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Optimal Design of Failure-Censored Constant-Stress Life Test Plans for the Inverse Weibull Distribution

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This paper presents the optimal design of constant-stress partially accelerated life tests (CSPALT) using type-II censored data from the inverse Weibull distribution. The maximum likelihood approach is applied to estimate the distribution parameters and the corresponding factor. In addition, the corresponding confidence interval estimates are obtained. Moreover, optimal CSPALT plans are developed using the D-optimality criterion. That is, the proportion of test units that should be allocated to run under accelerated condition is optimally determined. This proportion is obtained such that the generalized asymptotic variance of the maximum likelihood estimators of the model parameters is minimized. To illustrate the theoretical results presented in this paper, simulation studies are conducted.

Keywords: reliability, inverse Weibull distribution, constant stress, maximum likelihood estimation, Fisher information matrix, generalized asymptotic variance, optimum test plans.

Notation

PALT	–	partially accelerated life test
δ_{ui}, δ_{aj}	–	indicator functions: $\delta_{ui} = I(T_i \le P)$, $\delta_{aj} = I(X_j \le P)$ in case of type-II censoring
X	–	lifetime of an item under accelerated conditions
T	–	lifetime of an item under normal (use) conditions
r	–	number of failures
n_u, n_a	–	number of items failed under use and accelerated conditions, respectively
π^*	–	optimum proportion of sample units allocated to accelerated condition
π	–	proportion of units allocated to run under accelerated condition
P	–	time of the rth failure at which the test is terminated
n	–	total number of test items in PALTs

Introduction. Today, the majority of manufacturers in numerous fields do their best to optimize the quality and performance of their products before releasing them to the market to improve the product demand and the trust of their clients. The manufacturing field is developing all the time. This motivates firms' owners to continuously optimize the performance of their products. Indeed, firms are vulnerable to many problems in their businesses. So, the most recommended solution to these problems is the use of accelerated life tests (ALT) or partially accelerated life tests (PALT). The difference between these two types of tests is that items are tested only under accelerated conditions in ALTs, while in PALTs, the units are tested under both use (normal) and accelerated conditions. PALTs outperform ALTs in their ability to get more failure data in a shorter time without high levels of stress in all test units. Therefore, PALTs are being used by many experimenters due to these advantages that are combined also with their simplicity, time saving, adaptability, and economical benefits.

Over years, the optimal design of PALTs plans under several types of distribution models and data has been developed. For example, Bai and Chung [1] have discussed the optimal design of CSPALTs plans for test items exponentially distributed using type-I censored data. Bai et al. [2] have studied the optimal design of PALTs plans for test items having lognormal distribution using time-censored data. Ismail et al. [3] have developed the optimal CSPALTs plans for the case of Pareto distribution under type-I censoring. Ismail [4] has estimated the Weibull parameters and the acceleration factor under CSPALTs using the maximum likelihood (ML) method with type-II censoring. Ismail [5] has developed optimal CSPALTs plans considering Weibull distribution using time-censored data. Ismail and Al-Babtain [6] have explored the optimal failure-censored CSPALTs plans under Pareto distribution. In addition, Ismail [7] has considered Bayesian estimation under constant-stress partially accelerated life tests for Pareto distribution with type-I censoring. Ismail and Al Tamimi [8] have developed the optimum constant-stress partially accelerated life test plans using time-censored data from the inverse Weibull distribution. Now, this paper will spotlight on constant-stress PALTs using type-II censored data from the inverse Weibull (IW) distribution.

The rest of this paper is structured as follows. Section 1 describes the used model and the test methodology. Section 2 introduces the ML method applied to estimate the model parameters. Section 3 presents the confidence intervals of the model parameters. Section 4 discusses the use of the D-optimality criterion to address the optimal design of the test. Section 5 presents simulation studies needed to illustrate theoretical results.

1. **Model and Test Procedure**.

1.1. ***The Inverse Weibull Distribution***. As indicated by many authors, the IW distribution can be used to model a variety of failure characteristics such as useful life and wear-out periods. It has a main role in numerous applications to describe and illustrate the degradation incidents of mechanical components. Moreover, it gives a suitable fit to survival data. More details about this distribution were presented by Nelson [9], Drapella [10], and Jiang et al. [11]. The IW distribution has three basic functions; probability density function (PDF), reliability function (RF) and failure rate function (FRF) which are given, respectively, as

$$f(t)=\alpha\theta(t)^{-(\alpha+1)}e^{-\theta(t)^{-\alpha}}, \qquad t\ge 0,\ \theta>0,\ \alpha>0, \tag{1}$$

$$R(t)=1-e^{-\theta t^{-\alpha}}, \qquad t>0, \tag{2}$$

$$h(t)=\frac{f(t)}{R(t)}=\frac{\alpha\theta t^{-(\alpha+1)}e^{-\theta t^{-\alpha}}}{1-e^{-\theta t^{-\alpha}}}=\frac{\alpha\theta t^{-(\alpha+1)}}{e^{\theta t^{-\alpha}}-1}. \tag{3}$$

1.2. ***Constant-Stress PALTs (CSPALTs)***. The CSPALTs procedure involves two steps described as follows.

Step one. The total number of test items (n) is divided into two groups with a certain proportion of units π allocated to run under accelerated condition. The first group includes $n\pi$ items selected randomly from n items to be tested under accelerated condition, while the second group includes the remaining items that are tested under use condition.

Step two. Each item in both groups is run until the test is terminated according to the type-II censoring scheme.

Some basic assumptions:

The lifetimes T_i, $i = 1, \ldots, n(1-\pi)$ of items allocated to run under use condition, are i.i.d. r.v.'s.

The lifetimes X_j, $j = 1, \ldots, n\pi$ of items allocated to run under accelerated condition, are i.i.d r.v.'s.

The PDF under accelerated condition is given by

$$f(x) = \alpha\theta\beta(x\beta)^{-(\alpha+1)} e^{-\theta(x\beta)^{-\alpha}}, \qquad x > 0,\ \alpha, \theta > 0,\ \beta > 1, \tag{4}$$

where $X = \beta^{-1}T$.

2. **Maximum Likelihood Estimates (MLE)**. The ML method is applied in this section to find the MLEs using type-II censored data. Numerical techniques and some computer programs are preferred to be used with the ML method to make statistical inference about the complicated models that contains more than two parameters.

Now, let us define the indicator functions:

$$\delta_{ui} = \begin{cases} 1 & t_i \le P, \\ 0 & \text{otherwise}, \end{cases} \qquad i = 1, 2, \ldots, n(1-\pi)$$

and

$$\delta_{aj} = \begin{cases} 1 & x_j \le P, \\ 0 & \text{otherwise}, \end{cases} \qquad j = 1, 2, \ldots, n\pi.$$

Thus, the likelihood functions for (t_i, δ_{ui}) and (x_j, δ_{aj}) are formulated, respectively, as follows:

$$L_{ui}(t_i, \delta_{ui}) = \prod_{i=1}^{n(1-\pi)} [f(t)]^{\delta_{ui}} [R(P)]^{\bar{\delta}_{ui}} =$$

$$= \prod_{i=1}^{n(1-\pi)} [\alpha\theta t_i^{-(\alpha+1)} e^{-\theta t_i^{-\alpha}}]^{\delta_{ui}} [1 - e^{-\theta P^{-\alpha}}]^{\bar{\delta}_{ui}}, \tag{5}$$

$$L_{aj}(x_j, \delta_{aj}) = \prod_{j=1}^{n\pi} [f(x)]^{\delta_{aj}} [R(\beta P)]^{\bar{\delta}_{aj}} =$$

$$= \prod_{j=1}^{n\pi} [\alpha\theta\beta(x\beta)^{-(\alpha+1)} e^{-\theta(x\beta)^{-\alpha}}]^{\delta_{aj}} [1 - e^{-\theta(\beta P)^{-\alpha}}]^{\bar{\delta}_{aj}}, \tag{6}$$

where $\bar{\delta}_{ui} = 1 - \delta_{ui}$ and $\bar{\delta}_{aj} = 1 - \delta_{aj}$.

Therefore, the total likelihood function for $(t_1;\ \delta_{u1}, \ldots, t_{n(1-\pi)}\delta_{un(1-\pi)}, x_1;\ \delta_{a1}, \ldots$ $\ldots, x_{n\pi}\delta_{an\pi})$ is formulated by

$$L(t, x, \alpha, \theta, \beta) = \prod_{i=1}^{n(1-\pi)} L_{ui}(t_i, \delta_{ui}) \prod_{j=1}^{n\pi} L_{aj}(x_j, \delta_{aj}) =$$

$$= \prod_{i=1}^{n(1-\pi)} [\alpha\theta t_i^{-(\alpha+1)} e^{-\theta t_i^{-\alpha}}]^{\delta_{ui}} [1 - e^{-\theta P^{-\alpha}}]^{\bar{\delta}_{ui}} \times$$

$$\times \prod_{j=1}^{n\pi} [\alpha\theta\beta(x_j\beta)^{-(\alpha+1)} e^{-\theta(x_j\beta)^{-\alpha}}]^{\delta_{aj}} [1-e^{-\theta(\beta P)^{-\alpha}}]^{\bar{\delta}_{aj}}. \tag{7}$$

For simplicity, it should maximize the natural logarithm of the likelihood function instead of the likelihood function itself. The natural logarithm of the likelihood function can be expressed by

$$\ln L = \sum_{i=1}^{n(1-\pi)} \delta_{ui} [\ln \alpha + \ln \theta - (\alpha+1)\ln t_i - \theta t_i^{-\alpha}] + \sum_{i=1}^{n(1-\pi)} \bar{\delta}_{ui} \ln[1-e^{-\theta P^{-\alpha}}] +$$

$$+ \sum_{j=1}^{n\pi} \delta_{aj} [\ln \alpha\theta\beta - (\alpha+1)\ln(x_j\beta) - \theta(x_j\beta)^{-\alpha}] + \sum_{j=1}^{n\pi} \bar{\delta}_{aj} \ln[1-e^{-\theta(\beta P)^{-\alpha}}]. \tag{8}$$

Equation (8) is then derived with respect to α, β, and θ, respectively, as

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n_u + n_a}{\alpha} - \sum_{i=1}^{n(1-\pi)} \delta_{ui} \ln t_i + \theta \sum_{i=1}^{n(1-\pi)} \delta_{ui} t_i^{-\alpha} \ln t_i + \frac{\theta P^{-\alpha} \ln P}{e^{\theta P - \alpha} - 1} (n(1-\pi) - n_u) +$$

$$- \sum_{j=1}^{n\pi} \delta_{aj} \ln(x_j\beta) + \theta \sum_{j=1}^{n\pi} \delta_{aj} (x_j\beta)^{-\alpha} \ln(x_j\beta) + \frac{\theta(\beta P)^{-\alpha} \ln(\beta P)}{e^{\theta(\beta P)^{-\alpha}} - 1} (n\pi - n_a), \tag{9}$$

where $n_u = \sum_{i=1}^{n(1-\pi)} \delta_{ui}$ and $n_a = \sum_{j=1}^{n\pi} \delta_{aj}$,

$$\frac{\partial \ln L}{\partial \beta} = -\frac{\alpha n_a}{\beta} + \theta\alpha \sum_{j=1}^{n\pi} \delta_{aj} x_j (x_j\beta)^{-\alpha-1} + \frac{\theta\alpha P^{-\alpha} (\beta)^{-\alpha-1}}{e^{\theta(\beta P)^{-\alpha} - 1}} (n\pi - n_a), \tag{10}$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{n_u + n_a}{\theta} - \sum_{i=1}^{n(1-\pi)} \delta_{ui} t_i^{-\alpha} + \frac{P^{-\alpha} (n(1-\pi) - n_u)}{e^{\theta P^{-\alpha}} - 1} - \sum_{i=1}^{n\pi} \delta_{aj} (x_j\beta)^{-\alpha} +$$

$$+ \frac{(\beta P)^{-\alpha} (n\pi - n_a)}{e^{\theta(\beta P)^{-\alpha}} - 1}. \tag{11}$$

Consequently, the MLEs of α, β, and θ are obtained by solving the system of equations: $\frac{\partial \ln L}{\partial \alpha} = 0$, $\frac{\partial \ln L}{\partial \beta} = 0$, and $\frac{\partial \ln L}{\partial \theta} = 0$, respectively. But, since the previous three equations are nonlinear, the Newton–Raphson method is used to solve this system of equations numerically.

To find the variance-covariance matrix of the MLEs of α, β, and θ, the second derivatives of the natural logarithm of the likelihood function are obtained as follows:

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = -\frac{n_u + n_a}{\alpha^2} - \theta \sum_{i=1}^{n(1-\pi)} \delta_{ui} t_i^{-\alpha} (\ln t_i)^2 + (n(1-\pi) - n_u) \times$$

$$\times \frac{\theta P^{-\alpha} (\ln P)^2 (e^{\theta P^{-\alpha}} - 1) + \theta^2 P^{-2\alpha} (\ln P)^2 e^{\theta P^{-\alpha}}}{(e^{\theta P^{-\alpha}} - 1)^2} -$$

$$-\theta \sum_{j=1}^{n\pi} \delta_{aj} (x_j \beta)^{-\alpha} (\ln x_j \beta)^2 + (n\pi - n_a) \times$$

$$\times \frac{(-\theta(\beta P)^{-\alpha} (\ln \beta P)^2 (e^{\theta(\beta P)^{-\alpha}} - 1) + \theta^2 (\beta P)^{-2\alpha} (\ln(\beta P))^2) e^{\theta(\beta P)^{-\alpha}}}{(e^{\theta(\beta P)^{-\alpha}} - 1)^2}, \qquad (12)$$

$$\frac{\partial^2 \ln L}{\partial \theta^2} = -\frac{n_u + n_a}{\theta^2} - \frac{P^{-2\alpha} e^{\theta P^{-\alpha}} (n(1-\pi) - n_u)}{(e^{\theta P^{-\alpha}} - 1)^2} - \frac{(\beta P)^{-2\alpha} e^{\theta(\beta P)^{-\alpha}} (n\pi - n_a)}{(e^{\theta(\beta P)^{-\alpha}} - 1)^2}, \qquad (13)$$

$$\frac{\partial^2 \ln L}{\partial \beta^2} = \frac{\alpha n_a}{\beta^2} + \theta\alpha \sum_{j=1}^{n\pi} \delta_{aj} (-\alpha - 1) x_j^{-\alpha} (\beta)^{-\alpha-2} +$$

$$+ \frac{\theta\alpha P^{-\alpha} (-\alpha - 1) \beta^{-\alpha-2} (e^{\theta(\beta P)^{-\alpha}} - 1) + \theta^2 \alpha^2 P^{-2\alpha} (\beta)^{2(-\alpha-1)} e^{\theta(\beta P)^{-\alpha}}}{(e^{\theta(\beta P)^{-\alpha}} - 1)^2} (n\pi - n_a), \qquad (14)$$

$$\frac{\partial^2 \ln L}{\partial \alpha \partial \theta} = \sum_{i=1}^{n(1-\pi)} \delta_{ui} t_i^{-\alpha} \ln t_i + (n(1-\pi) - n_u) \times$$

$$\times \frac{P^{-\alpha} \ln P (e^{\theta P^{-\alpha}} - 1) - \theta P^{-2\alpha} \ln P e^{\theta P^{-\alpha}}}{(e^{\theta P^{-\alpha}} - 1)^2} +$$

$$+ \sum_{j=1}^{n\pi} \delta_{aj} (x_j \beta)^{-\alpha} \ln(x_j \beta) + (n\pi - n_a) \times$$

$$\times \frac{(\beta P)^{-\alpha} \ln(\beta P) (e^{\theta(\beta P)^{-\alpha}} - 1) - \theta(\beta P)^{-2\alpha} \ln(\beta P) e^{\theta(\beta P)^{-\alpha}}}{(e^{\theta(\beta P)^{-\alpha}} - 1)^2}, \qquad (15)$$

$$\frac{\partial^2 \ln L}{\partial \beta \partial \theta} = \alpha \sum_{j=1}^{n\pi} \delta_{aj} x_j (x_j \beta)^{-\alpha-1} +$$

$$+ (n\pi - n_a) \frac{\alpha P^{-\alpha} (\beta)^{-\alpha-1} (e^{\theta(\beta P)^{-\alpha}} - 1) - \theta\alpha P^{-2\alpha} (\beta)^{-2\alpha-1} e^{\theta(\beta P)^{-\alpha}}}{(e^{\theta(\beta P)^{-\alpha}} - 1)^2}, \qquad (16)$$

$$\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} = -\frac{1}{\beta} \sum_{j=1}^{n\pi} \delta_{aj} + \theta \sum_{j=1}^{n\pi} \delta_{aj} \left(-\alpha x_j^{-\alpha} \beta^{-\alpha-1} \ln(x_j \beta) + \frac{(x_j \beta)^{-\alpha}}{\beta} \right) +$$

$$+\frac{(-\alpha\theta P^{-\alpha}\beta^{-\alpha-1}\ln(\beta P)+\theta\beta^{-\alpha-1}P^{-\alpha})(e^{\theta(\beta P)^{-\alpha}}-1)}{(e^{\theta(\beta P)^{-\alpha}}-1)^2}+$$

$$+\frac{\alpha\theta^2 P^{-2\alpha}\beta^{-2\alpha-1}\ln(\beta P)e^{\theta(\beta P)^{-\alpha}}}{(e^{\theta(\beta P)^{-\alpha}}-1)^2}. \qquad (17)$$

The asymptotic variance-covariance matrix of the MLEs of β, α, and θ is given by

$$V=F^{-1}=\begin{bmatrix}-\frac{\partial^2 \ln L}{\partial^2\beta^2} & -\frac{\partial^2 \ln L}{\partial\beta\partial\alpha} & -\frac{\partial^2 \ln L}{\partial\beta\partial\theta}\\ -\frac{\partial^2 \ln L}{\partial\alpha\partial\beta} & -\frac{\partial^2 \ln L}{\partial^2\alpha^2} & -\frac{\partial^2 \ln L}{\partial\alpha\partial\theta}\\ -\frac{\partial^2 \ln L}{\partial\theta\partial\beta} & -\frac{\partial^2 \ln L}{\partial\theta\partial\alpha} & -\frac{\partial^2 \ln L}{\partial^2\theta^2}\end{bmatrix}^{-1}\downarrow(\hat\beta,\hat\alpha,\hat\theta)=$$

$$=\begin{bmatrix}A\operatorname{var}(\hat\beta) & A\operatorname{cov}(\hat\beta\hat\alpha) & A\operatorname{cov}(\hat\beta\hat\theta)\\ A\operatorname{cov}(\hat\alpha\hat\beta) & A\operatorname{var}(\hat\alpha) & A\operatorname{cov}(\hat\alpha\hat\theta)\\ A\operatorname{cov}(\hat\theta\hat\beta) & A\operatorname{cov}(\hat\theta\hat\alpha) & A\operatorname{var}(\hat\theta)\end{bmatrix}. \qquad (18)$$

3. **Interval Estimates for Model Parameters**. As indicated by Vander Wiel and Meeker [12], since the MLEs have the asymptotic normal distribution and consistency properties, the approximate $100(1-\gamma)\%$ two-sided confidence bounds for β, α, and θ can be, respectively, given by

$$\hat\beta\mp Z_{\gamma/2}\sqrt{F_{11}^{-1}(\hat\beta,\hat\alpha,\hat\theta)},\qquad \hat\alpha\mp Z_{\gamma/2}\sqrt{F_{22}^{-1}(\hat\beta,\hat\alpha,\hat\theta)},\qquad \hat\theta\mp Z_{\gamma/2}\sqrt{F_{33}^{-1}(\hat\beta,\hat\alpha,\hat\theta)}, \qquad (19)$$

where $Z_{\gamma/2}$ is the upper $(\gamma/2)$th percentile of a standard normal distribution.

4. **Optimum Constant-Stress Test Plans**. The optimum plan is to obtain the optimal proportion of sample units π^* allocated to run under accelerated condition based on the outputs of the stage of parameter estimation that are at the same time considered inputs to the optimal design stage of the test. It is worth noting that the proportion of sample units π allocated to run under accelerated condition is pre-specified for the stage of parameter estimation. But this proportion for the optimal design stage of the test is considered a division parameter that has to be optimally determined according to a certain optimality criterion.

The optimality criterion is to find the optimal proportion of sample units π^* allocated to run under accelerated condition such that the generalized asymptotic variance (GAV) of the MLE of the model parameters is minimized. The GAV of the MLEs of the model parameters as an optimality criterion is commonly used and defined as the reciprocal of the determinant of the Fisher information matrix F [13]. That is,

$$GAV(\hat\beta,\hat\alpha,\hat\theta)=\frac{1}{|F|}. \qquad (20)$$

The observed Fisher information matrix can be written asymptotically by dropping the expectations as follows [14]:

$$F=\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}=\begin{bmatrix} -\frac{\partial^2 \ln L}{\partial^2\beta^2} & -\frac{\partial^2 \ln L}{\partial\beta\partial\alpha} & -\frac{\partial^2 \ln L}{\partial\beta\partial\theta} \\ -\frac{\partial^2 \ln L}{\partial\alpha\partial\beta} & -\frac{\partial^2 \ln L}{\partial^2\alpha^2} & -\frac{\partial^2 \ln L}{\partial\alpha\partial\theta} \\ -\frac{\partial^2 \ln L}{\partial\theta\partial\beta} & -\frac{\partial^2 \ln L}{\partial\theta\partial\alpha} & -\frac{\partial^2 \ln L}{\partial^2\theta^2} \end{bmatrix}.$$

So, the determinant of the matrix F is

$$|F|=f_{11}(f_{22}f_{33}-f_{23}^2)-f_{12}(f_{12}f_{33}-f_{13}f_{23})+f_{13}(f_{12}f_{23}-f_{13}f_{22}). \tag{21}$$

Consequently, the partial derivative of $|F|$ with respect to π is

$$\frac{\partial|F|}{\partial\pi}=f_{11}(f'_{22}f_{33}+f_{22}f'_{33}-2f_{23}f'_{23})+f'_{11}(f_{22}f_{33}-f_{23}^2)-$$

$$-f_{12}(f'_{12}f_{33}+f_{12}f'_{33}-f'_{13}f_{23}-f_{13}f'_{23})-f'_{12}(f_{12}f_{33}-f_{13}f_{23})+$$

$$+f_{13}(f'_{12}f_{23}+f_{12}f'_{23}-f'_{13}f_{22}-f_{13}f'_{22})+f'_{13}(f_{12}f_{23}-f_{13}f_{22}), \tag{22}$$

$$f'_{11}=-\frac{\alpha n}{\beta^2}+\theta\alpha n(\alpha+1)(\beta)^{-\alpha-2}x_{n\pi}^{-\alpha}, \tag{23}$$

where

$$n_u=\sum_{i=1}^{n(1-\pi)}\delta_{ui}=\begin{Bmatrix} \sum_{i=1}^{n(1-\pi)}1 & t_i\le P \\ 0 & o.w \end{Bmatrix}=\begin{Bmatrix} n(1-\pi) & t_i\le P \\ 0 & o.w \end{Bmatrix}.$$

Consequently,

$$\frac{\partial n_u}{\partial\pi}=-n, \qquad t_i\le P.$$

Correspondingly, for $n_a=\sum_{j=1}^{n\pi}\delta_{aj}$, $\frac{\partial n_a}{\partial\pi}=n$, $t_i\le P$.

$$f'_{22}=-n\theta t_{n(1-\pi)}^{-\alpha}(\ln t_{n(1-\pi)})^2+\theta n(x_{n\pi}\beta)^{-\alpha}(\ln x_{n\pi}\beta)^2, \tag{24}$$

$$f'_{23}=nt_{n(1-\pi)}^{-\alpha}\ln t_{n(1-\pi)}-n(x_{n\pi}\beta)^{-\alpha}\ln(x_{n\pi}\beta), \tag{25}$$

$$f'_{12}=\frac{n}{\beta}-\theta n\left(\alpha x_{n\pi}^{-\alpha}\beta^{-\alpha-1}\ln(x_{n\pi}\beta)-\frac{(x_{n\pi}\beta)^{-\alpha}}{\beta}\right), \tag{26}$$

$$f'_{13}=-\alpha n x_{n\pi}(x_{n\pi}\beta)^{-\alpha-1}, \tag{27}$$

$$f'_{33}=0. \tag{28}$$

The optimal fraction of test units π^* that minimizes the GAV is determined numerically. Accordingly, the expected optimal numbers of items that failed under use and accelerated conditions are obtained as

$$n_u^* = n(1-\pi^*)P_u \quad \text{and} \quad n_a^* = n\pi^* P_a,$$

where P_u is probability that an item tested only under use condition failed by P and P_a is probability that an item tested only under accelerated condition failed by P.

5. **Simulation Studies**. A simulation study is conducted using the R software statistical package to evaluate the performance of the MLEs and to develop optimal CSPALTs plans using type-II censored data from the IW distribution. The mean square errors (MSEs) for various choices of the true parameter values are calculated. Also, the 95% asymptotic confidence bounds based on the asymptotic distribution of the MLEs are constructed. To achieve high levels of precision, 20,000 replications are used. In this section, several data sets generated from the IW distribution under type-II censoring are considered with sample sizes of 20, 25, 30, 45, 50, 75, and 100.

Tables 1, 3, and 5 summarize the results of the ML estimates of the parameters. Results of simulation studies provide insight into the sampling behavior of the estimators.

Table 1

Average Values of the Estimates, Variances, MSEs, and Confidence Limits, Respectively, when (α, β, θ) Set at (1, 3.5, 2) Given $\pi = 0.3$ and $r = 0.75n$

n	Parameters	Estimate	Variance	MSE	95%	
					LCL	UCL
20	α	1.1189	0.2272	0.0658	0.9834	1.8742
	β	3.7475	1.9468	3.8512	0.8180	6.8136
	θ	2.3059	0.8886	0.8832	0.5603	4.0437
25	α	1.0961	0.0404	0.0496	0.4473	1.2353
	β	3.7103	3.0043	3.0484	0.2675	7.0620
	θ	2.2419	0.5123	0.5707	0.0797	2.8854
30	α	1.0724	0.1690	0.0338	0.5946	1.2572
	β	3.6852	1.5220	2.3508	0.4466	5.5198
	θ	2.1773	0.5895	0.3789	1.2679	3.5786
45	α	1.0481	0.1330	0.0199	0.5718	1.0930
	β	3.6026	1.2139	1.4839	2.0414	6.7998
	θ	2.1123	0.4282	0.1960	1.1160	2.7946
50	α	1.0420	0.1217	0.0166	0.7846	1.2616
	β	3.6035	1.1432	1.3176	1.1434	5.6248
	θ	2.0969	0.3983	0.1680	1.3825	2.9438
75	α	1.0273	0.0970	0.0102	0.7829	1.1633
	β	3.5768	0.9194	0.8512	0.8861	4.4902
	θ	2.0634	0.3076	0.0986	1.4362	2.6419
100	α	1.0206	0.0820	0.0071	0.8839	1.2054
	β	3.5489	0.7849	0.6184	3.0976	6.1744
	θ	2.0464	0.2627	0.0712	1.1588	2.1887

Note. Here and Tables 3 and 5: LCL and UCL are lower and upper confidence limits, respectively.

T a b l e 2

The Results of Optimal Design of the Life Test for Different Sized Samples under Type-II Censoring in CSPALTs Based on the Estimation Results in Table 1

n	π^*	n_u^*	n_a^*	Optimal GAV
20	0.4254	6	9	0.04181
25	0.5517	8	11	0.02799
30	0.5708	10	13	0.00522
45	0.6030	13	21	0.00064
50	0.6070	15	23	0.00024
75	0.6680	18	38	0.00017
100	0.6945	23	52	0.00005

T a b l e 3

Average Values of the Estimates, Variances, MSEs, and Confidence Limits, Respectively, when (α, β, θ) set at (1.5, 3.5, 2) Given $\pi = 0.3$ and $r = 0.75n$

n	Parameters	Estimate	Varince	MSE	95%	
					LCL	UCL
20	α	1.6737	0.1101	0.1403	0.7464	2.0471
	β	3.5838	1.4373	1.4443	0.1737	4.8734
	θ	2.3018	0.7399	0.8310	0.3601	3.7321
25	α	1.6405	0.0848	0.1045	1.2135	2.3550
	β	3.5512	1.1970	1.1996	1.5304	5.8192
	θ	2.2335	0.5000	0.5545	0.6587	3.4307
30	α	1.6108	0.0631	0.0754	1.1587	2.1436
	β	3.5480	0.9416	0.9439	1.8814	5.6853
	θ	2.1772	0.3456	0.3770	1.7306	4.0352
45	α	1.5705	0.0387	0.0437	1.2032	1.9742
	β	3.5254	0.6199	0.6205	1.5585	4.6448
	θ	2.1058	0.1820	0.1932	1.1146	2.7868
50	α	1.5638	0.0333	0.0373	1.1564	1.8714
	β	3.5345	0.5332	0.5343	1.5849	4.4472
	θ	2.1012	0.1583	0.1686	0.8951	2.4550
75	α	1.5408	0.0206	0.0223	1.4494	2.0129
	β	3.5220	0.3607	0.3611	1.6547	4.0089
	θ	2.0649	0.0989	0.1031	0.8166	2.0492
100	α	1.5287	0.0148	0.0157	1.2083	1.6859
	β	3.5200	0.2639	0.2643	2.6756	4.6894
	θ	2.0451	0.0677	0.0697	1.3720	2.3919

The numerical results indicate that the ML estimates approximate the true values of the parameters as the sample size n increases. In addition, as seen from the numerical results, the asymptotic variances of the estimators decrease as the sample size n increases. Also, as the sample size n increases, both the MSEs and the interval lengths decrease.

Table 4

The Results of Optimal Design of the Life Test for Different Sized Samples under Type-II Censoring in CSPALTs Based on the Estimation Results in Table 3

n	π^*	n_u^*	n_a^*	Optimal GAV
20	0.4653	6	9	0.16052
25	0.4695	8	11	0.00541
30	0.5567	10	13	0.00481
45	0.5893	14	20	0.00276
50	0.6565	13	25	0.00088
75	0.6838	18	38	0.00034
100	0.7153	20	55	0.00018

Table 5

Average Values of the Estimates, Variances, MSEs, and Confidence Limits, Respectively, when (α, β, θ) set at (0.8, 3.5, 2) Given $\pi = 0.3$ and $r = 0.75n$

n	Parameters	Estimate	Variance	MSE	95%	
					LCL	UCL
20	α	0.8981	0.0348	0.0444	0.5795	1.3107
	β	3.9847	6.8384	7.0729	1.9120	12.1629
	θ	2.3172	0.9200	1.0205	1.5817	5.3416
25	α	0.8804	0.0264	0.0328	0.6791	1.3157
	β	3.8958	5.4276	5.5840	0.1834	9.3159
	θ	2.2527	0.5542	0.6181	0.5659	3.4842
30	α	0.8596	0.0190	0.0225	0.5414	1.0816
	β	3.8058	4.0327	4.1261	1.7379	9.6099
	θ	2.1781	0.3517	0.3834	1.3328	3.6575
45	α	0.8377	0.0116	0.0131	0.6111	1.0340
	β	3.7014	2.5047	2.5451	0.4977	6.7016
	θ	2.1077	0.1924	0.2040	1.3696	3.0890
50	α	0.8331	0.0096	0.0107	0.5661	0.9513
	β	3.6886	2.1474	2.1829	0.7554	6.4998
	θ	2.0965	0.1645	0.1738	1.3427	2.9325
75	α	0.8221	0.0061	0.0066	0.6213	0.9278
	β	3.6056	1.3742	1.3853	1.6664	6.2618
	θ	2.0613	0.0980	0.1017	1.2670	2.4941
100	α	0.8163	0.0044	0.0047	0.6439	0.9051
	β	3.5779	0.9963	1.0023	1.4265	5.3392
	θ	2.0449	0.0686	0.0706	1.2178	2.2445

Tables 2, 4, and 6 present the results of the test design. That is, the optimal sample-proportion π^* allocated to run under accelerated condition, the expected fraction failing at each stress, represented by n_u^* and n_a^*, and the optimal GAV of the MLEs of the model parameters are obtained numerically for each sample size. One can observe from the

Table 6

The Results of Optimal Design of the Life Test for Different Sized Samples under Type-II Censoring in CSPALTs Based on the Estimation Results in Table 5

n	π^*	n_u^*	n_a^*	Optimal GAV
20	0.4020	9	6	0.04065
25	0.4423	10	9	0.01991
30	0.4843	13	10	0.01425
45	0.5142	18	16	0.00652
50	0.5455	25	13	0.00452
75	0.5818	35	21	0.00072
100	0.6142	48	27	0.00004

numerical results listed in Tables 2, 4, and 6 via sample-proportion π^* that the optimum test plans do not allocate the same number of test units to each stress. In practice, the optimum test plans are important for improving precision in parameter estimation and thus enhancing the statistical inference quality. Also, these tables present the optimal GAV of the MLEs of the model parameters, which is obtained numerically with π^* in place of π for different-sized samples. As anticipated, the optimal GAV decreases with the sample size n.

Conclusions. In this paper the optimal design problem of the CSPALTs was discussed under the IW distribution assuming the type-II censoring. The minimization of the GAV of the MLEs of the model parameters was used as the optimality criterion. The optimal design problem is to determine the proportion of test units that should be allocated to run under accelerated condition such that the GAV of the MLEs of the model parameters is minimized. It is noteworthy that these plans provide not only the statistical optimum but also the economical one. The optimum test plans improve the quality of the statistical inference and achieve more savings in time and cost. The results obtained strongly indicate that the CSPALT model is quite instrumental in this respect, since it envisages tests not only under use conditions but also under accelerated ones.

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