y, z), x) = B^{R_A^{-1}}(A(y, x), A(z, x)). \tag{9}$$

In addition, if $Q(B)$ is a quasigroup, then $Q(H)$ also is a quasigroup.

According to [43, 45–47, 49], a hyperidentity (or $\forall(\forall)$-identity) is a universal second-order formula of the following type:

$$\forall X_1, \ldots, X_m \forall x_1, \ldots, x_n \ (w_1 = w_2),$$

where $X_1, \ldots, X_m$ are functional variables, and $x_1, \ldots, x_n$ are object variables in the words (terms): w_1, w_2. Hyperidentities are usually written without quantifiers: $w_1 = w_2$. We say that the hyperidentity $w_1 = w_2$ is satisfied in the algebra $(Q; \Sigma)$ (or the algebra $(Q; \Sigma)$ satisfies the hyperidentity $w_1 = w_2$) if this equality is valid, whenever every object variable x_i and every functional variable X_j in it is replaced by any element from Q and by any operation of the corresponding arity from Σ respectively (supposing the possibility of such replacement) (also see [10, 12–14, 19, 22, 23, 25, 33, 42, 65, 67, 70]). If $m > 1$, the hyperidentity is called non-trivial. The number m is called functional rank of the hyperidentity. (For the second order formulae see [18, 33–35].)

For example, a binary idempotent algebra satisfies the hyperidentity of idempotency:

$$X(x, x) = x.$$

The mode [61] is an idempotent algebra with the hyperidentity of mediality:

$$X(Y(x, y), Y(u, v)) = Y(X(x, u), X(y, v)).$$

A distributive bisemilattice (multisemilattice) [24] is an algebra with semilattice operations satisfying the hyperidentity of distributivity:

$$X(x, Y(y, z)) = Y(X(x, y), X(x, z)).$$

Binary algebras with the hyperidentity of associativity:

$$X(x, Y(y, z)) = Y(X(x, y), z)$$

under the name of Γ-semigroups (or gamma-semigroups), doppelsemigroups and doppelalgebras also were considered by various authors [4, 8, 32, 53, 56, 60, 62–64, 73, 74]. In addition, the hyperidentity of associativity is satisfied in commutative dimonoids (see, e.g., [75], [76]).

For classification of hyperidentities in invertible and related algebras see [43, 45–47, 50].

For categorical definition of a hyperidentity, in [43] (bi)homomorphisms between two algebras $(Q; \Sigma)$ and $(Q'; \Sigma')$ are defined as the pairs $(\varphi; \tilde{\psi})$ of mappings:

$$\varphi : Q \to Q', \tilde{\psi} : \Sigma \to \Sigma', |A| = |\tilde{\psi}A|,$$

with the following condition:

$$\varphi A(a_1, \ldots, a_n) = (\tilde{\psi}A)(\varphi a_1, \ldots, \varphi a_n)$$

for any $A \in \Sigma, a_1, \ldots, a_n \in Q, |A| = n$. Hyperidentities are "identities" of algebras in the category of (bi)homomorphisms $(\varphi; \tilde{\psi})$. (More about the application of such morphisms in the cryptography can be found in [7].)

The set of all binary operations defined on the set Q is denoted by $\mathcal{F}_Q^2$, and we consider the following two operations on this set:

$$A \cdot B(x, y) = A(x, B(x, y)),$$
$$A \circ B(x, y) = A(B(x, y), y),$$

where $A, B \in \mathcal{F}_Q^2$, $x, y \in Q$. These operations $(\cdot)$ and $(\circ)$ are called the right and left multiplications of binary operations (functions), and they were studied in the works of various authors [15, 21, 36, 44, 46, 48, 51, 52, 54, 59, 69, 71, 72].

Lemma 3. *The set $\mathcal{F}_Q^2$ forms a monoid under the right (and left) multiplication of binary operations. These two semigroups are isomorphic. The identity element of the semigroup $\mathcal{F}_Q^2(\cdot)$ is $E \in \mathcal{F}_Q^2$ and it is defined by the rule: $E(x, y) = y$ for all $x, y \in Q$, and the identity element of the semigroup $\mathcal{F}_Q^2(\circ)$ is $F \in \mathcal{F}_Q^2$, and it is defined by the rule: $F(x, y) = x$ for all $x, y \in Q$. The mapping $A \to A^*$ is the isomorphism of these two semigroups, where $A^*(x, y) = A(y, x)$ for all $x, y \in Q$.* □

Corollary 1. *The set of idempotent binary operations on Q is a subsemigroup in the semigroups $\mathcal{F}_Q^2(\cdot)$ and $\mathcal{F}_Q^2(\circ)$.* □

The binary operation $A \in \mathcal{F}_Q^2$ is the right (left) invertible one if the equation $A(a, x) = b$ $(A(y, a) = b)$ has the unique solution $x \in Q$ $(y \in Q)$ for every $a, b \in Q$. Unique solutions $x, y \in Q$ are usually denoted by $x = A^{-1}(a, b)$ and $y = {}^{-1}A(b, a)$. Hence,

$$A \cdot A^{-1} = A^{-1} \cdot A = E\,,$$

for the right invertible operation A, and we have:

$${}^{-1}A \circ A = A \circ {}^{-1}A = F$$

for the left invertible operation A. The operation A^{-1} (or ${}^{-1}A$) is a right (or left) invertible for a right (or left) invertible operation $A \in \mathcal{F}_Q^2$ and:

$$\left(A^{-1}\right)^{-1} = A = {}^{-1}\left({}^{-1}A\right);$$

It is evident that if A is right (or left) invertible then A^* is left (right) invertible and:

$$\left(A^{-1}\right)^{*} = {}^{-1}(A^*), \quad \left({}^{-1}A\right)^{*} = (A^*)^{-1}.$$

The binary operation $A \in \mathcal{F}_Q^2$ is invertible if it is right and left invertible, i.e. $Q(A)$ is a quasigroup. In this case:

$$\left({}^{-1}\left(A^{-1}\right)\right)^{-1} = {}^{-1}\left(\left({}^{-1}A\right)^{-1}\right) = A^*.$$

The set of all right (left) binary invertible operations on the set Q is denoted by $\mathcal{F}_Q^r$ (and $\mathcal{F}_Q^\ell$).

Lemma 4. *The set $\mathcal{F}_Q^r$ is a group under the right multiplication of binary operations. The set $\mathcal{F}_Q^\ell$ is a group under the left multiplication of binary operations. These two groups are isomorphic too.* □

The concept of the right (left) invertibility can be defined via orthogonality of operations, as well [17, 20]. For applications of right (left) invertible operations in geometry and topology (knot theory) see [40, 55]. For right (left) loops see [68].

2. Main results

An invertible algebra $(Q;\Sigma)$ is called D_l-algebra (D_r-algebra) if it satisfies the following two conditions:

a) In the algebra $(Q;\Sigma)$ the following hyperidentity of the left (right) distributivity is satisfied:

$$X(x, X(y,z)) = X(X(x,y), X(x,z)) \tag{10}$$

$$(X(X(y,z),x) = X(X(y,x), X(z,x))); \tag{11}$$

b) In the algebra $(Q;\Sigma)$ the following $\forall\exists^*\exists^{**}(\forall)$-identity of the left (right) distributivity is satisfied:

$$\forall X, Y \exists^* X' \exists^{**} Y' \forall x, y, z (X(Y(y,z),x) = X'(Y'(y,x), Y'(z,x))) \tag{12}$$

$$(\forall X, Y \exists^* X' \exists^{**} Y' \forall x, y, z (X(x, Y(y,z)) = X'(Y'(x,y), Y'(x,z)))), \tag{13}$$

where $\forall X, Y$ means "for every values of $X, Y \in \Sigma$", $\exists^* X'$ means "there exists an operation on Q" and $\exists^{**} Y'$ means "there exists a quasigroup operation on Q". Thus, (12) and (13) are the second order formulae with specialized quantifiers (see [33]).

Examples. 1) Let $Q(+,\cdot)$ be a field and for every $a \in Q$:

$$A_a(x,y) = (1-a)x + ay.$$

If $\Sigma = \{A_a | a \in Q\}$ then the algebra $(Q;\Sigma)$ is an D_l- and D_r-algebra.

2) Let $Q(A)$ be a distributive quasigroup.
If $\Sigma = \{A, A^{-1}, {}^{-1}A, {}^{-1}(A^{-1}), ({}^{-1}A)^{-1}, A^*\}$, where $A^*(x,y) = A(y,x)$ for all $x, y \in Q$, then the algebra $(Q;\Sigma)$ is an D_l- and D_r-algebra (also see [52]).

3) Let $Q(+,\cdot)$ be a field and:

$$A_i(x,y) = a_i x + b_i y + c_i,$$

where $a_i, b_i, c_i \in Q$ and $a_i \neq 0$, $b_i \neq 0$, $a_i + b_i \neq 0$. If $\Sigma_I = \{A_i | \, i \in I\}$, then the algebra $(Q; \Sigma_I)$ satisfies the formulae (12) and (13).

Special cases of D_l-algebras and D_r-algebras were considered in [46] (Theorems 4.3, 4.3'). Namely, in [46], the invertible algebras with the following hyperidentities are characterized:

$$X(x, X(y,z)) = X(X(x,y), X(x,z)),$$
$$X(Y(y,z),x) = Y(X(y,x), X(z,x)),$$

and the invertible algebras with the following hyperidentities are characterized:

$$X(X(y,z),x) = X(X(y,x), X(z,x)),$$
$$X(x, Y(y,z)) = Y(X(x,y), X(x,z)).$$

In [46] is considered the dual case too.

Now we prove the following more general results.

Theorem 1. *1) If $(Q;\Sigma)$ is a D_l-algebra, then the quasigroups $Q(A)$, $A \in \Sigma$, are distributive and hence are isotopic to commutative Moufang loops. Every D_l-algebra satisfies the following non-trivial hyperidentity:*

$$X(Y(X(y,z),z),x) = X(Y(X(y,x), X(z,x)), X(z,x)). \qquad (14)$$

2) If $(Q;\Sigma)$ is a D_r-algebra, then the quasigroups $Q(A)$, $A \in \Sigma$ are distributive and hence are isotopic to commutative Moufang loops. Every D_r-algebra satisfies the following non-trivial hyperidentity:

$$X(x, Y(y, X(y,z))) = X(X(x,y), Y(X(x,y)X(x,z))). \qquad (15)$$

Proof. 1) From the hyperidentity (10) of the left distributivity it follows that every quasigroup $Q(A)$, $A \in \Sigma$, is idempotent. If in the formula (12) $X = A \in \Sigma$, $Y = B \in \Sigma$, $X' = H$, $Y' = K$, we obtain:

$$A(B(y,z),x)) = H(K(y,x), K(z,x)).$$

According to Lemma 2 (equation (9)) we have:

$$A(B(y,z),x) = B^{R_A^{-1}}(A(y,x), A(z,x)). \qquad (16)$$

If we substitute $z = x$ in (16), we obtain:

$$A(B(y,x),x) = B^{R_A^{-1}}(A(y,x),x). \qquad (17)$$

In particular, by putting $B = A$, we have:

$$A(A(y,x),x) = A^{R_A^{-1}}(A(y,x),x),$$

or

$$A(u,x) = A^{R_A^{-1}}(u,x),$$

i.e. $A^{R_A^{-1}} = A$. Then substituting $B = A$ in (16), we obtain a right distributive identity for any quasigroup operation $A \in \Sigma$:

$$A(A(y,z),x) = A(A(y,x),A(z,x)).$$

Hence, the quasigroup $Q(A)$ is distributive for every operation $A \in \Sigma$.

According to [9] every distributive quasigroup $Q(A)$ is isotopic to certain commutative Moufang loop $Q(\underset{A}{+})$:

$$x \underset{A}{+} y = A\left(R_A^{-1}x, L_A^{-1}y\right),$$

where R_A and L_A are automorphisms of $Q(A)$ and automorphisms of the loop $Q(\underset{A}{+})$, $R_A L_A = L_A R_A$ and $R_A(x) \underset{A}{+} L_A(x) = x$ for any $x \in Q$.

Now we prove that hyperidentity (14) is satisfied in the D_ℓ-algebra $(Q;\Sigma)$.

From equality (17) we have:

$$A \circ B = B^{R_A^{-1}} \circ A,$$

and

$$B^{R_A^{-1}} = A \circ B \circ {}^{-1}A.$$

Hence, according to equality (16), we obtain:

$$\begin{aligned}
&A(B(y,z),x) = (A \circ B \circ {}^{-1}A)(A(y,x),A(z,x)),\\
&A(B(y,z),x) = A(B \circ {}^{-1}A(A(y,x),A(z,x)),A(z,x)),\\
&A(B(y,z),x) = A(B({}^{-1}A(A(y,x),A(z,x)),A(z,x)),A(z,x)),\\
&A(B(y,z),x) = A(B(A({}^{-1}A(A(y,z),x),A(z,x)),A(z,x)).\\
&A(B(A(y,z),z),x) = A(B(A({}^{-1}A(A(y,z),z),x)A(z,x)),A(z,x)),\\
&A(B(A(y,z),z),x) = A(B(A(y,x),A(z,x)),A(z,x));
\end{aligned}$$

Thus, in the algebra $(Q;\Sigma)$, the hyperidentity (14) is satisfied.

2) Analogically, by using the equality $B^{L_A^{-1}} = A \cdot B \cdot A^{-1}$, we prove property 2). □

Corollary 2. *If quasigroups $Q(A)$, $Q(K)$ and groupoid $Q(H)$ satisfies the following identities:*

$$A(x, A(y, z)) = A(A(x, y), A(x, z)),$$
$$A(A(y, z), x) = H(K(y, x), K(z, x)),$$

then $Q(A)$ and $Q(K)$ are isotopic to a commutative Moufang loop.

Proof. We applied Theorem 1 for $\Sigma = \{A\}$, and used Lemma 2. □

Corollary 3. *If quasigroups $Q(A)$, $Q(K)$ and groupoid $Q(H)$ satisfies the following identities:*

$$A(A(y, z), x) = A(A(y, x), A(z, x)),$$
$$A(x, A(y, z)) = H(K(x, y), K(x, z)),$$

then $Q(A)$ and $Q(K)$ are isotopic to a commutative Moufang loop.

Proof. We applied Theorem 1 when $\Sigma = \{A\}$, and used Lemma 1. □

Corollary 4. *If quasigroups $Q(A)$, $Q(K)$, $Q(K')$ and groupoids $Q(H)$ and $Q(H')$ satisfies the following identities:*

$$A(x, x) = x,$$
$$A(x, A(y, z)) = H(K(x, y), K(x, z)),$$
$$A(A(y, z), x) = H'(K'(y, x), K'(z, x)),$$

then $Q(A)$, $Q(K)$ and $Q(K')$ are isotopic to a commutative Moufang loop.

Proof. We applied the proof of Theorem 1 when $\Sigma = \{A\}$, and use Lemma 1 and Lemma 2. Note that in the proof of Theorem 1 we used the idempotency of the operations of Σ. □

Let Σ_l be the set of commutative Moufang loop operations corresponding to the quasigroup operations from D_l-algebra $(Q; \Sigma)$ according to the previous Theorem. We obtain a new algebra $(Q; \Sigma_l)$.

Let Σ_r be the set of commutative Moufang loop operations corresponding to the quasigroup operations from D_r-algebra $(Q; \Sigma)$ according to the previous Theorem, too. We obtain an algebra $(Q; \Sigma_r)$.

Our final result shows the connection (through the hyperidentity) between the operations from Σ_l and Σ_r.

Theorem 2. *If* $(Q;\Sigma)$ *is a* D_l*-algebra* (D_r*-algebra*), *then the algebra* $(Q;\Sigma_l)$ (*algebra* $(Q;\Sigma_r)$) *satisfies the following non-trivial hyperidentity:*

$$X(x,Y(x,X(y,z))) = X(Y(x,y),Y(x,z)). \tag{18}$$

Proof. 1) Let $(Q;\Sigma)$ be a D_l-algebra. First, we prove the following equality:

$$B^{R_A^{-1}R_A(a)} = B$$

for every $A,B\in\Sigma$, where $R_A(a)$ and R_A are defined for any element $a\in Q$ and the arbitrary fixed element $0\in Q$ by the rule:

$$\begin{aligned} R_A(a)(x) &= A(x,a),\\ R_A(x) &= A(x,0). \end{aligned}$$

By putting $x=a$ in identity (16), we have:

$$\begin{gathered} A(B(y,z),a) = B^{R_A^{-1}}(A(y,a),A(z,a)),\\ R_A(a)B(y,z) = B^{R_A^{-1}}(R_A(a)y,R_A(a)z),\\ R_A(a)B\left(R_A^{-1}(a)y,R_A^{-1}(a)z\right) = B^{R_A^{-1}}(y,z),\\ B^{R_A^{-1}(a)} = B^{R_A^{-1}},\\ B^{R_A^{-1}R_A(a)} = B; \end{gathered}$$

Thus, the mapping $\varphi = R_A^{-1}R_A(a)$ is an automorphism of quasigroup $Q(B)$ for any operation $B\in\Sigma$. We have:

$$\begin{aligned} &R_A\varphi x = R_A(a)(x) = A(x,a) = R_A(x)\underset{A}{+}L_A(a),\\ &\varphi(x) = x\underset{A}{+}R_A^{-1}L_A(a) = x\underset{A}{+}b, \end{aligned}$$

where $b=R_A^{-1}L_A(a)$. Then:

$$\begin{aligned} &\varphi B(x,y) = B(\varphi x,\varphi y),\\ &B(x,y)\underset{A}{+}b = B(x\underset{A}{+}b,y\underset{A}{+}b),\\ &B(x,y)\underset{A}{+}z = B(x\underset{A}{+}z,y\underset{A}{+}z), \end{aligned}$$

where $z=b$ is an arbitrary element of Q. Thus,

$$(R_Bx\underset{B}{+}L_By)\underset{A}{+}z = R_B(x\underset{A}{+}z)\underset{B}{+}L_B(y\underset{A}{+}z). \tag{19}$$

The identity element of the loop $Q(\underset{A}{+})$ is the element $A(0,0) = 0$, and the identity element of the loop $Q(\underset{B}{+})$ is the element $B(0,0) = 0$. Substituting $x = 0$ or $y = 0$ in equality (19), we obtain:

$$\begin{aligned}
&L_B y \underset{A}{+} z = R_B z \underset{B}{+} L_B(y \underset{A}{+} z),\\
&R_B x \underset{A}{+} z = R_B(x \underset{A}{+} z) \underset{B}{+} L_B z,\\
&L_B(y \underset{A}{+} z) = (-R_B z) \underset{B}{+}(L_B y \underset{A}{+} z),\\
&R_B(x \underset{A}{+} z) = (-L_B z) \underset{B}{+}(R_B x \underset{A}{+} z),
\end{aligned}$$

where $(-x) \underset{B}{+}(x \underset{B}{+} y) = y$. From equality (19) we obtain:

$$\begin{aligned}
&(R_B x \underset{B}{+} L_B y) \underset{A}{+} z = ((-L_B z) \underset{B}{+}(R_B x \underset{A}{+} z)) \underset{B}{+}((-R_B z) \underset{B}{+}(L_B y \underset{A}{+} z)),\\
&(x \underset{B}{+} y) \underset{A}{+} z = (L_B(-z) \underset{B}{+}(x \underset{A}{+} z)) \underset{B}{+}(R_B(-z) \underset{B}{+}(y \underset{A}{+} z)).
\end{aligned} \tag{20}$$

In the commutative Moufang loop $Q(\underset{B}{+})$ the following equality [9] is satisfied:

$$(L_B u \underset{B}{+} v) \underset{B}{+}(R_B u \underset{B}{+} w) = u \underset{B}{+}(v \underset{B}{+} w)$$

for any $u, v, w \in Q$.

Applying this identity in equality (20) we obtain:

$$\begin{aligned}
&(x \underset{B}{+} y) \underset{A}{+} z = (-z) \underset{B}{+}((x \underset{A}{+} z) \underset{B}{+}(y \underset{A}{+} z)),\\
&z \underset{B}{+}((x \underset{B}{+} y) \underset{A}{+} z) = (x \underset{A}{+} z) \underset{B}{+}(y \underset{A}{+} z),\\
&z \underset{B}{+}(z \underset{A}{+}(x \underset{B}{+} y)) = (z \underset{A}{+} x) \underset{B}{+}(z \underset{A}{+} y);
\end{aligned}$$

Hence, in the algebra $(Q; \Sigma_l)$ the hyperidentity (18) is satisfied.

2) Analogously, by using the equality:

$$B^{L_A^{-1} L_A(a)} = B,$$

it is proved that if $(Q; \Sigma)$ is a D_r-algebra, then the algebra $(Q; \Sigma_r)$ satisfies the hyperidentity (18). □

References

[1] J. Aczel, *Vorlesungen über Functionalgleichungen und ihre Anwendungem*, Berlin, VEB Deutsch. Verl. Wiss, 1961.

[2] J. Aczel, *Quelque problèmes non-résolus dans la théorie des équations fonctionelles*, in "Neki Nerešeni Problemi u Matematici," Beograd, 1963, pp. 149–152.

[3] J. Aczel, J. Dhombres, *Functional equations in several variables with applications to mathematics, information theory and to the natural and social sciences*, Cambridge University Press, Cambridge, 1991.

[4] M. Aguiar, M. Livernet, *The associative operad and the weak order on the symmetric groups*, J. Homotopy Relat. Struct., **2**(2007), 57-84.

[5] A. A. Albert, *Quasigroup I*, Trans. Amer. Math. Soc., **54**(1943), pp. 507-519.

[6] A. A. Albert, *Quasigroup II*, Trans. Amer. Math. Soc., **55**(1944), pp. 401-419.

[7] A. D. Anosov, *On homomorphisms of many-sorted algebraic systems in connection with cryptographic applications*, Discrete Math. Appl. **17(4)**(2007), pp. 331–347.

[8] W. E. Barnes, *On Γ-rings of Nobusawa.* Pacific J. Math. **3**, 411–422 (1966).

[9] V. D. Belousov, *The structure of distributive quasigroups*, Mat. Sb. (N.S.) **50(92)(3)**(1960), pp. 267-298.

[10] V. D. Belousov, *Systems of quasigroups with generalized identities*, Russian Math. Surveys, **20**(1965), pp. 73-143.

[11] V. D. Belousov, *Foundations of the theory of quasigroups and loops*, Nauka, Moscow (In Russian), 1967.

[12] V. D. Belousov , Yu. M. Movsisyan, *On the rank of generalized identities*, Izv. Akad. Nauk. Arm., ser. Mat. **9**(1974), pp. 135-142.

[13] G. M. Bergman, *An invitation on general algebra and universal constructions.* Second edition, Springer, 2015.

[14] G. M. Bergman, *Hyperidentities of groups and semigroups.* Aequationes Math., **23**(1981), pp.50–65.

[15] S. L. Bloom , Z. Esik , E. G. Manes , *A Cayley theorem for Boolean algebras*, Amer. Math. Monthly, **97**(1990), pp.831-833.

[16] R. H. Bruck, *A survey of binary systems*, Springer-Verlag, 1971.

[17] S. Burris, H. P. Sankappanavar, *A course in universal algebra*, Springer-Verlag, 1981.

[18] A. Church, *Introduction to mathematical logic*, vol. I, Princeton University Press, Princeton, NJ 1956.

[19] K. Denecke , S. L. Wismath, *Hyperidentities and Clones*, Gordon and Breach Science Publishers, 2000.

[20] J. Denes, A. D. Keedwell, *Latin squares and their applications*, Akademiai Kiado, Budapest 1974 (also Academic Press, New York,1974, and English Universities Press, London, 1974).

[21] P. S. Gevorgyan, *On binary G-spaces*, Math.Notes, **96(4)**(2014), pp.600-602.

[22] E. Graczynska, *Algebras of M-solid quasivarieties*, Siatras International Bookshop, Athens, 2014.

[23] G. Grätzer, *Universal Algebra*, 2nd edition, Springer- Verlag, New York, 1979.

[24] A. Knoebel, A. Romanowska, *Distributive Multisemilattices.* Dissertationes Mathematicae (Rozprawy Marhematyczne), Warszawa (1991).

[25] J. Koppitz, K. Denecke,*M-Solid Varieties of Algebras*, Springer, 2006.

[26] A. Krapež, *Functional equations of generalized associativity, bisymmetry, transitivity and distributivity*, Publ. Inst. Math.(Beograd),**30**(1981), pp. 81-87.

[27] A. Krapež, *Generalized associativity on groupoids*, Publ. Inst. Math. (Beograd), **28**(1980), pp. 105-112.

[28] A. Krapež, *Cryptographically Suitable Quasigroups via Functional Equations*, In: S.Markovski and M.Gusev (Eds.): ICT Innovations 2012, AISC 207,Springer-Verlag Berlin Heidelberg 2013, 265-274.

[29] A. Krapež, *Quadratic level quasigroup equations with four variables. I*, Publ. Inst. Math. (Beograd)(N.S.), **81**(95)(2007), pp. 53-67.

[30] A. Krapež, *Quadratic level quasigroup equations with four variables. II*, Publ. Inst. Math. (Beograd)(N.S.), **93**(107)(2013), pp. 29-47.

[31] A. G. Kurosh, *Lectures on general algebra*, Chelsea, New York, 1963.

[32] J.-L. Loday, M. O. Ronco, *Order structure on the algebra of permutations and of planar binary trees*, J. Algebraic Combin., 15(2002), 253-270.

[33] A. I. Mal'tsev, *Model correspondences*, Izv. Akad. Nauk SSSR, Ser. Mat. 23, **3**(1959), pp. 313-336.

[34] A. I. Mal'cev, *Some problems in the theory of classes of models*, Proceedings of IV All-Union Mathematical Congress, Leningrad, Publishing House of the USSR Academy of Sciences, Leningrad,1963, pp. 169-198.

[35] A. I. Mal'cev, *Algebraic systems*, Grundlehren der Mathematischen Wissenschaften, vol. 192, Springer, New-York 1973.

[36] H. B. Mann, *The construction of ortogonal Lattin squares*, Ann. Math. Statist., **13**(1942), pp.418-423.

[37] H. B. Marandjian, *On computable solutions to functional equations*, CSIT 2011, pp. 20-22.

[38] H. B. Marandjian, *Selected Topics in Recursive Function Theory in Computer Science*, DTH, Lyngby, 1990.

[39] H. B. Marandjian, *General Form Recursive Equations I*, in: Computer Science Logic. Selected Papers, Lecture Notes in Computer Science, vol. 933 , Springer, Berlin, Heidelberg, 1995.

[40] S.V.Matveev , *Distributive groupoids in knot theory*, Mat. Sb. (N.S.) **119(1)** (1982), pp.78-88.

[41] R. Moufang, *Zur Structur von Alternativkörpern*, Math. Ann., **110**(1935), pp. 416-430.

[42] Yu. M. Movsisyan, *Biprimitive classes of algebras of second degree*, Mat. Issled. Akad. Nauk. Mold., **9**(1974), pp.70-82.

[43] Yu. M. Movsisyan, *Introduction to the theory of algebras with hyperidentities*, Yerevan State University Press (in Russian), 1986.

[44] Yu. M. Movsisyan, *The multiplicative group of field and hyperidentities*, Math. USSR Izvestiya, **35** (1990), pp. 337-391.

[45] Yu. M. Movsisyan, *Hyperidentities and hypervarieties in algebras*, Yerevan State University Press (in Russian), 1990.

[46] Yu. M. Movsisyan, *Hyperidentities in algebras and varieties*, Russian Math. Surveys, **53**(1998), pp. 57-108.

[47] Yu. M. Movsisyan, *Hyperidentities and hypervarieties.* Sci. Math. Jpn., **54(3)**(2001), pp. 595–640.

[48] Yu. M. Movsisyan, *Binary representations of algebras with at most two binary operations. A Cayley theorem for distributive lattices*, International Journal of Algebra and Computation, **19(1)** (2009), pp. 97-106.

[49] Yu. M. Movsisyan, *Hyperidentities and related concepts, I*, Arm. J. Math., **2**(2017), pp. 146-222.

[50] Yu. M. Movsisyan, *Hyperidentities and related concepts, II*, Arm. J. Math., **1**(2018), pp. 1-90.

[51] Yu. M. Movsisyan, E. Nazari, *A Cayley theorem for the multiplicative semigroup of a field*, Journal of Algebra and Its Applications, **11(2)**(2012), pp.1-12.

[52] Yu. M. Movsisyan, E. Nazari, *Transitive Modes*, Demonstratio Math., **44(3)**(2011), pp. 511-522.

[53] N. Nobusawa, *On a generalization of the ring theory.* Osaka J. Math. **1**, pp. 81–89 (1964).

[54] R. Padmanabhan, P. Penner, *Lattice ordered polynomial algebras*, Order, **15** (1998), pp. 75-86.

[55] V. Pambuccian , *Euclidean geometry problems rephrased in terms of midpoints and point-reflections*, Elem. Math., **60**(2005), pp.19-24.

[56] T. Pirashvili, *Sets with two associative operations*, Cent. Eur. J. Math. **2**(2003), pp. 169-183.

[57] H. Pflugfelder, *Quasigroups and Loops, Introduction*, Heldermann Verlag, Berlin, 1990.

[58] B. I. Plotkin, *Universal algebra, algebraic logic, and databases* , Kluwer Academic Publisher, 1994.

[59] J. H. Przytycki, *Knot and distributive homology: from arc colorings to Yang-Baxter homology*, in: New Ideas in Low Dimensional Topology, 2015, pp. 413-488.

[60] B. Richter, *Dialgebren, Doppelalgebren und ihre Homologie*, Diplomarbeit, Universitat Bonn (1997). https://www.math.uni-hamburg.de/home/richter/publications.html.

[61] Romanowska, A., Smith, J.D.H.: Modes, *World Scientific* (2002).

[62] M. K. Sen, *On* Γ*-semigroup.* In: Algebra and Its Applications (New Delhi, 1981). Lecture Notes in Pure and Appl. Math. 91, Dekker, New York, (1984), pp.301-308.

[63] M. K. Sen, N. K. Saha, *On* Γ*-semigroup-I.* Bull. Calcutta Math. Soc. **78**,(1986), pp. 180-186.

[64] A. Seth, Γ*-group congruences on regular* Γ*-semigroups.* Int. J. Math. Math. Sci. **15(1)**, 103-106 (1992), pp. 103-106.

[65] J. Slominski, *On the satisfiabilities and varieties for abstract algebras induced by the cones and functor dynamics*, Demonstratio Math., **26(1)**(1993), pp.11-22.

[66] J. D. H. Smith, *An Introduction to Quasigroups and Their Representations*, Chapman & Hall/Crc, 2006.

[67] J. D. H. Smith, *On groups of hypersubstitutions*, Algebra Universalis, **64**(2010), pp.39–48.

[68] J. D. H. Smith, A. B. Romanowska, *Post-Modern Algerbra*, Wiley, New York, 1999.

[69] S.K.Stein , *On the foundation of quasigroups*, Trans. Amer. Math. Soc., **85**(1957), pp. 228–256.

[70] W. Taylor, *Hyperidentities and hypervarieties*, Aequationes Math., **23**(1981), pp. 30–49.

[71] K. Urbanik, *On algebraic operations in idempotent algebras*, Colloquium Math., **13**(1965), pp. 129-157.

[72] K. A. Zaretskii, *Abstract characteristics of bisemigroups of binary operations*, Math. Notes, **1(5)**(1967), pp. 347-350.

[73] A. V. Zhuchok, *Free products of doppelsemigroups*, Algebra Universalis, **77**(2017), pp. 361–374.

[74] A. V. Zhuchok, M. Demko, *Free n-dinilpotent doppelsemigroups*, Algebra Discrete Math., **22**(2016), pp. 304-316.

[75] A. V. Zhuchok, *Dimonoids and bar-units*, Sib. Math. J. **56**(5), (2015), pp. 827-840.

[76] A. V. Zhuchok, *Structure of relatively free dimonoids*, Commun. Algebra **45**(4), (2017), pp. 1639-1656.

Contact information

Yu. Movsisyan Yerevan State University, Alex Manoogian 1, Yerevan 0025, Armenia
E-Mail(s): movsisyan@ysu.am
Web-page(s): www.ysu.am

Received by the editors: 14.05.2017
and in final form 02.06.2018.