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Construction of a complementary quasiorder*

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Abstract. For a monounary algebra $\mathcal{A} = (A, f)$ we study the lattice $\operatorname{Quord} \mathcal{A}$ of all quasiorders of $\mathcal{A}$, i.e., of all reflexive and transitive relations compatible with f. Monounary algebras (A, f) whose lattices of quasiorders are complemented were characterized in 2011 as follows: $(*)$ $f(x)$ is a cyclic element for all $x \in A$, and all cycles have the same square-free number n of elements. Sufficiency of the condition $(*)$ was proved by means of transfinite induction. Now we will describe a construction of a complement to a given quasiorder of (A, f) satisfying $(*)$.

Introduction

If $\mathcal{A}$ is an algebra, then the set consisting of all reflexive and transitive relations on $\mathcal{A}$, which are compatible with all operations of $\mathcal{A}$ (i.e., quasiorders of $\mathcal{A}$), will be denoted $\operatorname{Quord} \mathcal{A}$. Then $\operatorname{Quord} \mathcal{A}$ is a lattice with respect to inclusion. It is easy to see that the latice $\operatorname{Con} \mathcal{A}$ of all congruences of $\mathcal{A}$ is a sublattice of $\operatorname{Quord} \mathcal{A}$.

We will deal with the lattice $\operatorname{Quord}(A, f)$ of all quasiorders of (A, f), where (A, f) is a monounary algebra. The necessary and sufficient conditions for a monounary algebra (A, f) under which the lattice $\operatorname{Quord}(A, f)$ is complemented were found in [4]. The sufficiency of the condition was proved by means of transfinite induction. Analogous conditions for the

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lattice Con (A, f) to be complemented were proved by Egorova and Skornyakov [2].

The aim of our paper is to describe a construction of a complement to a given quasiorder $\alpha \in \mathrm{Quord}(A, f)$ when the algebra (A, f) satisfies the condition $(*)$, i.e., when the lattice $\mathrm{Quord}(A, f)$ is complemented.

Another, still open question which is of interest is how to find a complement to a given quasiorder in an arbitrary monounary algebra provided the quasiorder has a complement.

1. Preliminaries

By a monounary algebra we will understand a pair $\mathcal{A} = (A, f)$ where A is a nonempty set and $f\colon A \to A$ is a mapping.

A monounary algebra $\mathcal{A}$ is called *connected* if for arbitrary $x, y \in A$ there are non-negative integers n, m such that $f^n(x) = f^m(y)$. A maximal connected subalgebra of a monounary algebra is called a *connected component.*

An element $x \in A$ is referred to as *cyclic* if there exists a positive integer n such that $f^n(x) = x$. Then the set $\{x, f^1(x), f^2(x), \dots, f^{n-1}(x)\}$ is said to be a *cycle.*

A *quasiorder* of an algebra $\mathcal{A} = (A, F)$ is a reflexive and transitive binary relation on A, which is compatible with all operations $f \in F$. A quasiorder is a congruence of $\mathcal{A}$ if it is symmetric. We will denote by $\mathrm{Quord}\,\mathcal{A}$ the lattice of all quasiorders ordered by inclusion and by $\mathrm{Con}\ \mathcal{A}$ its sublattice, the lattice of all congruences. The smallest and the greatest elements of $\mathrm{Quord}\,\mathcal{A}$ and of $\mathrm{Con}\ \mathcal{A}$ are denoted $I_A = \{(a, a) : a \in A\}$ and $A \times A$. If $\wedge_{\mathrm{Con}}, \vee_{\mathrm{Con}}, \wedge_{\mathrm{Quord}}, \vee_{\mathrm{Quord}}$ are the corresponding operations in the lattices $\mathrm{Con}\ \mathcal{A}$ and $\mathrm{Quord}\,\mathcal{A}$, then it is obvious, that $\wedge_{\mathrm{Con}} = \wedge_{\mathrm{Quord}} = \cap$ and $\vee_{\mathrm{Con}} = \vee_{\mathrm{Quord}}$ is the operation of the transitive hull. Therefore we will use the symbols $\wedge$ and $\vee$ for these operations.

A *complement* to a quasiorder α of (A, f) is a quasiorder β of (A, f) such that $\alpha \vee \beta = A \times A$ and $\alpha \wedge \beta = I_A$.

For $a, b \in A$ let $\alpha(a, b)$ and $\theta(a, b)$ be the smallest quasiorder and the smallest congruence, respectively, such that $(a, b) \in \alpha(a, b)$, $(a, b) \in \theta(a, b)$.

The symbol $\mathbb{N}$ is used for the set of all positive integers.

From the paper of Berman [1] concerning congruences, it follows that if $n \in \mathbb{N}$, then θ is a congruence relation of an n-element cycle (C, f) if and only if there is $d \in \mathbb{N}$ such that d divides n and for each $x \in C$, $[x]_\theta = \bigl\{x, f^d(x), \dots, f^{(\frac{n}{d}-1)\,d}(x)\bigr\} = \{f^k(x)\colon 0 \leqslant k \equiv d (\mathrm{mod}\ n)\}$.

The congruence with this property will be denoted θ_d^C (or simply θ_d). It is easy to verify that for each $x \in C$, θ_d^C is the smallest congruence containing the pair $(x, f^d(x))$.

It appears that even in a case when a quasiorder is congruence, finding a complementary quasiorder can prove to be difficult. E.g., let (A, f) be an algebra such that $A = \{0, 1, 2, 3, 4, 5, 0', 1', 2', 3', 4', 5'\}$ and

$$0 \xrightarrow{f} 1 \xrightarrow{f} 2 \xrightarrow{f} 3 \xrightarrow{f} 4 \xrightarrow{f} 5 \xrightarrow{f} 0 \text{ and } 0' \xrightarrow{f} 1' \xrightarrow{f} 2' \xrightarrow{f} 3' \xrightarrow{f} 4' \xrightarrow{f} 5' \xrightarrow{f} 0'.$$

Let us consider a congruence α such that $\alpha = \theta(0, 3) \cup \theta(0', 4')$. The lattice $\operatorname{Quord}(A, f)$ is complemented. However, to find a complementary quasiorder to α is not trivial. A general construction for finding a complementary quasiorder to a given quasiorder if the lattice $\operatorname{Quord}(A, f)$ is complemented could help with the task. In the next section, we will describe such a construction.

In [3] the following assertions were proved; we will use them often without any further quotation:

Lemma 1. *Let (A, f) be an n-element cycle, $n \in \mathbb{N}$. Then* $\operatorname{Quord}(A, f) = \operatorname{Con}(A, f) = \{\theta_d \colon d/n\}$.

Lemma 2. *Let (A, f) be an n-element cycle, $n \in \mathbb{N}$. If $a, b \in A$, $f^m(a) = b$, $d =$g.c.d.(n, m), then $\alpha(a, b) = \theta_d$.*

Corollary 1. *Let (A, f) be an n-element cycle, d/n, k/n. Then $\theta_d \vee \theta_k = \theta_{\text{g.c.d.}(d,k)}$ and $\theta_d \wedge \theta_k = \theta_{\text{l.c.m.}(d,k)}$.*

In the following, we will suppose that

- (A, f) is a monounary algebra,
- for each $a \in A$, the element $f(a)$ is cyclic,
- there is $n \in \mathbb{N}$ square-free, such that each cycle of (A, f) has n elements.

From Lemma 1 we get

Lemma 3. *Let (A, f) be a cycle, $\alpha = \theta_d$, d/n. Then β is a complement to α in the lattice* $\operatorname{Quord}(A, f)$ *if and only if $\beta = \theta_e$, $e = \frac{n}{d}$.*

For $a \in A$ let $C(a)$ be the cycle containing the element $f(a)$.

Lemma 4. *Assume that x is a noncyclic element of A, $\alpha \restriction C(x) = \theta_d^{C(x)}$, d/n. Next suppose that $k \in \mathbb{N}$ and either $(x, f^k(x)) \in \alpha$ or $(f^k(x), x) \in \alpha$. Then d/k.*

Proof. The assumption implies that either

$$(f(x), f^{k+1}(x)) \in \alpha \quad \text{or} \quad (f^{k+1}(x), f(x)) \in \alpha,$$

i.e., either $(f(x), f^{k+1}(x)) \in \theta_d^{C(x)}$ or $(f^{k+1}(x), f(x)) \in \theta_d^{C(x)}$. In both cases we obtain that d/k. □

Definition 1. Let $\alpha \in \mathrm{Quord}(A, f)$. We denote $\bar{\alpha}$ the dual quasiorder to α, i.e, such that, whenever $a, b \in A$,

$$(a, b) \in \alpha \iff (b, a) \in \bar{\alpha}.$$

It is easy to see that the relation $\alpha \cap \bar{\alpha}$ is an equivalence on A.

Definition 2. Let r_α be the binary relation (depending on α) defined on the set of all cycles of (A, f) as follows: If B, D are cycles of (A, f), then we put $B\ r_\alpha\ D$, if there are $k \in \mathbb{N}$, cycles $B = C_0, C_1, \ldots, C_k = D$, elements $c_0 \in C_0, c_1 \in C_1, \ldots, c_k \in C_k$ such that for each $i \in \{0, 1, \ldots, k-1\}$, $(c_i, c_{i+1}) \in \alpha \cup \bar{\alpha}$. If $a, b \in A$, then we set

$$a\ r_\alpha\ b \iff C(a)\ r_\alpha\ C(b).$$

It is apparent from the definition of r_α, that if C, D are cycles of (A, f) and $C\ r_\alpha\ D$, then $c\ r_\alpha\ d$ for $\forall c \in C, d \in D$.

Lemma 5. *Let $\alpha \in \mathrm{Quord}(A, f)$. The relation r_α is an equivalence on A.*

Proof. It is easy to see, that r_α is reflexive: to prove that $a\ r_\alpha\ a$, take $k = 1$, $c_0 = c_1 = f(a)$. Next, r_α is symmetric, since $\alpha \cup \bar{\alpha}$ is symmetric.

Now let us show transitivity. Assume that $c\ r_\alpha\ d$ and $d\ r_\alpha\ b$. Denote $C = C(c)$, $D = C(d)$, $B = C(b)$. There exist $m, l \in \mathbb{N}$, cycles $C = C_0, C_1, \ldots, C_m = D$, cycles $D = D_0, D_1, \ldots, D_l = B$, elements $c_0 \in C_0, c_1 \in C_1, \ldots, c_m \in C_m$, $d_0 \in D_0, d_1 \in D_1, \ldots, d_l \in D_l$ such that for each $i \in \{0, 1, \ldots, m-1\}$, $(c_i, c_{i+1}) \in \alpha \cup \bar{\alpha}$ and for each $j \in \{0, 1, \ldots, l-1\}$, $(d_j, d_{j+1}) \in \alpha \cup \bar{\alpha}$. Denote $k = m + l$ and for $j \in \{1, \ldots, l\}$ put

$$C_{m+j} = D_j.$$

Since $D = D_0 = C_m$ is a cycle and it contains the elements d_0, c_m, there is $t \in \{0, \ldots, n-1\}$ such that $d_0 = f^t(c_m)$. Further, the relation $(d_j, d_{j+1}) \in \alpha \cup \bar{\alpha}$ for $j \in \{0, 1, \ldots, l-1\}$ implies

$$(f^t(d_j), f^t(d_{j+1})) \in \alpha \cup \bar{\alpha}.$$

Now it suffices to denote $c_{m+j} = d_j$ for each $j \in \{1, \ldots, l\}$ and the proof is complete. □

Lemma 6. *Let* $\alpha \in \mathrm{Quord}(A, f)$*. If* $a, b \in A$ *belong to the same connected component, then* $a\ r_\alpha\ b$*.*

Proof. Similarly as in the proof of reflexivity of the relation r_α, let us take $C_0 = C_1 = C(a) = C(b)$, $k = 1$, $c_0 = f(a) = c_1$. □

Definition 3. Let $\alpha \in \mathrm{Quord}(A, f)$ and $A/r_\alpha = \{A_j : j \in J\}$. If J is a one-element set, then α is said to be connected.

Let us remark that this notion is natural: by drawing the quasiordered set, we obtain a graph G in which for every pair C_i, C_j cycles of (A, f), there exist elements $c_i \in C_i, c_j \in C_j$ such that there exists a path in G connecting vertices denoted c_i, c_j.

2. Construction of a complement to connected quasiorder

Now we will work with the classes of the equivalence r_α.The goal of the following construction is to define, for a given $j \in J$ and a given quasiorder $\alpha \in \mathrm{Quord}(A_j, f)$, some $\beta \in \mathrm{Quord}(A_j, f)$; later we show that β is a complement of α in $\mathrm{Quord}(A_j, f)$. In further, we will denote r_α by r.

For simplification, we will write A instead of A_j, i.e., till the main result about complements in $\mathrm{Quord}(A_j, f)$ (Theorem 2.2) of this section, we assume that J is a one-element set.

Notation 2.1. Let A' be the set of all noncyclic elements x of A such that

$$(x, f^n(x)) \notin \alpha \quad \text{and} \quad (f^n(x), x) \notin \alpha.$$

We define a binary relation ρ on A' as follows. Put $(a, b) \in \rho$ if $a, b \in A'$, $f(a) = f(b)$ and there are $k \in \mathbb{N}$ and $a = u_0, u_1, \ldots, u_k = b$ elements of A' such that

$$(\forall i \in \{0, \ldots, k-1\})\,(f(a) = f(u_i), (u_i, u_{i+1}) \in \alpha \cup \bar{\alpha}).$$

i.e., put $(a, b) \in \rho$ if $a, b \in A'$, $f(a) = f(b)$ and a, b belong to the same connected subcomponent of the quasiordered set of α, consisting of elements of A'.

It is easy to verify that the relation ρ is an equivalence and that the following assertion is valid.

Definition 4. Let $D \in A'/\rho$. We choose one fixed element t from each class $D/(\alpha \cap \bar{\alpha}) = T$ and denote the set of all these fixed elements t as D^*.

Lemma 7. *Let $D \in A'/\rho$. Then there exists a set $D^* \subseteq D$ such that*
1) $(\forall x \in D \setminus D^*)(\exists y \in D^*)((x, y) \in \alpha \cap \bar{\alpha})$;
2) $(\forall x, y \in D^*, x \neq y)((x, y) \in \alpha \Rightarrow (y, x) \notin \alpha)$.

For each $D \in A'/\rho$, there can be one or more sets D^* such as described in Lemma 7. We choose arbitrary one of them before we begin the construction (K). Then for each $D \in A'/\rho$, we choose a representative $d^* \in D^*$, again arbitrarily. By choosing different D^* and d^* for individual D, we can construct different complements to α.

The following example shows choosing of D^* and d^* in a particular case.

Example 1. Let us consider a monounary algebra (A, f) and a quasiorder α on (A, f) as we can see in Figures 1 and 2. By Notation 2.1, $A' = \{6, 7, 8, 9, 10\}$ and $A'/\rho = \{D_1^*, D_2^*\}$, where we can choose $D_1^* = \{6, 8, 9\}$ or $D_1^* = \{7, 8, 9\}$, and $D_2^* = \{10\}$.

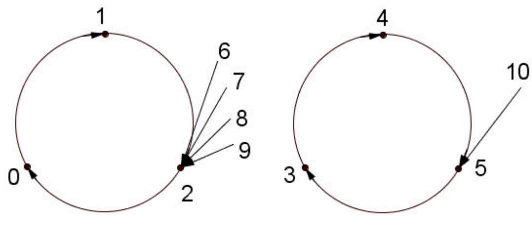

FIGURE 1. Algebra (A, f).

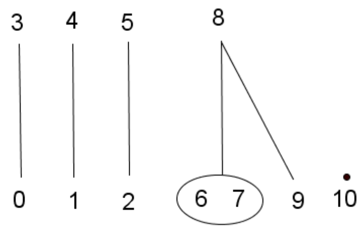

FIGURE 2. Quasiorder α.

If we choose $D_1^* = \{6, 8, 9\}$ and $D_2^* = \{10\}$, then d_1^* can be either $6, 8$ or 9 and $d_2^* = 10$. If we choose $D_1^* = \{7, 8, 9\}$ and $D_2^* = \{10\}$, then d_1^* can be either $7, 8$ or 9 and $d_2^* = 10$.

Now let us describe a relation β. Let $x, y \in A$. We put $(x, y) \in \beta$ if either $x = y$ or the pair (x, y) fulfils one of the steps of the construction. Let us remark that in (e) (and only there) we use some previous steps.

Construction (K)

Step (a). Let x, y belong to the same cycle C, $y = f^k(x)$, $\alpha \restriction C = \theta_d$, d/n and let $e = \frac{n}{d}$. We set $(x, y) \in \beta$ if and only if e/k.

Step (b). Let $x \in C_1$, $y \in C_2$, where C_1 and C_2 are distinct cycles. We put $(x, y) \in \beta$ if and only if there are $a \in C_1$ and $b \in C_2$ with $(b, a) \in \alpha, (a, b) \notin \alpha$.

Step (c). Suppose that $x, y \in D^*$ for some $D \in A'/\rho$. Then $(x, y) \in \beta$ if and only if and $(y, x) \in \alpha$.

Step (d1). Suppose that x belongs to a cycle C, y is noncyclic, $C(y) = C$. Further let $\alpha \restriction C = \theta_d$, d/n, $e = \frac{n}{d}$. If $y \notin A'$, then $(x, y) \in \beta$ if and only if $(f^n(y), y) \notin \alpha, (y, f^n(y)) \in \alpha, x = f^k(y), e/k$.

Step (d1′). Suppose that y belongs to a cycle C, x is noncyclic, $C(x) = C$. Further let $\alpha \restriction C = \theta_d$, d/n, $e = \frac{n}{d}$. If $x \notin A'$, then $(x, y) \in \beta$ if and only if $(f^n(x), x) \in \alpha, (x, f^n(x)) \notin \alpha, y = f^k(x), e/k$.

Step (d2). Suppose that x belongs to a cycle C, y is noncyclic, $C(y) = C$. Further let $\alpha \restriction C = \theta_d$, d/n, $e = \frac{n}{d}$. If $y \in A'$, then $(x, y) \in \beta$ if and only if there is $D \in A'/\rho$ such that $y \in D^*, x = f^k(y), e/k$ and $(y, d^*) \in \alpha$.

Step (d2′). Suppose that y belongs to a cycle C, x is noncyclic, $C(x) = C$. Further let $\alpha \restriction C = \theta_d$, d/n, $e = \frac{n}{d}$. If $x \in A'$, then $(x, y) \in \beta$ if and only if there is $D \in A'/\rho$ such that $x \in D^*, y = f^k(x), e/k$ and $(d^*, x) \in \alpha$.

Step (e). Suppose that x, y satisfy none of the assumptions of the previous steps. Then $(x, y) \in \beta$ if and only if $(x, f^n(x)) \in \beta$, $(f^n(y), y) \in \beta$, $(f^n(x), f^n(y)) \in \beta$.

We will show that $\beta \in \mathrm{Quord}(A, f)$ and that β is a complementary quasiorder to α.

Lemma 8. *Let $(x, y) \in \beta$. Then $(f(x), f(y)) \in \beta$.*

Proof. We can assume that $x \neq y$ and that the pair (x, y) is obtained according to the steps of the above construction.

(A) First x, y belong to the same cycle C, $y = f^k(x)$, $\alpha \restriction C = \theta_d$, d/n, $e = \frac{n}{d}$ and e/k. Then $(f(x), f(y)) = (f(x), f^k(f(x)))$, thus $(f(x), f(y)) \in \beta$ by the step (a).

(B) Now $x \in C_1$, $y \in C_2$, where C_1 and C_2 are distinct cycles and there are $a \in C_1$ and $b \in C_2$ with $(b,a) \in \alpha, (a,b) \notin \alpha$. Since $f(x) \in C_1$ and $f(y) \in C_2$, the above step (b) yields that $(f(x), f(y)) \in \beta$.

(C) In the step (c) the assumption implies that $f(x) = f(y)$.

(D1) We will not repeat all assumptions of (d1). We have

$$y \notin A', \qquad (f^n(y), y) \notin \alpha, \qquad (y, f^n(y)) \in \alpha, \qquad x = f^k(y), \qquad e/k.$$

For verifying that $(f(x), f(y)) \in \beta$ we need to apply (a), because $f(x)$ and $f(y)$ belong to the same cycle. We have $f(y) = f^{n-k}(f(f^k(y))) = f^{n-k}(f(x))$ and $e/n-k$, therefore $(f(x), f(y)) \in \beta$.

(D1′) Analogously as (D1).

(D2) We suppose that x belongs to a cycle C, y is noncyclic, $C(y) = C$. Further, $y \in A'$ and there is $D \in A'/\rho$ such that $y \in D^*, x = f^k(y), e/k$, $(y, d^*) \in \alpha$. The elements $f(x)$ and $f(y)$ belong to the same cycle, $f(y) = f(d^*)$, thus $f(y) = f^{n-k}(f(f^k(y))) = f^{n-k}(f(x))$ and $e/n-k$, therefore $(f(x), f(y)) \in \beta$.

(D2′) Analogously as (D2).

(E) In this case we have $(x, f^n(x)) \in \beta, (f^n(y), y) \in \beta, (f^n(x), f^n(y)) \in \beta$. The elements $f^n(x), f^n(y)$ are cyclic. Then (B), in the view of $(f^n(x), f^n(y)) \in \beta$, implies $(f(f^n(x)), f(f^n(y))) \in \beta$, i.e., $(f(x), f(y)) \in \beta$. □

Lemma 9. *Let $(x,y) \in \beta$, $(y,z) \in \beta$. Then $(x,z) \in \beta$.*

Proof. We can assume that x, y, z are mutually distinct.

1) First assume that $C(x) \neq C(y)$. By (e) we have

$$(x, f^n(x)) \in \beta, \tag{1}$$
$$(f^n(x), f^n(y)) \in \beta, \tag{2}$$
$$(f^n(y), y) \in \beta. \tag{3}$$

Then (b) yields

$$\text{there are } a \in C(x), b \in C(y) \text{ with } (b,a) \in \alpha, (a,b) \notin \alpha \tag{4}$$

Similarly suppose that $C(z) \neq C(y)$. Then

$$(y, f^n(y)) \in \beta, \tag{5}$$
$$(f^n(y), f^n(z)) \in \beta, \tag{6}$$
$$(f^n(z), z) \in \beta, \tag{7}$$
$$\text{there are } b' \in C(y), c' \in C(z) \text{ with } (c', b') \in \alpha, (b', c') \notin \alpha. \tag{8}$$

From (4) and (8) it follows that there is $m \in \mathbb{N}$ with $b = f^m(b')$. Denote $c = f^m(c')$. Then

$$c = f^m(c') \ \alpha \ f^m(b') = b \ \alpha \ a.$$

Since $(a, b) \notin \alpha$, we get $(a, c) \notin \alpha$. Therefore

$$(c_1, c_2) \in \beta \quad \text{for each } c_1 \in C(x), c_2 \in C(z),$$

according to (b). Then $(f^n(x), f^n(z)) \in \beta$. Thus (1) and (7), in view of (e), imply $(x, z) \in \beta$.

2) Suppose that $C(x) \neq C(y) = C(z)$. If z is cyclic, then $(x, z) \in \beta$ by (4). Let z be noncyclic. If the elements y, z satisfy (e), then $(x, z) \in \beta$ analogously as in the first part of the proof. Hence y is cyclic.

Let $\alpha \restriction C(y) = \theta_{\frac{n}{e}}$. If $z \notin A'$, then by (d1), $(f^n(z), z) \notin \alpha, (z, f^n(z)) \in \alpha$, $y = f^k(z)$, e/k. Thus again according to (d1), $(f^n(z), z) \in \beta$. If $z \in A'$, then by (d2) there is $D \in A'/\rho$ such that $z \in D^*, y = f^k(z), e/k$ and $(z, d^*) \in \alpha$. Thus $(f^n(z), z) \in \beta$ in view of (d2). This in view of (1), (2) and (e) yields that $(x, z) \in \beta$.

3) The case when $C(x) = C(y) \neq C(z)$ is similar to 2).

4) Finally we suppose that $C(x) = C(y) = C(z)$, $\alpha \restriction C(x) = \theta_{\frac{n}{e}}$.

First we show the assertion for cyclic elements x, y, z. There are k, m with $y = f^k(x)$, $z = f^m(y)$, e/k, e/m. Then $z = f^{k+m}(x), e/k + m$, hence $(x, z) \in \beta$. From the assumption $(x, y) \in \beta$, $(y, z) \in \beta$ it follows $(f^n(x), f^n(y)) \in \beta$, $(f^n(y), f^n(z)) \in \beta$, the elements $f^n(x), f^n(y), f^n(z)$ are cyclic, thus

$$(f^n(x), f^n(z)) \in \beta. \tag{9}$$

This implies that if $(x, f^n(x)) \in \beta$, $(f^n(z), z) \in \beta$ then the pair x, z satisfies (e) and then either $(x, z) \in \beta$ or x, z satisfy some of the assumptions of (a), (c), (d1), (d1′), (d2), (d2′). We will proceed according to this idea in the remaining part of the proof.

4.1) Let x, y be cyclic, z be noncyclic. By $(x, y) \in \beta$ we have $y = f^k(x), e/k$, thus also $x = f^n(x) = f^{k+i}(x) = f^i(f^k(x)) = f^i(y), e/i$. In view of (d1) or (d2), $y = f^m(z), e/m$. Then $x = f^{i+m}(z), e/i + m$ and $(x, z) \in \beta$ according to (d1) or (d2).

4.2) Let x, z be cyclic, y be noncyclic. For $y \notin A'$, then (d1′) by $(y, z) \in \beta$ implies that $(y, f^n(y)) \notin \alpha$ and (d1) by $(x, y) \in \beta$ implies that $(y, f^n(y)) \in \alpha$, a contradiction. If $y \in A'$, then (d2′) and $(y, z) \in \beta$ yield $y \in D^*$ for some $D \in A'/\rho$ and $z = f^m(y), e/m$. Similarly, if $y \in A'$, then (d2) and $(x, y) \in \beta$ yield that $x = f^k(y), e/k$. There is $t \in \mathbb{N}$ with $m - k + tn \geqslant 0$ and then

$$z = f^{m+tn}(y) = f^{m-k+tn}(f^k(y)) = f^{m-k+tn}(x), \qquad e/m - k + tn.$$

Therefore $(x, z) \in \beta$ in view of (a).

4.3) Let x be cyclic, y, z be noncyclic. First let $y, z \in D^*$ for some $D \in A'/\rho$. Then $(z, y) \in \alpha$ in view of (c). Next, $x = f^m(y)$, e/m, $(y, d^*) \in \alpha$, thus $(z, d^*) \in \alpha$. Since $f^m(y) = f^m(d^*) = f^m(z)$, we obtain by (d2) that $(x, z) \in \beta$. Now let $(y, z) \in \beta$ by (e). Then $(y, f^n(y)) \in \beta$, $(f^n(y), f^n(z)) \in \beta$, $(f^n(z), z) \in \beta$. The second relation implies that $y = f^k(z), e/k$. From (d1), (d2) for the elements x, y we get that $x = f^m(y), e/m$, thus $x = f^{m+k}(z), e/m + k$. If $z \notin A'$, then by (d1), $(f^n(z), z) \notin \alpha, (z, f^n(z)) \in \alpha$ and then $(x, z) \in \beta$. If $z \in A'$, then according to $(f^n(z), z) \in \beta$ by (d2) we obtain $z \in D^*$ for some $D \in A'/\rho$ and $(z, d^*) \in \alpha$, therefore $(x, z) \in \beta$.

4.4) The case when x, y are noncyclic, z is cyclic is dual to 4.3).

4.5) Let x, z be noncyclic, y be cyclic. From $(x, y) \in \beta$ and (d1$'$), (d2$'$) it follows that either $x \notin A', (f^n(x), x) \in \alpha, (x, f^n(x)) \notin \alpha, y = f^k(x), e/k$, or $x \in A'$, there is $D \in A'/\rho$ such that $x \in D^*, y = f^k(x), e/k$ and $(d^*, x) \in \alpha$. Next, (d1$'$), (d2$'$) yield $(x, f^n(x)) \in \beta$. It can be shown analogously that $(f^n(z), z) \in \beta$. Therefore we either obtain that $(x, z) \in \beta$ according to (e) or x, z satisfy the assumption of (c). Then $z \in D^*$. Since $(y, z) \in \beta$, (d2) implies that $y = f^m(z), e/m$ and $(z, d^*) \in \alpha$. Therefore

$$z \;\alpha\; d^* \;\alpha\; x,$$

hence $(x, z) \in \beta$ by (c).

4.6) Finally suppose that x, y, z are noncyclic. Then either x, y satisfy the assumption of (c) and

$$x, y \in D^*, \qquad D \in A'/\rho, \qquad (y, x) \in \alpha$$

or x, y satisfy the assumption of (e) and

$$(x, f^n(x)) \in \beta, \qquad (f^n(x), f^n(y)) \in \beta, \qquad (f^n(y), y) \in \beta.$$

Similarly, either y, z satisfy the assumption of (c) and

$$y, z \in D_1^*, \qquad D_1 \in A'/\rho, \qquad (z, y) \in \alpha$$

or y, z satisfy the assumption of (e) and

$$(y, f^n(y)) \in \beta, \qquad (f^n(y), f^n(z)) \in \beta, \qquad (f^n(z), z) \in \beta.$$

Let x, y satisfy the assumption of (c) and y, z satisfy the assumption of (c). Then $D_1 = D$, $z\ \alpha\ y\ \alpha\ x$, thus $(x, z) \in \beta$ by (c).

Let x, y satisfy the assumption of (c) and y, z satisfy the assumption of (e) (the case when x, y satisfy the assumption of (e) and y, z satisfy the assumption of (c) is analogous). We have $(y, f^n(y)) \in \beta$, thus by (d2′), $(d^*, y) \in \alpha$, which yields $d^*\ \alpha\ y\ \alpha\ x$. Then (d2′) implies that $(x, f^n(x)) \in \beta$, therefore (e) according to (9) yields $(x, z) \in \beta$.

Let x, y satisfy the assumption of (e) and y, z satisfy the assumption of (e). In view of (9), if $(x, z) \notin \beta$, then $x, z \in D_2^*, D_2 \in A'/\rho, (z, x) \notin \alpha$. Since $(f^n(z), z) \in \beta$, by (d2) we obtain $(z, d_2^*) \in \alpha$, and from (d2′) and $(x, f^n(x)) \in \beta$ it follows that $(d_2^*, x) \in \alpha$. Therefore $(x, z) \in \beta$, a contradiction. □

We have shown that β is a quasiorder on (A, f). Now, we will show that β is also complementary to α in $\mathrm{Quord}(A, f)$.

Lemma 10. *If $(x, y) \in \alpha \wedge \beta$, then $x = y$.*

Proof. Let $(x, y) \in \alpha \wedge \beta$, $x \neq y$.

(A) Assume that x, y belong to the same cycle C. There is $d \in \mathbb{N}$ such that $\alpha \restriction C = \theta_d$, d/n. Step (a) implies that $\beta \restriction C = \theta_e$, where $e = \frac{n}{d}$. We have $(x, y) \in \alpha \restriction C \cap \beta \restriction C = \theta_d \cap \theta_e$. Then according to Lemma 3, $x = y$.

(B) Suppose that $x \in C_1$, $y \in C_2$, where C_1 and C_2 are distinct cycles. There is $d \in \mathbb{N}$ such that $\alpha \restriction C_2 = \theta_d$, d/n. Then $(x, y) \in \beta$ if and only if there are $a \in C_1$ and $b \in C_2$ with $(b, a) \in \alpha, (a, b) \notin \alpha$. There are $k, m \in \mathbb{N}$ such that $a = f^k(x), b = f^m(y)$. Since $(x, y) \in \alpha$, also $(f^k(x), f^k(y)) \in \alpha$, hence

$$f^m(y) = b\ \alpha\ a = f^k(x)\ \alpha\ f^k(y).$$

The elements $f^m(y), f^k(y)$ belong to C_2 and $(f^m(y), f^k(y)) \in \theta_d$, which yields that $d/m - k$. Then

$$a\ \alpha\ f^{m-k}(a) = f^{m-k}(f^k(x)) = f^m(x)\ \alpha\ f^m(y) = b,$$

which is a contradiction.

(C) Let $x, y \in D^*$ for some $D \in A'/\rho$. Then $(x, y) \in \beta$ if and only if and $(y, x) \in \alpha$. We assumed that $(x, y) \in \alpha$, but this is a contradiction, because $x, y \in D^*$.

(D1) Suppose that x belongs to a cycle C, y is noncyclic, $C(y) = C$. Further let $\alpha \restriction C = \theta_d$, d/n, $e = \frac{n}{d}$ and let $y \notin A'$. Then $(f^n(y), y) \notin \alpha$, $(y, f^n(y)) \in \alpha$, $x = f^k(y)$, e/k. Next, $(f^{k+1}(y), f(y)) = (f(x), f(y)) \in \alpha$, which implies that d/k. The assumption about n at the beginning of the section yields ed/k, i.e., n/k and $x = f^n(y) = y$.

(D2) Suppose that x belongs to a cycle C, y is noncyclic, $C(y) = C$. Further let $\alpha \restriction C = \theta_d$, d/n, $e = \frac{n}{d}$ and $y \in D^*$ for $D \in A'/\rho$. Then $x = f^k(y), e/k$ and $(y, d^*) \in \alpha$. Similarly as in (D1), $(f^{k+1}(y), f(y)) = (f(x), f(y)) \in \alpha$, therefore we obtain $x = y$.

(D1′), (D2′) Analogously as (D1), (D2).

(E) Now x, y satisfy none of the assumptions of the previous steps and

$$(x, f^n(x)) \in \beta, (f^n(x), f^n(y)) \in \beta, \qquad (f^n(y), y) \in \beta.$$

From the assumption of the lemma it follows that $(f^n(x), f^n(y)) \in \alpha$. For the cyclic elements $f^n(x), f^n(y)$ we can apply (A) or (B), thus $f^n(x) = f^n(y)$. If y is cyclic, then $y = f^n(x)$, hence $(x, y) = (x, f^n(x)) \in \beta$, $(x, y) \in \alpha$ and $x = y$. Therefore we can assume that x and y are noncyclic. If $x \notin A'$, then $(x, f^n(x)) \in \beta$ by (d1′) implies $(f^n(x), x) \in \alpha$, thus

$$f^n(y) = f^n(x) \ \alpha \ x \ \alpha \ y,$$

a contradiction to $(f^n(y), y) \in \beta$. Similarly for y; therefore let $x, y \in A'$. From $f(x) = f^{n+1}(x) = f^{n+1}(y) = f(y)$ it follows that $x, y \in D^*$ for some $D \in A'/\rho$. This completes the proof according to (C). □

Lemma 11. $\alpha \vee \beta = A \times A$.

Proof. Let $x, y \in A$, $x \neq y$.

1) If x, y belong to the same cycle, then the assertion follows from Lemma 3.

2) Let x, y belong to distinct cycles. First let us prove that if C, D are distinct cycles, $c \in C$, $d \in D$ and $(c, d) \in \alpha \cup \bar{\alpha}$, then $(c', d') \in \alpha \vee \beta$ for each $c' \in C, d' \in D$. Let $c' \in C, d' \in D$. If $(c, d) \in \bar{\alpha}$, then $(d, c) \in \alpha$ and (b) implies $(c', d') \in \beta$. If $(c, d) \in \alpha$, then using the proved case 1) we get

$$c' \ (\alpha \vee \beta) \ c \ \alpha \ d \ (\alpha \vee \beta) \ d'.$$

By the assumption, $x \ r \ y$. Then $C(x) \ r \ C(y)$ and there are $k \in \mathbb{N}$, cycles $C(x) = C_0, C_1, \ldots, C_k = C(y)$ and elements $c_0 \in C_0$, $c_1 \in C_1, \ldots,$ $c_k \in C_k$ such that for each $i \in \{0, 1, \ldots, k-1\}$, $(c_i, c_{i+1}) \in \alpha \cup \bar{\alpha}$. Then by induction, $(x, y) \in \alpha \vee \beta$.

3) Let $C(x) = C(y)$ and either x is noncyclic, $x \notin A'$, y is cyclic, or x is cyclic, y is noncyclic, $y \notin A'$. We prove only the first case; the second one is analogous. Since $x \notin A'$, thus either $(x, f^n(x)) \in \alpha$ or $(f^n(x), x) \in \alpha$,

$(x, f^n(x)) \notin \alpha$, which by (d1$'$) implies $(x, f^n(x)) \in \beta$. Then $(x, y) \in \alpha \vee \beta$ by 1).

4) Assume that x, y belong to the same connected component, $x, y \notin A'$. Then $(x, y) \in \alpha \vee \beta$ in view of 3). From this and from 1) it follows, that the condition that x, y belong to the same connected component can be omitted.

5) Let $x, y \in D$, $D \in A'/\rho$. Then there are $k \in \mathbb{N}$ and $x = u_0, u_1, \dots, u_k = y$ elements of $D^* \subseteq D$ such that $f(x) = f(y) = f(u_i)$, $(u_i, u_{i+1}) \in \alpha \cup \bar{\alpha}$ for each $i \in \{0, \dots, k-1\}$. It can be shown analogously as in 2) that $(x, y) \in \alpha \vee \beta$.

6) Let $D \in A'/\rho$. In view of (d2$'$) we obtain $(d^*, f^n(d^*)) \in \beta$. This, together with the previous steps, implies that if $x \in A'$, then $x \in D$ for some $D \in A'/\rho$, thus $(x, d^*) \in \alpha \vee \beta$ and $(f^n(d^*), y) \in \alpha \vee \beta$ for each $y \notin A'$. So then $(x, y) \in \alpha \vee \beta$.

7) Let $D \in A'/\rho$. Then $(f^n(d^*), d^*) \in \beta$ by (d2). Thus if x is cyclic, $y \in A'$, then $y \in D$ for some $D \in A'/\rho$ and we get by (2) that $(x, f^n(d^*)) \in \alpha \vee \beta$, $(f^n(d^*), d^*) \in \beta$ and by (5) that $(d^*, y) \in \alpha \vee \beta$. It follows from the previous steps that $(x, y) \in \alpha \vee \beta$ for arbitrary $x, y \in A$, so the claim is proved. □

In the view of Construction (K) and Lemmas 8–11 we obtain:

Theorem 2.2. *Let (A, f) be a monounary algebra such that for each $a \in A$, the element $f(a)$ is cyclic, and there is a square-free $n \in \mathbb{N}$ such that each cycle of (A, f) has n elements. Let $\alpha \in$ Quord(A, f) be connected. If a binary relation β on A is formed by Construction (K), then β is a complementary quasiorder to α in the lattice* Quord(A, f).

Example 2. The converse is not true. Let us consider the algebra (A, f), such that $A = \{0, 1, 2, 3\}$, $f(0) = 1, f(1) = 0, f(2) = 3, f(3) = 2$ and a quasiorder $\alpha = I_A \cup \{(0, 2), (1, 3)\}$. It is easy to verify that a quasiorder $\gamma = I_A \cup \{(2, 1), (3, 0)\}$ is a complement in Quord(A, f) to α. However, a complementary quasiorder in Quord(A, f) to α formed by the construction (K) is $\beta = I_A \cup \{(1, 0), (0, 1), (2, 3), (3, 2), (2, 0), (2, 1), (3, 0), (3, 1)\}$.

3. Construction of a complement to a quasiorder — the general case

The aim of this section is to find a complementary quasiorder to a non-connected quasiorder if the lattice Quord(A, f) is complemented.

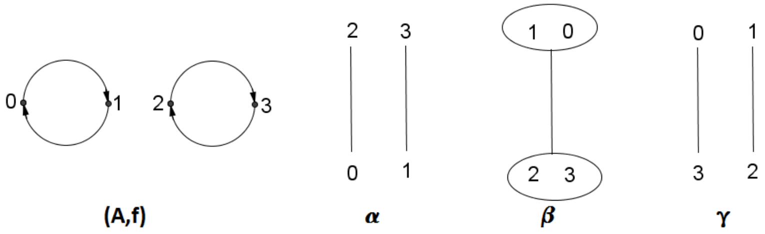

FIGURE 3. Converse of Theorem 3.8 is not true.

Suppose that $\alpha \in \mathrm{Quord}(A, f)$ and that r_α is as above. According to the previous section the case $|J| = 1$ is solved; now let us suppose that $|J| > 1$. We will describe the Construction (K′) in the following section.

For $i \in J$ let c_i be a fixed cyclic element of some chosen cycle C_i in A_i. We denote by γ the following relation:

$$\gamma = \{(f^k(c_i), f^k(c_j)) \colon i, j \in J, k \in \mathbb{N}\}.$$

It can be easily shown that $\gamma \in \mathrm{Quord}(A, f)$.

For each $i \in J$, the relation $\alpha \restriction C_i$ is a congruence of the cycle C_i, thus there is $d_i \in \mathbb{N}$ such that $\alpha \restriction C_i$ is the smallest congruence containing the pair $(c_i, f^{d_i}(c_i))$. The set of all d_i is finite, denote it by $\{d_1, d_2, \ldots, d_s\}$. Without loss of generality, let $\{1, 2, \ldots, s\} \subseteq J$.

Notice that, for $i \in J$, $d, l, k \in \mathbb{N}$, $(f^l(c_i), f^k(c_i)) \in \theta(c_i, f^d(c_i))$ if and only if d divides $l - k$. In what follows, let d will be the greatest common divisor of $d_1, d_2, \ldots,\ d_s$. This implies the following.

Lemma 12. *There exist positive integers $q_1, q_2, \ldots, q_s$ and q such that*

$$1 + qn = q_1 \frac{d_1}{d} + q_2 \frac{d_2}{d} + \cdots + q_s \frac{d_s}{d}.$$

Let $i \in J$. Put

$$\alpha_i' = \theta(c_i, f^d(c_i)) \vee \alpha_i.$$

If $\alpha' = \bigcup_{j \in J} \alpha_i'$, then $\alpha' \in \mathrm{Quord}(A, f)$ and it easy to see that $r_{\alpha'} = r_\alpha$. By the results of the previous section there exists a complement β_i' of α_i' in $\mathrm{Quord}(A_i, f)$. Further, from the construction of a complement on A_i we obtain

$$\beta_i' \restriction C_i = \theta(c_i, f^{\frac{n}{d}}(c_i)).$$

Lemma 13. *Let $i \in J$, $l, k \in \mathbb{N}$. Then $(f^l(c_i), f^k(c_i)) \in \alpha_i \vee \beta_i'$ if and only if $\frac{d_i}{d}/l - k$.*

Proof. From the notation above, $(f^l(c_i), f^k(c_i)) \in \alpha_i$ if and only if $d_i/l-k$ and $(f^l(c_i), f^k(c_i)) \in \beta'_i$ if and only if $\frac{n}{d}/l-k$. Then $(f^l(c_i), f^k(c_i)) \in \alpha_i \vee \beta'_i$ if and only if $\mathrm{g.c.d}(\mathrm{d_i}, \frac{\mathrm{n}}{\mathrm{d}})/\mathrm{l}-\mathrm{k}$, i.e., if and only if $\frac{d_i}{d}/l-k$. □

Now we define the relation β by putting

$$\beta = \gamma \ \vee \ \bigvee_{j\in J} \beta'_j.$$

We are going to show that β is a complement to the quasiorder α in the lattice $\mathrm{Quord}(A, f)$. Since β is a join of quasiorders, it is clear that it is also a quasiorder.

Lemma 14. *If $(x, y) \in \alpha \wedge \beta$, then $x = y$*

Proof. Let $(x, y) \in \alpha \wedge \beta$, $x \neq y$. The relation $(x, y) \in \alpha$ implies that there is $i \in J$ such that $x, y \in A_i$, $(x, y) \in \alpha_i$. Then $(x, y) \in \alpha'_i$. We have $\alpha_i \cap \beta'_i = \alpha'_i \cap \beta'_i$, which, since β'_i is a complement to α'_i, is the smallest quasiorder of (A_i, f). The assumption $x \neq y$ yields that $(x, y) \notin \beta'_i$. There is the shortest chain of elements $x = u_0, u_1, \ldots, u_m = y$ with $m > 1$ such that either $(u_k, u_{k+1}) \in \gamma$ or $(u_k, u_{k+1}) \in \bigvee_{j\in J} \beta'_j$, for any k. Obviously, the elements $u_0, u_1, \ldots, u_m$ are distinct and if $(u_k, u_{k+1}) \in \gamma$, then $(u_{k+1}, u_{k+2}) \in \bigvee_{j\in J} \beta'_j$, and similarly for the second possibility. For each k there is $i_k \in J$ with $u_k \in A_{i_k}$. From the definition of β we get

$$\begin{aligned} (u_k, u_{k+1}) \in \gamma \implies u_k = f^{t_k}(c_{i_k}),\\ u_{k+1} = f^{t_{k+1}}(c_{i_{k+1}}), \quad i_k \neq i_{k+1},\\ t_k = t_{k+1}, \qquad (10)\\ (u_k, u_{k+1}) \in \beta'_j \implies i_k = i_{k+1}, \qquad (11)\\ u_k = f^{t_k}(c_{i_k}), u_{k+1} = f^{t_{k+1}}(c_{i_{k+1}}), (u_k, u_{k+1}) \in \beta'_j \implies i_k = j, \qquad (12)\\ \frac{n}{d}/t_k - t_{k+1}. \qquad (13) \end{aligned}$$

We have either

$$x = u_0 \ \gamma \ u_1 \ \beta'_j \ u_2 \ \gamma \ u_3 \ \beta'_j \ u_4 \ \ldots, \qquad (14)$$

or

$$x = u_0 \ \beta'_j \ u_1 \ \gamma \ u_2 \ \beta'_j \ u_3 \ \gamma \ u_4 \ \ldots. \qquad (15)$$

We have $m > 1$, thus between the elements of the chain, the quasiorder γ is used at least twice.

Assume that (15) holds. Also, assume that $u_{m-1} \in A_i$. (The remaining cases are similar, but more simple.) Then m is odd. By the definition of γ, for each $0 < k \leqq m$ there exists a positive integer t_k such that $u_k = f^{t_k}(c_{i_k})$. In view of (10)–(13), $t_1 = t_2$, $\frac{n}{d}/t_2 - t_3$, $t_3 = t_4$, $\frac{n}{d}/t_4 - t_5$, $\dots, t_{m-2} = t_{m-1}$. Then $\frac{n}{d}/(t_1-t_2)+(t_2-t_3)+(t_4-t_5)+\dots+(t_{m-3}-t_{m-2})+(t_{m-2}-t_{m-1}) = t_1 - t_{m-1}$, hence $(u_1, u_{m-1}) \in \beta'_{i_0}$. This, together with the relations $(u_0, u_1) \in \beta'_{i_0}$, $(u_{m-1}, u_m) \in \beta'_{i_0}$ implies $(x, y) = (u_0, u_m) \in \beta'_{i_0}$, which is a contradiction. □

Lemma 15. *$\alpha \vee \beta = A \times A$.*

Proof. We must show that $(x, y) \in \alpha \vee \beta$ for every $x, y \in A$. We will prove that there are $m \in \mathbb{N} \cup \{0\}$ and a chain of elements $x = u_0$, u_1, u_2, $\dots$, $u_m = y$ of the set A such that either

$$(u_k, u_{k+1}) \in \gamma \quad \text{or} \quad (u_k, u_{k+1}) \in \alpha_j \vee \beta'_j \quad \text{for some } j \in J \tag{16}$$

is valid for each $0 \leqslant k < m$. Assume that $x \neq y$. We will investigate the following four cases and we will use the previous cases for the proof of a new one (we omit the case symmetric to the third one, because these cases are similar):

1) $x \in C_1$, $y = f(x)$,
2) $i \in J$, $x, y \in C_i$,
3) $i \in J$, $x \in A_i$, $y \in C_i$,
4) $i, j \in J$, $x \in A_i$, $y \in A_j$.

Let the case 1) be valid. There is $k \in \mathbb{N}$ with $x = f^k(c_1)$. In view of Lemmas 13 and 12 we obtain

$$\begin{aligned} x = f^k(c_1)\ (\alpha_1 \vee \beta'_1)\ f^{k+q_1\frac{d_1}{d}}(c_1)\ \gamma\ f^{k+q_1\frac{d_1}{d}}(c_2)\ (\alpha_2 \vee \beta'_2) \\ f^{k+q_1\frac{d_1}{d}+q_2\frac{d_2}{d}}(c_2)\ \dots\ (\alpha_s \vee \beta'_s)\ f^{k+q_1\frac{d_1}{d}+q_2\frac{d_2}{d}+\dots+q_s\frac{d_s}{d}}(c_s) \\ = f^{k+1+qn}(c_s) = f^{k+1}(c_s)\ \gamma\ f^{k+1}(c_1) = f(x) = y. \end{aligned}$$

Hence $x\ (\alpha \vee \beta)\ y$. Assume that the case 2) occurs. Then $x = f^k(c_i)$, $y = f^l(c_i)$. By Lemma 13 and by the case 1),

$$\begin{aligned} x = f^k(c_i)\ \gamma\ f^k(c_1)\ (\alpha \vee \beta)\ f(f^k(c_1))\ (\alpha \vee \beta)\ f(f^{k+1}(c_1))\ \dots \\ (\alpha \vee \beta) f^l(c_1)\ \gamma\ f^l(c_i) = y. \end{aligned}$$

Now let the case 3) be valid. Since β'_i is a complement to α'_i, it yields that $(x, y) \in \alpha'_i \vee \beta'_i$ and there exist $m \in \mathbb{N}$ and a chain $x = v_0, v_1, \dots, v_m = y$ such that for each $0 \leqslant k < m$ either $(v_k, v_{k+1}) \in \alpha'_i$ or

$(v_k, v_{k+1}) \in \beta'_i$ holds. If k is such that $(v_k, v_{k+1}) \in \alpha'_i$ and $(v_k, v_{k+1}) \notin \alpha_i$, then $v_{k+1} \in C_i$ and there is $v'_{k+1} \in C_i$ such that $(v_k, v'_{k+1}) \in \alpha_i$. By the case 2), $(v'_{k+1}, v_{k+1}) \in \alpha \vee \beta$. This implies that $x\ (\alpha \vee \beta)\ y$. Finally, suppose that the case 4) holds. Using the case 3) (and the dual to it) we obtain

$$x\ (\alpha \vee \beta)\ c_i\ \gamma\ c_j\ (\alpha \vee \beta)\ y,$$

therefore $x\ (\alpha \vee \beta)\ y$. □

According to Lemmas 12–15 and the Construction (K′) we obtain:

Theorem 3.1. *Let (A, f) be a monounary algebra such that for each $a \in A$, the element $f(a)$ is cyclic, and there is a square-free $n \in \mathbb{N}$ such that each cycle of (A, f) has n elements. Let $\alpha \in \mathrm{Quord}(A, f)$ be disconnected. If a binary relation β on A is formed by Construction (K′), then β is a complementary quasiorder to α in the lattice $\mathrm{Quord}(A, f)$.*

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