Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume **25** (2018). Number 1, pp. 56–72

On application of linear algebra in classification cubic s-regular graphs of order $28p$

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Communicated by V. V. Kirichenko

Abstract. A graph is s-regular if its automorphism group acts regularly on the set of s-arcs. In this paper, by applying concept linear algebra, we classify the connected cubic s-regular graphs of order $28p$ for each $s \geqslant 1$, and prime p.

1. Introduction

In this study, all graphs considered are assumed to be finite, simple, and connected, unless stated otherwise. For a graph X, $V(X)$, $E(X)$, and $\mathrm{Aut}(X)$ denote its vertex set, edge set, and full automorphism group, respectively. Let G be a subgroup of $\mathrm{Aut}(X)$. For $u, v \in V(X)$, $\{u, v\}$ denotes the edge incident to u and v in X, and $N_X(u)$ denotes the neighborhood of u in X, that is, the set of vertices adjacent to u in X.

A graph $\widetilde{X}$ is called a covering of a graph X with projection $p : \widetilde{X} \to X$ if there is a surjection $p : V(\widetilde{X}) \to V(X)$ such that $p|_{N_{\widetilde{X}}(\widetilde{v})} : N_{\widetilde{X}}(\widetilde{v}) \to N_X(v)$ is a bijection for any vertex $v \in V(X)$ and $\widetilde{v} \in p^{-1}(v)$. A permutation group G on a set Ω is said to be semiregular if the stabilizer G_v of v in G is trivial for each $v \in \Omega$, and is regular if G is transitive, and semiregular. Let K be a subgroup of $\mathrm{Aut}(X)$ such that K is intransitive on $V(X)$. The quotient graph X/K induced by K is defined as the graph such that the set Ω of K-orbits in $V(X)$ is the vertex set of X/K and B,

2010 MSC: 05C25, 20B25.

Key words and phrases: S-regular graphs, Homology group, Coxeter graph, Symmetric graphs, Regular covering.

$C \in \Omega$ are adjacent if and only if there exists a $u \in B$ and $v \in C$ such that $\{u, v\} \in E(X)$. A covering $\widetilde{X}$ of X with a projection p is said to be regular (or N-covering) if there is a semiregular subgroup N of the automorphism group $\mathrm{Aut}(\widetilde{X})$ such that graph X is isomorphic to the quotient graph $\widetilde{X}/N$, say by h, and the quotient map $\widetilde{X} \to \widetilde{X}/N$ is the composition ph of p and h. If N is a cyclic or an elementary Abelian, then, $\widetilde{X}$ is called a cyclic or an elementary Abelian covering of X, and if $\widetilde{X}$ is connected, N becomes the covering transformation group.

An s-arc in a graph X is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices of X such that v_{i-1} is adjacent to v_i for $1 \leqslant i \leqslant s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leqslant i < s$; in other words, a directed walk of length s that never includes a backtracking. For a graph X and a subgroup G of $\mathrm{Aut}(X)$, X is said to be G-vertex-transitive, G-edge-transitive, or G-s-arc-transitive if G is transitive on the sets of vertices, edges, or s-arcs of X, respectively, and G-s-regular if G acts regularly on the set of s-arcs of X. A graph X is called vertex-transitive, edge-transitive, s-arc-transitive, or s-regular if X is $\mathrm{Aut}(X)$-vertex-transitive, $\mathrm{Aut}(X)$-edge-transitive, $\mathrm{Aut}(X)$-s-arc-transitive, or $\mathrm{Aut}(X)$-s-regular, respectively. In particular, 1-arc-transitive means arc-transitive, or symmetric.

Covering techniques have long been known as a powerful tool in topology, and graph theory. Regular covering of a graph is currently an active topic in algebraic graph theory. Tutte [23, 24] showed that every finite cubic symmetric graph is s-regular for some $s \geqslant 1$, and this s is at most five. It follows that every cubic symmetric graph has an order of the form $2mp$ for a positive integer m and a prime number p. In order to know all cubic symmetric graphs, we need to classify the cubic s-regular graphs of order $2mp$ for a fixed positive integer m and each prime p. Conder and Dobcsányi [5, 6] classified the cubic s-regular graphs up to order 2048 with the help of the "Low index normal subgroups" routine in MAGMA system [3]. Cheng and Oxley [4] classified the cubic s-regular graphs of order $2p$. By using the covering technique, cubic s-regular graphs with order

$$2p^2,\quad 2p^3,\quad 4p,\quad 4p^2,\quad 6p,\quad 6p^2,\quad 8p,\quad 8p^2,\quad 10p,\quad 10p^2,$$
$$12p,\quad 12p^2,\quad 14p,\quad 36p,\quad 44p,\quad 52p,\quad 66p,\quad 68p,\quad 76p,\quad 22p,$$
$$22p^2,\quad 10p^3,\quad \text{and}\quad 8p^3$$

were classified in $[1, 8-13, 19, 20, 22]$.

In this paper, by employing the covering technique, group-theoretical construction, and concept linear algebra, is investigated the connected cubic s-regular graphs of order $28p$ for each $s \geqslant 1$, and each prime p.

2. Preliminaries related to covering, Voltage graphs, lifting problems and the first homology group

Let X be a graph and K be a finite group. By a^{-1} we mean the reverse arc to an arc a. A voltage assignment (or K-voltage assignment) of X is a function $\xi : A(X) \to K$ with the property that $\xi(a^{-1}) = \xi(a)^{-1}$ for each arc $a \in A(X)$. The values of ξ are called voltages, and K is the voltage group. The graph $X \times_\xi K$ ($\mathrm{Cov}(X, \xi)$) derived from a voltage assignment $\xi : A(X) \to K$ has vertex set $V(X) \times K$ and edge set $E(X) \times K$, so that an edge (e, g) of $X \times_\xi K$ joins a vertex (u, g) to $(v, \xi(a)g)$ for $a = (u, v) \in A(X)$ and $g \in K$, where $e = \{u, v\}$. [21] The voltage assignment ξ on arcs extends to a voltage assignment on walks in a natural way, that is, the voltage on a walk W, say with consecutive incident arcs $a_1, a_2, \ldots, a_n$, is $\xi(a_1)\xi(a_2)\ldots\xi(a_n)$.

Clearly, the derived graph $X \times_\xi K$ is a covering of X with the first coordinate projection $p : X \times_\xi K \to X$, which is called the natural projection. By defining $(u, g')^g = (u, g'g)$ for any $g \in K$ and $(u, g') \in V(X \times_\xi K)$, K becomes a subgroup of $\mathrm{Aut}(X \times_\xi K)$ which acts semiregularly on $V(X \times_\xi K)$. Therefore, $X \times_\xi K$ can be viewed as a K-covering. For each $u \in V(X)$ and $uv \in E(X)$, the vertex set $\{(u, g)|g \in K\}$ is the fibre of u and the edge set $\{(u, g)(v, \xi(a)g)| \in K\}$ is the fibre of $\{u, v\}$, where $a = (u, v)$. Conversely, each regular covering $\widetilde{X}$ of X with a covering transformation group K can be derived from a K-voltage assignment. Given a spanning tree T of the graph X, a voltage assignment ξ is said to be T-reduced if the voltages on the tree arcs are the identity. Gross and Tucker [15] showed that every regular covering $\widetilde{X}$ of a graph X can be derived from a T -reduced voltage assignment $\widetilde{X}$ with respect to an arbitrary fixed spanning tree T of X.

Let $\widetilde{X}$ be a K-covering of X with a projection p. If $\alpha \in \mathrm{Aut}(X)$ and $\widetilde{\alpha} \in \mathrm{Aut}(\widetilde{X})$ satisfy $\widetilde{\alpha}p = p\alpha$, we call $\widetilde{\alpha}$ a lift of α, and α the projection of $\widetilde{\alpha}$. Concepts such as a lift of a subgroup of $\mathrm{Aut}(X)$ and the projection of a subgroup of $\widetilde{X}$ are self-explanatory [17]. The lifts and projections of such subgroups are of course subgroups in $\mathrm{Aut}(\widetilde{X})$ and $\mathrm{Aut}(X)$, respectively. In particular, if the covering graph $\widetilde{X}$ is connected, then the covering transformation group K is the lift of the trivial group, that is,

$$K = \{\widetilde{\alpha} \in \mathrm{Aut}(\widetilde{X}) : p = \widetilde{\alpha}p\}.$$

Let T be a spanning tree of a graph X. A closed walk W that contains only one cotree arc is called a fundamental closed walk. Similarly, a cycle W that contains only one cotree arc is called a fundamental cycle. Observe that a voltage assignment on arcs extends to a voltage assignment on walks in a natural way. Given $\alpha \in \text{Aut}(X)$, we define a function $\bar{\alpha}$ from the set of voltages on fundamental closed walks based at a fixed vertex $v \in V(X)$ to the voltage group K by

$$(\xi(C))^{\bar{\alpha}} = \xi(C^{\alpha}),$$

where C ranges over all fundamental closed walks at v, and $\xi(C)$ and $\xi(C^{\alpha})$ are the voltages on C and C^{α}, respectively. Note that if K is abelian, $\bar{\alpha}$ does not depend on the choice of the base vertex, and the fundamental closed walks at v can be substituted by the fundamental cycles generated by the cotree arcs of X.

Two coverings $\widetilde{X}_1$ and $\widetilde{X}_2$ of X with projection p_1 and p_2, respectively, are said to be isomorphic if there exist an automorphism $\alpha \in \text{Aut}(X)$ and an isomorphism $\tilde{\alpha} : \widetilde{X}_1 \to \widetilde{X}_2$ such that $\tilde{\alpha}\text{p}_2 = \text{p}_1\alpha$. In particular, if α is the identity automorphism of X, then we say $\widetilde{X}_1$ and $\widetilde{X}_2$ are equivalent.

For a graph X, $D(X)$ is a set of darts, which is required to be disjoint from $V(X)$, I is a mapping of $D(X)$ onto $V(X)$, called the incidence function, and λ is an involutory permutation of $D(X)$, called the dart-reversing involution. For convenience or if λ is not explicitly specified we sometimes write x^{-1} instead of λx. Intuitively, the mapping I assigns to each dart its initial vertex, and the permutation λ interchanges a dart and its reverse. The terminal vertex of a dart x is the initial vertex of λx. The set of all darts initiated at a given vertex u is denoted by D_u, called the neighborhood of u. The cardinality $|D_u|$ of D_u is the valency of the vertex u. The orbits of λ are called edges; thus each dart determines uniquely its underlying edge. An edge is called a semiedge if $\lambda x = x$, a loop if $\lambda x \neq x$ and $I\lambda x = Ix$, and it is called a link otherwise. A walk of length $n \geqslant 1$ is a sequence of n darts $W = x_1 x_2 \ldots x_n$ such that, for each index $1 \leqslant k \leqslant n-1$, the terminal vertex of x_k coincides with the initial vertex of x_{k+1}. Moreover, we define each vertex to be a trivial walk of length 0. The initial vertex of W is the initial vertex of x_1, and the terminal vertex of W is the terminal vertex of x_n. The walk is closed if the initial and the terminal vertex coincide. In this case we say that the walk is based at that vertex. If W has initial vertex u and terminal vertex v, then we usually write $W : u \to v$. Let W_1 and W_2 be two walks such that the terminal vertex of W_1 coincides with the initial vertex of W_2. We define the product W_1W_2 as the juxtaposition of the two sequences. A walk W is reduced if it contains no subsequence of the form xx^{-1}.

By $\pi(X)$ we denote the fundamental groupoid of a graph X, that is, the set of all reduced walks equipped with the product W_1W_2. The group

$\pi(X, u)$ is called the fundamental group of X at u. The fundamental group is not a free group in general. Consequently, the first homology group $H_1(X)$, obtained by abelianizing $\pi(X, u)$, is not necessarily a free Z-module. Namely, let $r_e + r_s$ be the minimal number of generators of $\pi(X, u)$, where r_s is the number of semiedges and r_e is the number of cotree loops and links relative to some spanning tree. Then $H_1(X) \cong Z^{r_e} \times Z_2^{r_s}$. [18] The first homology group $H_1(X, Z_p) \cong H_1(X)/pH_1(X)$ with Z_p as the coefficient ring can be considered as a vector space over the field Z_p. Observe that

$$H_1(X, Z_p) \cong \begin{cases} Z_p^{r_e+r_s} & p = 2 \\ Z_p^{r_e} & p \geqslant 3. \end{cases}$$

We start by introducing five propositions for later applications in this paper. The following proposition is necessary to classify s-regular graph.

Proposition 2.1. [16] Let X be a connected symmetric graph of prime valency and G a s-regular subgroup of $\mathrm{Aut}(X)$ for some $s \geqslant 1$. If a normal subgroup N of G has more than two orbits, then it is semiregular and G/N is an s-regular subgroup of $\mathrm{Aut}(X_N)$, where X_N is the quotient graph of X corresponding to the orbits of N. Furthermore, X is a N-regular covering of X_N.

Proposition 2.2. [24] If X is an s-arc regular cubic graph, then $s \leqslant 5$.

Proposition 2.3. [9] Let X be a connected cubic symmetric graph of order $4p$ or $4p^2$ for a prime p. Then X is isomorphic to the 2-regular hypercube Q_3 of order 8, the 2-regular generalized Petersen graphs $P(8, 3)$ or $P(10, 7)$ of order 16 or 20 respectively, the 3-regular Dodecahedron of order 20 or the 3-regular Coxeter graph C_{28} of order 28.

Proposition 2.4. [19] Let p be a prime and let X be a cubic symmetric graph of order $14p$. Then, X is 1-, 2- or 3-regular. Furthermore,

(1) X is 1-regular if and only if X is isomorphic to one of the graphs $F42$, $F98A$, $CF14p$ and $DF14p$ where $p > 7$ and $p \equiv 1 \mod 6$.
(2) X is 2-regular if and only if X is isomorphic to one of the graphs $F98B$ and $F182C$.
(3) X is 3-regular if and only if X is isomorphic to one of the graphs $F28$ and $F182D$.

The next proposition[9, Theorem 6.1] is shown the cyclic or elementary abelian coverings of the complete graph K_4.

Proposition 2.5. Let K be a cyclic or an elementary abelian group and let $\widetilde{X}$ be a connected K-covering of the complete graph K_4 whose fibre-preserving group is arc-transitive. Then, X is 2-regular. Moreover,

(1) if K is cyclic then $\widetilde{X}$ is isomorphic to the complete graph K_4, the 3-dimensional hypercube Q_3, or the generalized Petersen graph $P(8,3)$.

(2) If K is elementary abelian but not cyclic, then $\widetilde{X}$ is isomorphic to one of EC_{p^3} for a prime p (defined in Example 3.2 in [9]).

3. Coxeter graph

In the mathematical field of graph theory, the Coxeter graph is a 3-regular graph with 28 vertices and 42 edges.

$$\begin{aligned} V(X) = [&0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,\\ &22,23,24,25,26,27],\\ E(X) = [&\{0,1\},\{0,23\},\{0,24\},\{1,2\},\{1,12\},\{2,3\},\{2,25\},\{3,4\},\\ &\{3,21\},\{4,5\},\{4,17\},\{5,6\},\{5,11\},\{6,7\},\{6,27\},\{7,8\},\\ &\{7,24\},\{8,9\},\{8,25\},\{9,10\},\{9,20\},\{10,11\},\{10,26\},\\ &\{11,12\},\{12,13\},\{13,14\},\{13,19\},\{14,15\},\{14,27\},\\ &\{15,16\},\{15,25\},\{16,17\},\{16,26\},\{17,18\},\{18,19\},\\ &\{18,24\},\{19,20\},\{20,21\},\{21,22\},\{22,23\},\{22,27\},\{23,26\}]. \end{aligned}$$

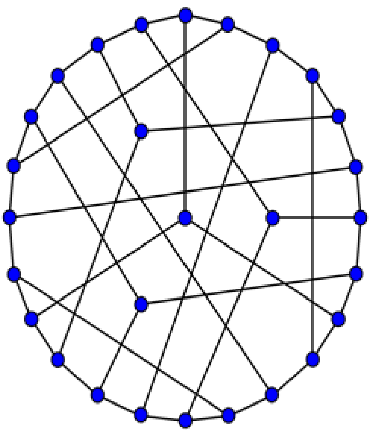

Figure 1. Coxeter graph.

We choose

$$\alpha = (2,12)(3,11)(4,5)(6,17)(7,18)(8,19)(9,20)(10,21)(13,25)(14,15)(16,27)(22,26),$$
$$\beta = (2,12)(3,13)(4,14)(5,15)(6,16)(7,26)(8,10)(11,25)(17,27)(18,22)(19,21)(23,24),$$
$$\gamma = (1,23)(2,26)(3,10)(4,9)(5,20)(6,19)(7,18)(8,17)(11,21)(12,22)(13,27)(16,25),$$
$$\sigma = (0,1)(2,24)(3,18)(4,17)(5,16)(6,15)(7,25)(11,26)(12,23)(13,22)(14,27)(19,21),$$

as automorphisms of Coxeter graph. Then $\mathrm{Aut}(F28A) = \langle \alpha, \beta, \gamma, \sigma \rangle$. The automorphism group of the Coxeter graph is a group of order 336. It acts transitively on the vertices, on the edges and on the arcs of the graph. Therefore the Coxeter graph is a symmetric graph. It has automorphisms that take any vertex to any other vertex and any edge to any other edge. According to the Foster census, the Coxeter graph, referenced as $F28A$, is the only cubic symmetric graph on 28 vertices. By sage[2] the automorphism group of Coxeter graph has one proper arc-transitive subgroup $G = \langle \beta, \gamma, \sigma \rangle$.

We choose a spanning tree T of Coxeter graph consisting of the edges

$$(0,1),(0,23),(0,24),(1,2),(1,12),(2,3),(2,25),(3,4),$$
$$(3,21),(4,5),(4,17),(5,6),(5,11),(6,7),(6,27),(7,8),(8,9),(9,10),$$
$$(9,20),(10,26),(12,13),(13,14),(13,19),(14,15),(15,16),(17,18),(21,22).$$

By choosing T, we can define a T-reduced voltage assignment.

We show the cotree arcs by setting

$$x_1 = (7,24), \quad x_2 = (8,25), \quad x_3 = (10,11), \quad x_4 = (11,12),$$
$$x_5 = (14,27), \quad x_6 = (15,25), \quad x_7 = (16,17), \quad x_8 = (16,26),$$
$$x_9 = (18,19), \quad x_{10} = (18,24), \quad x_{11} = (19,20), \quad x_{12} = (20,21),$$
$$x_{13} = (22,23), \quad x_{14} = (22,27), \quad x_{15} = (23,26).$$

4. Classifying cubic s-regular graphs of order $28p$

In this section, by applying concept linear algebra, the connected cubic s-regular graphs of orders $28p$, where p is a prime, is investigated. Assume that a connected graph X and a subgroup $G \leqslant \mathrm{Aut}(X)$ are given. Choose a spanning tree T of X and a set of arcs $\{x_1, \ldots, x_r\} \subseteq A(X)$

containing exactly one arc from each edge in $E(X \setminus T)$. Let B_T be the corresponding basis of the first homology group $H_1(X, Z_p)$ determined by $\{x_1, \ldots, x_r\}$. Further, denote by $G^{*h} = \{\alpha^{*h} | \alpha \in G\} \leqslant GL(H_1(X, Z_p))$ the induced action of G on $H_1(X, Z_p)$, and let $M_G \leqslant Z_p^{r\times r}$ be the matrix representation of G^{*h} with respect to the basis B_T. By M_G^t we denote the dual group consisting of all transposes of matrices in M_G.

The following proposition is a special case of [18, Proposition 6.3, Corollary 6.5].

Proposition 4.1. Let T be a spanning tree of a connected graph X and let the set $\{x_1, x_2, \ldots, x_r\} \subseteq A(X)$ contain exactly one arc from each cotree edge. Let $\xi : A(X) \to Z_p$ be a voltage assignment on X which is trivial on T, and let $Z(\xi) = [\xi(x_1), \xi(x_2), \ldots, \xi(x_r)]^t \in Z_p^{r\times 1}$. Then the following hold.

(a) A group $G \leqslant \mathrm{Aut}(X)$ lifts along $p_\xi : \mathrm{Cov}(X, \xi) \to X$ if and only if the induced subspace $\langle Z(\xi)\rangle$ is an M_G^t-invariant 1-dimensional subspace.

(b) If $\xi' : A(X) \to Z_p$ is another voltage assignment satisfying (a), then $\mathrm{Cov}(X, \xi')$ is equivalent to $\mathrm{Cov}(X, \xi)$ if and only if $\langle Z(\xi)\rangle = \langle Z(\xi')\rangle$, as subspaces. Moreover, $\mathrm{Cov}(X, \xi')$ is isomorphic to $\mathrm{Cov}(X, \xi)$ if and only if there exists an automorphism $\alpha \in \mathrm{Aut}(X)$ such that the matrix M_α^t maps $\langle Z(\xi')\rangle$ onto $\langle Z(\xi)\rangle$.

We have the following theorem, by [5, 6].

Theorem 4.2. Let $p < 79$ be a prime. Then, there are cubic symmetric graphs of order $28p$. We classify all cubic symmetric graphs in Table 1.

Graph	order	s-regular
F056A	28*2	1
F056B	28*2	2
F056C	28*2	3
F084A	28*3	2
F364A	28*13	2
F364B	28*13	2
F364C	28*13	2
F364D	28*13	2
F364E	28*13	2
F364F	28*13	2
F364G	28*13	3

Table 1. Cubic symmetric graphs of order $28p$ with $p < 79$.

Remark 4.3. To find all arc-transitive G-admissible Z_p-covering projections of $F28A$, we have to find, by proposition 4.1, all invariant 1-dimensional subspaces of the transpose of the matrix M_G.

For this purpose, we express the following lemma.

Lemma 4.4. Let B, C and D be the transposes of the matrices which represent the linear transformations β^{*h}, γ^{*h} and σ^{*h} relative to $B_T = \{C_{x_i} | 1 \leqslant i \leqslant 15\}$; the standard ordered basis of $H_1(F28A, Z_p)$ associated with the spanning tree T and the arcs $x_i (i = 1, \ldots, 15)$, respectively. Then

$$B = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$C = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & -1 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & -1 \\
0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & 1 & -1 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1
\end{pmatrix}$$

$$D = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\
0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Proof. The rows of these matrices are obtained by letting the automorphisms β, γ and σ act on B_T. For example, the permutation β maps the cycle

$$[0, 1, 2, 3, 4, 5, 6, 7, 24, 0]$$

corresponding to x_1, to the cycle

$$[0, 1, 12, 13, 14, 15, 16, 26, 23, 0].$$

Since the latter is the sum of the base cycles corresponding to x_8 and x_{15}^{-1}, the first row of B is

$$(0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, -1).$$

By similar computations we can get the matrices B, C and D. □

By Sage [2] we have the following lemma.

Lemma 4.5. The minimal polynomials of B, C and D are

$m_B(x) = (x-1)(x+1)$, $m_C(x) = (x-1)(x+1)$ and $m_D(x) = (x-1)(x+1)$,

respectively.

By a straightforward calculation, lemma 4.4 and lemma 4.5, we have

$$\begin{aligned}
\ker(B - I) &= \langle u_1, u_2, u_3, u_4, u_5, u_6, u_7 \rangle, \\
\ker(B + I) &= \langle u_8, u_9, u_{10}, u_{11}, u_{12}, u_{13}, u_{14}, u_{15} \rangle, \\
\ker(C - I) &= \langle v_1, v_2, v_3, v_4, v_5, v_6, v_7 \rangle, \\
\ker(C + I) &= \langle v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15} \rangle, \\
\ker(D - I) &= \langle w_1, w_2, w_3, w_4, w_5, w_6 \rangle, \\
\ker(D + I) &= \langle w_7, w_8, w_9, w_{10}, w_{11}, w_{12}, w_{13}, w_{14}, w_{15} \rangle,
\end{aligned}$$

where

$$u_1 = \begin{pmatrix} 1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\-1\\0\\0\\-1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0\\1\\1\\0\\0\\-1\\0\\0\\0\\0\\1\\-1\\0\\0\\0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0\\0\\0\\1\\0\\1\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 0\\0\\0\\0\\1\\0\\0\\0\\0\\1\\0\\0\\1\\-1\\0 \end{pmatrix}, \quad u_5 = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\1\\0\\0\\1\\0\\0\\1\\-1\\0 \end{pmatrix},$$

$$u_6 = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\-1\\1\\0\\0\\0 \end{pmatrix}, \quad u_7 = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\1\\-1\\0\\0\\-1\\0\\0 \end{pmatrix}, \quad u_8 = \begin{pmatrix} 1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1 \end{pmatrix}, \quad u_9 = \begin{pmatrix} 0\\1\\-1\\0\\0\\-1\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}, \quad u_{10} = \begin{pmatrix} 0\\0\\0\\1\\0\\-1\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix},$$

$$u_{11} = \begin{pmatrix} 0\\0\\0\\0\\1\\0\\0\\0\\0\\-1\\0\\0\\1\\-1\\0 \end{pmatrix}, \quad u_{12} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\1\\0\\0\\1\\0\\0\\-1\\1\\0 \end{pmatrix}, \quad u_{13} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}, \quad u_{14} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\1\\-1\\0\\0\\1\\0\\0 \end{pmatrix}, \quad u_{15} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\1\\0\\0\\0 \end{pmatrix},$$

$$v_1 = \begin{pmatrix} 1\\0\\0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0\\1\\0\\0\\0\\0\\-1\\1\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\0\\-1\\0\\0\\0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\1\\1\\0\\1 \end{pmatrix}, \quad v_5 = \begin{pmatrix} 0\\0\\0\\0\\1\\-1\\0\\-1\\0\\0\\0\\0\\1\\-1\\1 \end{pmatrix},$$

$$v_6 = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\1\\-1\\1 \end{pmatrix}, \quad v_7 = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\0\\-1\\1\\-1 \end{pmatrix}, \quad v_8 = \begin{pmatrix} 1\\0\\0\\0\\0\\0\\0\\0\\0\\-1\\0\\0\\0\\0\\1 \end{pmatrix}, \quad v_9 = \begin{pmatrix} 0\\1\\0\\0\\0\\0\\1\\-1\\0\\0\\0\\0\\0\\0\\2 \end{pmatrix}, \quad v_{10} = \begin{pmatrix} 0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\2 \end{pmatrix},$$

$$v_{11} = \begin{pmatrix} 0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\-1\\-1\\0\\-1 \end{pmatrix}, \quad v_{12} = \begin{pmatrix} 0\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\1\\-1\\0 \end{pmatrix}, \quad v_{13} = \begin{pmatrix} 0\\0\\0\\0\\0\\1\\0\\-1\\0\\0\\0\\0\\0\\0\\1 \end{pmatrix}, \quad v_{14} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\-1\\1\\-1 \end{pmatrix}, \quad v_{15} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\0\\1\\-1\\-1 \end{pmatrix},$$

$$w_1=\begin{pmatrix}1\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\-1\\1\\0\end{pmatrix},\quad w_2=\begin{pmatrix}0\\1\\0\\0\\0\\-1\\0\\0\\0\\0\\1\\-1\\0\\-1\\0\end{pmatrix},\quad w_3=\begin{pmatrix}0\\0\\1\\0\\0\\0\\0\\-1\\0\\0\\1\\-1\\0\\-1\\0\end{pmatrix},\quad w_4=\begin{pmatrix}0\\0\\0\\1\\0\\0\\0\\1\\0\\0\\0\\0\\-1\\1\\-1\end{pmatrix},\quad w_5=\begin{pmatrix}0\\0\\0\\0\\0\\0\\1\\0\\0\\1\\0\\0\\-1\\1\\0\end{pmatrix},$$

$$w_6=\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\\1\\-1\\0\\0\\1\\0\\0\end{pmatrix},\quad w_7=\begin{pmatrix}1\\0\\0\\0\\0\\-1\\0\\0\\0\\0\\0\\0\\-1\\1\\0\end{pmatrix},\quad w_8=\begin{pmatrix}0\\1\\0\\0\\0\\-1\\0\\0\\0\\0\\0\\0\\-1\\1\\0\end{pmatrix},\quad w_9=\begin{pmatrix}0\\0\\1\\0\\0\\0\\0\\1\\0\\0\\0\\0\\1\\-1\\0\end{pmatrix},\quad w_{10}=\begin{pmatrix}0\\0\\0\\1\\0\\0\\0\\-1\\0\\0\\0\\0\\-1\\1\\1\end{pmatrix},$$

$$w_{11}=\begin{pmatrix}0\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\1\\-1\\0\end{pmatrix},w_{12}=\begin{pmatrix}0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0\\1\\-1\\0\end{pmatrix},w_{13}=\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\-1\\0\\0\end{pmatrix},\quad w_{14}=\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0\end{pmatrix},\quad w_{15}=\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\1\\1\\0\\0\end{pmatrix}$$

Now, we have $\ker(B \pm I) \cap \ker(C \pm I) \cap \ker(D \pm I) = 0$.

Due to the above description, we have the following result.

Corollary 4.6. There is not $\langle B, C, D\rangle$-invariant 1-dimensional subspaces in Z_p^{15}.

Remark 4.7. If X is a regular graph with valency k on m vertices and $s \geqslant 1$, then there are exactly $mk(k-1)^{s-1}$ s-arcs. It follows that if X is s-arc transitive then $|\operatorname{Aut}(X)|$ must be divisible by $mk(k-1)^{s-1}$, and if X is s-regular, then $|\operatorname{Aut}(X)| = mk(k-1)^{s-1}$. In particular, a cubic arc-transitive graph X is s-regular if and only if $|\operatorname{Aut}(X)| = (3m)2^{s-1}$.

Theorem 4.8. Let $p \geqslant 79$ be a prime. Then, there is no cubic symmetric graph of order $28p$.

Proof. Suppose that X is a connected cubic symmetric graph of order $28p$, where $p \geqslant 79$ is a prime. Set $A := \operatorname{Aut}(X)$. let P be a Sylow p-subgroup of A. If P is normal in A then by Proposition 2.1 X is a regular covering of the graph $F28A$ with the covering transformation group Z_p. On the contrary, suppose that P is not normal in A. Assume that $N_A(P)$ is the normalizer of P in A. By Sylow's theorem, the number of Sylow p-subgroups of A is $1 + np = |\frac{A}{N_A(P)}|$, for a positive integer n. By Proposition 2.2 X is at most 5-regular and hence $|A|$ is a divisor of $3 \cdot 7 \cdot 2^6 p$. Then $1 + np$ is a divisor of $3 \cdot 7 \cdot 2^6$. Since $p \geqslant 79$, we have $(n, p) = (1, 83), (1, 167), (17, 79), (1, 223), (3, 149)$. We consider the following cases.

Case I: $(n, p) = (17, 79), (1, 223), (3, 149)$.

First, we suppose that $(n, p) = (17, 79), (1, 223), (3, 149)$. Then $3 \cdot 7 \cdot 2^5 \mid |A|$, implying that X is at least 4-regular. Assume that A is nonsolvable. Its composition factors would have to be a non-abelian simple $\{2, 3, 7, p\}$-group where $p = 79, 149, 223$. Now, we can get a contradiction, by the classification of finite simple groups [14, pp. 12-14] and [7]. Let N be a minimal normal subgroup of A and X/N the quotient graph of X corresponding to the orbits of N. Then N is an elementary abelian. By Proposition 2.1 X/N is at least 4-regular with order 28, $14p$, $7p$ or $4p$. If $|X/N| = 28, 14p, 4p$, by [3,4], Proposition 2,3 and 2.4 a contradiction can be obtained. If $|X/N| = 7p$, then the quotient graph X_N corresponding to orbits of N has odd number($7p$) of vertices and valency 3. It is a contradiction.

Case II: $(n, p) = (1, 83), (1, 167)$.

Now, we assume that $(n, p) = (1, 83), (1, 167)$. Then $3 \cdot 7 \cdot 2^2 \mid |A|$, implying that X is at least 1-regular. With the same reasoning as Case I A is solvable. Let M be a minimal normal subgroup of A and X/M the quotient graph of X corresponding to the orbits of M. Then M is elementary abelian. If $|M| \neq p$ then by Proposition 2.1 X/M is at least 1-regular with order $14p$, $7p$ or $4p$. By the same argument as above, there is no s-regular $(s \geqslant 1)$ with order $14p$, $7p$ or $4p$. If $|M| = p$, then the quotient graph X/M has order 28. The automorphism group of the Coxeter graph contains no 1-regular subgroup [2]. Therefore the quotient graph X/M is at least 2-regular. Since $M \lhd A$, A/M is solvable. Let T/M be a minimal normal subgroup of A/M. Hence T/M is an elementary abelian 2-, 7-group. By Proposition 2.3 X/T is at least 2-regular with order 4 or 14. We arrive at a contradiction with Proposition 2.5 and [19, Proposition 2.1 and Corollary 2.2]. Therefore P is normal in A. Then X is a regular covering of the graph $F28A$ with the covering transformation group Z_p. By sage [2] the automorphisms of Coxeter graph has one proper arc-transitive subgroup $G = \langle \beta, \gamma, \sigma \rangle$. By Remark 4.3, we have to find all invariant one-dimensional subspaces of the transpose of the matrix M_G. In other words, we need to look for $\langle B, C, D \rangle$-invariant 1-dimensional subspaces in Z_p^{15}. By Corollary 4.6 there is not $\langle B, C, D \rangle$-invariant one-dimensional subspaces in Z_p^{15}. Then by Proposition 4.1.a $G \leqslant \mathrm{Aut}(F28A)$ cannot lift and hence there is no cubic symmetric graph of order $28p$ where $p \geqslant 79$. □

Corollary 4.9. Let p be a prime and let X be a connected cubic symmetric graph of order $28p$. Then

(1) X is 1-regular if and only if X is isomorphic to the graph $F056A$.

(2) X is 2-regular if and only if X is isomorphic to one of the eight graphs $F056B$, $F084A$, $F364A$, $F364B$, $F364C$, $F364D$, $F364E$ and $F364F$.

(3) X is 3-regular if and only if X is isomorphic to one of the two graphs $F056C$ and $F364G$.

Proof. By Theorems 4.2 and 4.8, the proof is complete. □

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Received by the editors: 16.02.2016.