J. J. R u s h c h i t s k y

NONLINEARITY OF ELASTIC DEFORMATIONS AND MODERATENESS OF STRAINS AS A FACTOR EXPLAINING THE AUXETICITY OF MATERIALS

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Abstract. A theoretical attempt is proposed to explain of auxeticity of elastic materials by use of nonlinear models of elastic deformations for wide range of strain values up to moderate level. The analytical expressions are obtained that corresponds to three kinds of universal deformations (simple shear, uniaxial tension, omniaxial tension) within the framework of three well-known in the nonlinear theory of elasticity models – two-constant Neo-Hookean model, three-constant Mooney-Rivlin model, five-constant Murnaghan model. A most interesting novelty consists in that the sample from elastic material is deformed as the conventional material for small values of strains whereas as the auxetic with increasing to moderate values of strains.

Key words: auxeticity, universal deformation, moderate strain, nonlinear hyperelastic model.

Motivation and essential.

The object of study can be defined as the auxetic materials as some subclass of nontraditional materials. This class includes the metamaterials, which include the mechanical metamaterials, which in turn include the auxetic materials. It should be noted that metamaterials (the materials engineered to have a property that is not found in naturally occurring materials; they derive their properties not from the properties of the base materials, but from their newly designed structures) [30] and smart materials (the designed materials that have one or more properties that can be significantly changed in a controlled fashion by external stimuli; smart materials are the basis of many applications, including sensors and actuators) [11] are the modern kinds of materials. To the point, the journal "Prikladnaya mekhanika – International Applied Mechanics" is publishing regularly the articles on this topic [23, 46]. Auxetic materials are deformed elastically exhibiting the unconventional property of increasing the cross-section (growing swollen) of cylindrical or prismatic sample under uniaxial tension, whereas in the conventional materials this cross-section decreases (grows thin). This is shown in Figure 1 [3]. The point is that the property of the decrease is described in the linear theory of elasticity by use of the Poisson's ratio. A change of the decrease of cross-section on the increase of one means a change of positive values of Poisson's ratio on the negative ones.

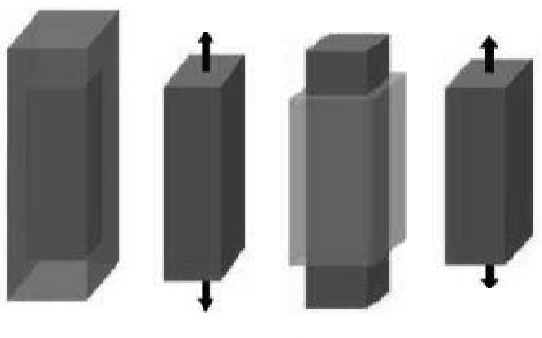

Fig. 1

Note. This article corresponds to results reported partially at three international scientific conferences (Poland, 2016; Ukraine, 2017; China, 2018) and the short communication in the scientific journal of the NAS of Ukraine "Dopovidi NAN Ukrainy" (2018, N7).

But the auxetic materials can not be associated only with description in the framework of linear theory of elasticity, where the Poisson's ratio is the elastic constant. Recently, the term "negative ratio of the trans-verse strain to the longi tudinal one" is often used instead the term "negative Poisson's ratio".

The main partial component of author's motivation in this study is the often used (described verbally or by the picture) demonstration of auxeticity of a foam as increasing the volume of sample from a foam under tension. It is shown in Fig. 2 [10 (left), 20 (right)].

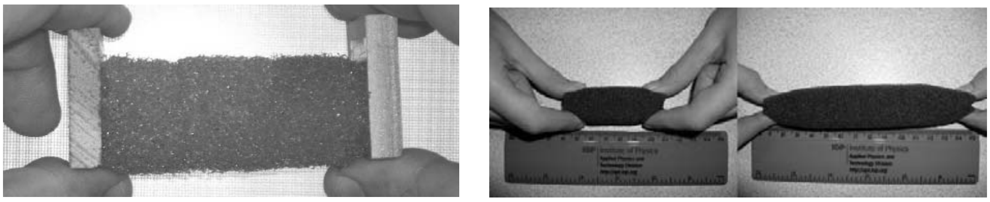

Fig. 2

These pictures are really very demonstrative, because they show two basic features. The feature 1 consists in that the sample length is possibly not sufficient to create the classical conditions of the test on the universal deformation of uniaxial tension-compression. The feature 2 can be meant as follows: the longitudinal and transverse strains are seemingly not sufficiently small.

An essential part of studies of auxetics consists in finding of diverse variants of internal structure that is further studied by methods of molecular physics and computational simulations. The most popular is so called hexagonal system (It is shown in Figure 2; left – before stretching, right – after stretching). Just this structure is given by different authors to illustrate the auxeticity.

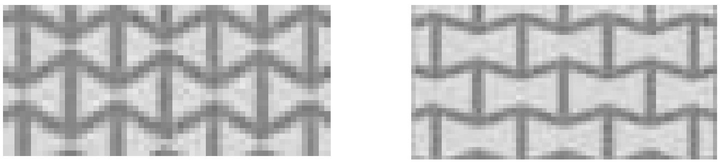

Fig. 3

The linear theory admits the swelling of sample only in the case of negative values of Poisson's ratio. But materials with such characteristics nobody observed during centuries and nobody recalled in textbooks on elasticity a possibility of their existence. The exception can be seen in only two classical books – Love's [27] and Lurie's [28].

Let us cite first [27, page 244]: "If σ were < -1, μ would be negative, and the function W would not be a positive quadratic function. We may show that this would also be the case if k were negative. Negative values for σ are not exclude by the condition of stability, but such values have not been found for any isotropic material."

And now a few sentences from the Lurie's book [28, page 117]: "A tension of the rod with negative ν (but the more than -1) would be accompanied by increasing of transverse sizes. Energetically, an existence of such elastic materials is not excluded." "In hypothetic material with $\nu < -1$, the hydrostatic compression of the cube would accompanied by increasing its volume."

Note that the Poisson/s ratio is denoted in the theory of elasticity by σ "sigma" and ν "nu". Love uses σ, whereas Lurie uses ν. In this paper, the Love's denotation is used.

The first publications on auxetics linked the auxeticity with negativity of Poisson's ratio. A prevalent majority of scientists identify up to this time auxeticity with negativity of Poisson/s ratio. A few publications only exist, where the nonlinear models are used (for example, [39] and experiments with large strains in auxetics (for example, [4, 9, 43, 45]).

But the linear theory (model) has an important restriction on the value of rod deformation: it must be small (for traditional materials, it is restricted to 3% from the initial length of rod). The experiment shown in Fig. 2 demonstrates that deformations of rod are not small and elongation reach tens percents of the initial length of rod. Such elongations can be correctly described only in the nonlinear theory of elasticity.

Let us turn on once again the fact that the first observed auxetic materials were the foams which are characterized by the small value of density and the porous internal structure. In the next studies, the new auxetics were revealed, the density of which was also small. But it was shown later that small density is not the defining property of auxetics, because the significant part of foams has not the property of auxeticity. The defining characteristics of auxetics is the special internal structure of this material.

Note that in mechanics the internal structure of materials can be appeared on two different stages of modeling the materials. First, on the stage of changing the discrete structure of material by the continuous one (that is, when the notion of continuum is introduced according to the principle of continualization). Second, on the stage of modeling the piece-wise inhomogeneous continuum by the homogeneous continuum (that is, when the principle of homogenization is applied).

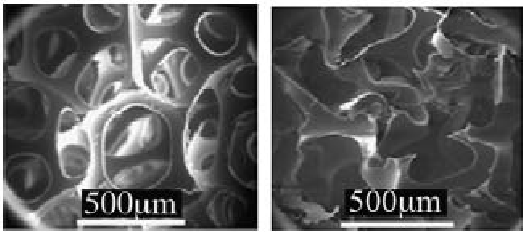

Fig. 4

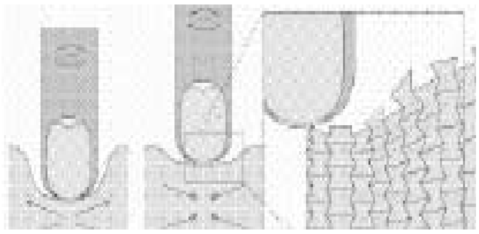
Fig. 5

The first stage is usually associated with methods of molecular physics, whereas the second stage is standard one in mechanics od composite materials. This is peculiar to all the materials that are studied in mechanics. For the presence in material a property of auxeticity, its internal structure have to change under deformation by the special way exhibiting the un-usual (nontraditional) mechanical effects. Note here that mechanics of materials studied traditionally first the elastic deformation and this concern both traditional (non-auxetic), and nontraditional (auxetic) materials. As far as the number of known nonauxetic materials exceeds the number of auxetic ones on many orders, then the term “unusual effect” is looking appropriate. In contrast to the traditional effects that count tens, the effects of auxeticity are observed as now in the identical me chanical problems in three types of such problems that are realized experimentally and described theoretically. An identity consists in that the samples from material must be compared, when the internal structure of material in cases “auxetic-nonauxetic” is differing by only geometrical shape of pores. This case is shown on Figure 4 for the sample from the polyuretane foam (left – traditional structure, right – auxetic structure) [3].

Thus, the auxeticity is generated by the special kind of internal structure of material and appears in three basic mechanical tests on deformation of material – swelling under tension, hardening under indentation, synclastic and anticlastic deformation of thin flexible plate). The test 1 is described above and shown in Figure 1.

The test 2 on indentation (statical Hertz’s problem, problem on hardness by Rockwell - Brinell-Wikkers) and impact (dynamical Hertz’s problem) shows the effect of hard-ness of auxetics in the contact zone. Within the framework of the theory of elasticity, this problem is solving nume rically with the given exactness. A scheme of test that exhibits the essential difference in the degree of indentation of the spherical indentor into the traditional (left) and auxetic (right)materials is shown in Figure 5 [3].The test 3 on synclastic and anticlastic deformation of deformation of flexible elastic plate is stated within the assumption that the plate is quadratic in plan and is loaded by the balanced system of three forces – one force is applied at the center of plate and directed upward, whereas two other identical forces are applied at the centers of two opposite ends of plate and directed downward. Within the framework of theory of flexible plates, this problem is solving numerically with the given exactness. The simple experiment that exhibits the essential difference in deformation of plate from the traditional and auxetic materials is shown in Figure 6 [2] (left- traditional material, right – auxetic material).

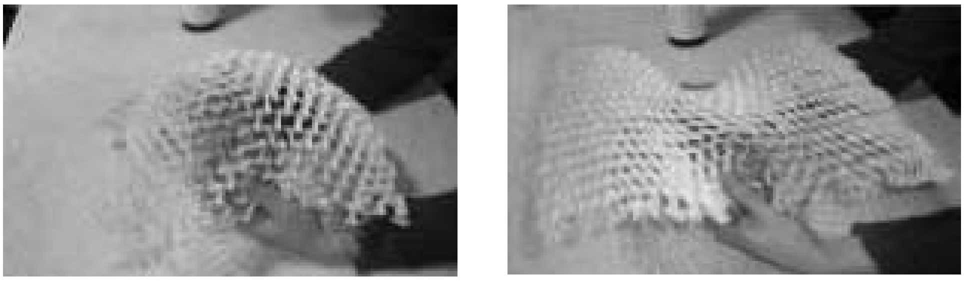

Fig. 6

The shown above information on auxetics permits to state that their definition is based on the secondary fact – negativity of Poisson's ratio that in addition is the term from only the linear theory of elasticity. The primary fact consists in existence of the special internal structure and shown above basic mechanical effects.

The term "auxetic material" was proposed by Evans in 1991 [12] for materials with negative Poisson's ratio. Nowadays, the short term "auxetics" is used. Both names come from the Greek word $\alpha\upsilon\xi\eta\tau\iota\kappa o\zeta$ (that which tends to increase). One of the first mentions (1982) of materials with negative Poisson's ratio can be found in [17, 18]. The works of Lakes [24] and Wojciechowski [44] of 1987 can be considered as the fundamental ones in the area of the auxetic materials. In the following years, the scientific journals regularly reported on new mate rials with Poisson's ratio with value less than -1. The state-of-the-art in the science on auxetic materials is shown in the review articles [3, 5, 6, 19, 23, 30, 31, 39 – 41] and the monograph [26].

It is worth to note that the auxetic materials are studies mainly by methods of molecular physics, computer simulations and material science. The contribution of specialists from the mechanics of materials looks very small. Further more, only the methods of linear models of mechanics of materials are used. This situation allows to predict that using the notions and methods of nonlinear mechanics of materials can be promising what constitute the main general component of the author's motivation.

The facts stated above permit to formulate the following goal: let us try to use the nonlinear models of deformation of elastic materials that will allow to take into account the non-small strains within the framework of classical experiments on three universal deformations: simple shear, uniaxial tension, multiaxial tension.

Therefore the short information on the universal deformation and the basic notions of non linear theory of elasticity seems to be expedient.

1. Universal deformations.

Universal deformations (uniform deformations, universal states) occupy the special place in the theory of elasticity just owing to their universality. It consists in that the theoretically and experimentally determined elastic constants of material in samples, in which the universal deformation are created purposely, are valid also for all other deformed states both samples and any different production made of this material. It is considered therefore that the particular importance of universal deformation (their fundamentality) consists in a possibility to use them in determination of properties of materials from tests [9, 13, 15, 21, 26, 28]. To realize the universal deformation, two conditions have to be fulfilled: 1. Uniformity of deformation must not depend on the choice of material. 2. Deformation of material has to occur by using only the surface loads.

In the theory of infinitesimal deformations, the next kinds of universal deformations are studied in detail: simple shear, simple (uniaxial) tension-compression, uniform volume (omniaxial) tension-compression. In the linear theory of elasticity, the experiment with a sample, in which the simple shear is realized, allows to determine the elastic shear modulus μ. The experiment with a sample, in which the uniaxial tension is realized, allows to determine the Young's elastic modulus E and Poisson's ratio ν. The experiment with a sample, in which the uniform compression is realized, allows determining the elastic bulk modulus k.

While being passed from the linear model of very small deformations to the models of non-small (moderate or large) ones, that is, from the linear mechanics of materials to nonlinear mechanics of materials, the universal states permits to describe theoretically and experimentally many nonlinear phenomena. The history of mechanics testifies the experimental observation in XIX century of the nonlinear effects that arose under the simple shear and were named later by names of Poynting and Kelvin [12, 37]. After about hundred years in XX century, these effects were described theoretically within the framework of the nonlinear Mooney-Rivlin model [16, 31, 33, 35].

The mechanics of composite materials is one more area of application of universal deformations. The simplest and most used model in this case is the model of averaged (effective, reduced) moduli. In the theory of effective moduli, the composite materials of the complex internal structure with internal links are treated usually as the homogeneous elastic media. A possibility to create in such media the states with universal deformations was used in evaluation of effective moduli by different authors and different methods. It was found that it is sufficient for isotropic composites to study the energy stored in the elementary volumes of composites under only two kinds of universal deformations: simple shear and omniaxial compression [36 – 38, 42].

2. Theoretical description of experiments where the universal deformations of simple shear, uniaxial tension, and omniaxial compression are realized.

2.1 Essentials of nonlinear mechanics of materials [16, 21, 22, 28, 37]. A body is termed some area V of 3D space R^3, in each point of which the density of mass ρ is given (the area occupied by the material continuum). In this way, a real body, the shape of which coincides with V, is changed on a fictitious body. This fictitious body is the basic notion of mechanics. The Lagrangian $\{x_k\}$ or Eulerian $\{X_k\}$ coordinate systems can be given in R^3. In the theory of deformation of a body as a change of its initial shape, the notions are utilized that are associated with a geometry of body (kinematic notions) and with the forces acting on body from outside and inside (kinetic notions). The notions of the configuration χ, the vector of dis- placement $\vec{u} = \{u_k\}$, the principal extensions λ_k, the strain tensor ε_{ik} are referred to the notions of kinematics. The external and internal forces as well as the tensors of internal stresses refer to the notions of kinetics,

The configuration of body at a moment t is called the actual one, whereas the configuration of body at arbitrarily chosen initial moment $t°$ is called the reference one. The co ordinates of the body point before deformation is denoted by x_k. It is assumed that after deformation this point is displaced on $u_k(x_1, x_2, x_3, t)$. Then the vector with components u_k is called the displacement vector and the coordinates of the point after deformation are presented in the form $\xi_k = x_k + u_k(x_1, x_2, x_3, t)$. The frequently used Cauchy-Green strain tensor is given by the known displacement vector of $\vec{u}(x_k, t)$ in the Lagrangian coordinates $\{x_k\}$ and in the reference configuration

$$\varepsilon_{nm}(x_k, t) = (1/2)\left(u_{n,m} + u_{m,n} + u_{n,i}u_{m,i}\right). \tag{1}$$

As a result, the deformation of body is given by nine components of displacement gradients $u_{i,k}$. Such a description of deformation is used in the most part of models of the theory of elasticity. But the process of deformation can be described also by other parameters of the geometry change of the body. It seems meaningful to use the first three algebraic invariants of tensor (1) $A_1 = \varepsilon_{mn}\delta_{mn}$, $A_2 = (1/2)\left[(\varepsilon_{mn}\delta_{mn})^2 - \varepsilon_{ik}\varepsilon_{ik}\right] \times \left[(\varepsilon_{mn}\delta_{mn})^2 - \varepsilon_{ik}\varepsilon_{ik}\right]$, $A_3 = \det \varepsilon_{mn}$, which can be rewritten through the principal values of tensor (1) ε_k by the formulas $A_1 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$, $A_2 = \varepsilon_1\varepsilon_2 + \varepsilon_1\varepsilon_3 + \varepsilon_2\varepsilon_3$, $A_3 = \varepsilon_1\varepsilon_2\varepsilon_3$. The often used invariants I_1, I_2, I_3 of tensor ε_{ik} are linked with the algebraic invariants of the same tensor by relations

$$I_1 = 3 + 2\varepsilon_{nn} = 3 + 2A_1;\ I_2 = 3 + 4\varepsilon_{nn} + 2(\varepsilon_{nn}\varepsilon_{mm} - \varepsilon_{nm}\varepsilon_{mn}) = 3 + 4A_1 + 2(A_1^2 - A_2);$$

$$I_3 = \det\left\|\delta_{pq} + 2\varepsilon_{pq}\right\| = 1 + 2A_1 + 2(A_1^2 - A_2) + (4/3)(2A_3 - 3A_2A_1 + A_1^3).$$

In a number of models of nonlinear deformation of materials, the elongation coefficients (principal extensions) defined as a change of length of the conditional linear elements (the infinitesimal segments that are directed arbitrarily) are used

$$\lambda_k = \sqrt{1+2\varepsilon_k} \; .$$

A simpler formula $\lambda_k - 1 \approx \varepsilon_k$ is valid for the case of linear theory. Additionally to three parameters above, in threes parameters should be introduced that characterize a change of the angles between linear elements and areas of elements of coordinate surfaces.

It seems to be necessary to show the very often used notation of the displacements gradient

$$\mathbf{F} = \begin{bmatrix} 1+u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & 1+u_{2,2} & u_{2,3} \\ u_{3,1} & u_{3,2} & 1+u_{3,3} \end{bmatrix}$$

and notation of the left Cauchy-Green strain tensor $\mathbf{B} = \mathbf{F}\ \mathbf{F}^{\mathrm{T}}$ associated with it. The most used are two tensors of internal stresses: the symmetric Cauchy-Lagrange tensor σ_{ik}, that is measured on the unit of area of deformed body, and the nonsymmetric Kirchhoff tensor t_{ik}, that is measured on the unit area of un-deformed body.

2.2. Universal deformation of simple shear. The experiments on simple shear are realized on the sufficiently long beam of quadratic cross-section, in which the uniform deformation is created on some distance from the ends. The lower side of beam is fixed rigidly and the surface tangential constant load T_2 is applied to the upper side. The deformation of beam can be described by one component of the deformation gradient $u_{1,2} = (\partial u_1 / \partial x_2)$. The component $u_{1,2}$ and the shear angle γ are linked as follows

$$u_{1,2} = \tan\gamma = \tau > 0.$$

In the linear theory, the shear angle is assumed to be small and then $\gamma \approx \tan\gamma = \tau$ [15, 28, 37].

The Cauchy-Green strain tensor is characterized by only three nonzero components

$$\varepsilon_{11} = (1/2)\left(u_{1,1} + u_{1,1} + u_{1,k}u_{1,k}\right) = (1/2)\left(u_{1,2}u_{1,2} + u_{1,3}u_{1,3}\right) = \tau^2 \; ;$$

$$\varepsilon_{12} = \varepsilon_{21} = (1/2)\left(u_{1,2} + u_{2,1} + u_{1,k}u_{2,k}\right) = (1/2)\tau.$$

The principal extensions are written through the shear angle by formulas $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = \tau$.

2.3. Universal deformation of uniaxial tension. A rod in the form of a straight long (of circular or quadratic cross-section) cylinder with the axis in direction of axis Ox_1 is considered when the lateral surface of rod is free of stresses. The rod is stretched in the axial direction. Then the uniform stress-strain state is formed in the rod except for the area near the ends. It is characterized by only one nonzero component σ_{11} of the stress tensor and two nonzero components $\varepsilon_{11}, \varepsilon_{22} = \varepsilon_{33}$ of the strain tensor (or two principal extensions $\lambda_1, \lambda_2 = \lambda_3$).

On the Young modulus and Poisson's ratio. Perhaps, the most old and exhausting procedures are shown in the classical Love's book [27]. Let us use the adopted at that time notations and write according to [9] the standard representation of the Hooke law through the Lame moduli λ, μ

$$X_x = \lambda\Delta + 2\mu\varepsilon_{xx}; \; Y_y = \lambda\Delta + 2\mu\varepsilon_{yy}; \; Z_z = \lambda\Delta + 2\mu\varepsilon_{zz};$$

$$X_y = 2\mu\varepsilon_{xy}; \; Z_x = 2\mu\varepsilon_{zx}; \; Y_z = 2\mu\varepsilon_{yz}, \tag{2}$$

where the notation of dilatation is used $\Delta = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$.

Let us repeat the classical procedure of introducing the Young modulus and the Poisson's ratio. Toward this end, the universal deformation of uniaxial tension is considered, when the axis is chosen in direction Ox and the prism is stretched at the ends by an uniform tension T. The stress state of prism is uniform and is characterized by only the stress $X_x = T$ (other stresses are zero ones). In this case, the Hooke law becomes simpler $T = \lambda\Delta + 2\mu\varepsilon_{xx}$, $0 = \lambda\Delta + 2\mu\varepsilon_{yy}$, $0 = \lambda\Delta + 2\mu\varepsilon_{zz}$. The expression for dilatation is obtained by adding all three equalities above $T = (3\lambda + 2\mu)\Delta \to \Delta = T/(3\lambda + 2\mu)$.

The substitution of the last expression for dilatation into the first equality (2) gives relations

$$T = \left[\lambda/(3\lambda + 2\mu)\right]T + 2\mu\varepsilon_{xx} \to T = \left[\mu(3\lambda + 2\mu)/(\lambda + \mu)\right]\varepsilon_{xx}.$$

The last expression represents the elementary law $T = E\varepsilon_{xx}$ of link between tension and deformation of prism, in which the Young modulus E is used. Comparison of this law with relation (3) gives the classical expression for the Young modulus through the Lame moduli

$$E = \left[\mu(3\lambda + 2\mu)/(\lambda + \mu)\right]. \tag{3}$$

The substitution of expression for dilatation into the second and third equalities (2) gives relations $-\varepsilon_{yy} = -\varepsilon_{zz} = \left[(\lambda/2)/(\lambda + \mu)\right]\varepsilon_{xx}$, which express the classical Poisson's law on the transverse compression under the longitudinal extension and permit to introduce the Poisson's ratio

$$\sigma = \left(-\varepsilon_{yy}/\varepsilon_{xx}\right) = \left(-\varepsilon_{zz}/\varepsilon_{xx}\right) = (\lambda/2)/(\lambda + \mu). \tag{4}$$

Thus, the Poisson's ratio is one of characteristics of linear deformation of elastic material and is considered as the basic notion of linear elasticity. But the ratio of transverse strain to the longitudinal one can be used in any model of nonlinear elasticity (and not only elasticity). In this case, this ratio will have its own representation in each model and possibly will not be constant quantity for any level of strains.

2.4. Universal deformation of uniform (omniaxial) compression-tension. A sample has the shape of a cube, to sides of which the uniform surface load (hydrostatic compression) is applied. Then the uniform stress state is formed in the cube. The normal stresses are equal with each other $\sigma_{11} = \sigma_{22} = \sigma_{33}$, and the shear stresses $\sigma_{ik}\ (i \neq k)$ are absent. This type of universal deformation is defined by the following components of displacement gradients

$$u_{1,1} = u_{2,2} = u_{3,3} = \varepsilon > 0; \quad u_{1,1} + u_{2,2} + u_{3,3} = 3\varepsilon = e; \quad u_{k,m} = \left(\partial u_k/\partial x_m\right) = 0 \ (k \neq m). \tag{5}$$

The Cauchy-Green strain tensor is as follows

$$\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = \varepsilon + (1/2)\varepsilon^2; \ \varepsilon_{ik} = 0 \quad (i \neq k), \tag{6}$$

and the algebraic invariants are written in the form

$$I_1 = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = e; \quad I_2 = (\varepsilon_{11})^2 + (\varepsilon_{22})^2 + (\varepsilon_{33})^2; \quad I_3 = (\varepsilon_{11})^3 + (\varepsilon_{22})^3 + (\varepsilon_{33})^3. \tag{7}$$

The principal extensions are equal with each other

$$\lambda_1 = \lambda_2 = \lambda_3. \tag{8}$$

3. Three nonlinear models of hyperelastic deformation.

These models are related to the models of hyperelastic materials. This class of materials is characterized by the way of introduction of constitutive equations. First the function of kinematic parameters (elastic potential, internal energy) is defined, from which later the constitutive equations are derived mathematically and substantiated physically.

Two-constant Neo-Hookean model (model 1). The elastic potential of Neo-Hookean mo del is defined as follows [16, 22, 34, 37]

$$W = C_1\left(\overline{I}_1 - 3\right) + D_1\left(J-1\right)^2;\ \overline{I}_1 = J^{-2/3} I_1;\ J = \det u_{i,k};$$

$$W\left(\lambda_1, \lambda_2, \lambda_3\right) = C_1\left[\left(\lambda_1 \lambda_2 \lambda_3\right)^{-2/3}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right) - 3\right] + D_1\left(\lambda_1 \lambda_2 \lambda_3 - 1\right)^2. \tag{9}$$

where the elastic constants of model are linked with the classical elastic constants by relations $2C_1 = \mu;\ 2D_1 = k$.

The constitutive equations have the form

$$\sigma_{nm} = 2C_1 J^{-5/3}\left[B_{nm} - (1/3) I_1 \delta_{nm}\right] + 2D_1\left(J-1\right)\delta_{nm}; \tag{10}$$

$$\sigma_{nn} = 2C_1 J^{-5/3}\left(\lambda_n - (1/3) I_1\right) + 2D_1\left(J-1\right).$$

It is considered that this model describes well the deformation of rubber under the principal extensions up to 20% from the initial state. Since these extensions are linked with the principal values of the strain tensor by relation (2) $\lambda_k = \sqrt{1+2\varepsilon_k}$, then it is assumed $\lambda_k - 1 \approx \varepsilon_{kk}$ approximately with exactness to $\leq 1\%$ in the cases of universal deformations for Neo-Hookean model, what is true in the case of linear theory too. Because the extensions in the linear theory are two orders less, then this observation testifies the fact that the Neo-Hookean model extends essentially the area of allowable values of strains as compared with the Hookean model.

3.2. Three-constant Mooney-Rivlin model (model 2). The elastic potential of the Mo- oney -Rivlin model is defined as follows [16, 22, 31, 35, 37]

$$W = C_{10}\left(\overline{I}_1 - 3\right) + C_{01}\left(\overline{I}_2 - 3\right) + D_1\left(J-1\right)^2;\ \overline{I}_2 = J^{-4/3} I_2;$$

$$W\left(\lambda_1, \lambda_2, \lambda_3\right) = C_{10}\left[\left(\lambda_1 \lambda_2 \lambda_3\right)^{-2/3}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right) - 3\right] + \tag{11}$$

$$+ C_{01}\left[\left(\lambda_1 \lambda_2 \lambda_3\right)^{-4/3}\left(\lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2\right) - 3\right] + D_1\left(\lambda_1 \lambda_2 \lambda_3 - 1\right)^2,$$

where the elastic constants of the model are linked with the classical constants by relations $2(C_{10} + C_{01}) = \mu; 2D_1 = k.$.

The stresses are determined by formulas

$$\sigma = 2J^{-5/3}\left(C_{10} + C_{01} J^{-2/3} I_1\right) B - 2J^{-7/3} C_{01} BB +$$

$$+\left[2D_1\left(J-1\right) - (2/3) J^{-5/3}\left(C_{10} I_1 + 2C_{01} J^{-2/3} I_2\right)\right]1; \tag{12}$$

$$\sigma_{kk} = \lambda_k \frac{\partial W}{\partial \lambda_k} = 2C_{10}\left(\lambda_1 \lambda_2 \lambda_3\right)^{-5/3}\left[\lambda_k^2 - (1/3)\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right)\right] + 2C_{01}\left(\lambda_1 \lambda_2 \lambda_3\right)^{-7/3}\left[\lambda_k^2\left(\lambda_n^2 + \lambda_m^2\right) - \right.$$

$$\left. - (2/3)\lambda_k\left(\lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2\right)\right] + D_1\left(\lambda_1\lambda_2\lambda_3 - 1\right). \tag{13}$$

Here the indexes knm form the cyclic permutation from numbers 123.

The Mooney-Rivlin model is classical one. This can be seen from the next historical information.

Information. An effect of nonlinear dependence of decreasing the shear stresses when the torsion angle (deformation) to the level of nonsmall values is called "the Poynting effect" owing to his publication of 1909, where this effect was described. At that, Poynting does not mentioned the results of Coloumb (1784), Wertheim (1857), Kelvin (1865), Bauschinger (1881), Tomlinson (1883), where this effect was also described in one way or an other.

But only within the framework of finite elastic deformations, that was developed in 20 century, this effect was satisfactorily explained by Rivlin in 1951. He used the model of nonlinear deformation which now is termed "the Mooney-Rivlin model".

Five-constant Murnaghan model (model 3). The elastic potential in the Murnaghan model has the form [14, 22, 32, 33, 37]

$$W(\varepsilon_{ik}) = (1/2)\lambda(\varepsilon_{mm})^2 + \mu(\varepsilon_{ik})^2 + (1/3)A\varepsilon_{ik}\varepsilon_{im}\varepsilon_{km} + B(\varepsilon_{ik})^2\varepsilon_{mm} + (1/3)C(\varepsilon_{mm})^3; \qquad (14)$$

$$W(I_1, I_2, I_3) = (1/2)\lambda I_1^2 + \mu I_2 + (1/3)AI_3 + BI_1I_2 + (1/3)I_1^3 .$$

The Cauchy-Green strain tensor ε_{ik} and five elastic constants (two Lame elastic constants λ, μ and three Murnaghan elastic constants A, B, C) are used in this potential.

The Murnaghan model can be considered as the classical one in the nonlinear theory of hyperelastic materials. It takes into account all the quadratic and cubic summands from expansion of the internal energy and describes the deformation of big class of engineering and other materials. If to unite the data on the constants of Murnaghan model, shown in books [14, 22, 28], then the sufficiently full information can be obtained on many tens of materials.

4. Simple shear.

The following materials are used in the numerical evaluations below (elastic constants are shown): 1. Rubber – $\mu = 20$ MPa, $k = 2{,}0$ GPa. 2. Foam – $\lambda = 0{,}58 \cdot 10^9$, $\mu = 0{,}39 \cdot 10^9$, $k = 0{,}84 \cdot 10^9$. 3. Foam – $\lambda = 0{,}58 \cdot 10^9$, $\mu = 0{,}39 \cdot 10^9$, $A = -1{,}0 \cdot 10^{10}$, $B = -0{,}9 \cdot 10^{10}$, $C =$ $= -1{,}1 \cdot 10^{10}$. 4. Polystyrene – $\lambda = 3{,}7 \cdot 10^9$, $\mu = 1{,}14 \cdot 10^9$, $A = -1{,}1 \cdot 10^{10}$, $B = -0{,}79 \cdot 10^{10}$, $C = -0{,}98 \cdot 10^{10}$.

Description by model 1. In this case $J = (1+\tau)^2$, $I_1 = 1 + 2\tau^2$. Then expressions for displacement gradients **F** and components of tensor **B** are simplified

$$\mathbf{F} = \begin{bmatrix} 1 & \tau & \tau \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \ \mathbf{B} = \begin{bmatrix} 1+2\tau^2 & \tau & \tau \\ \tau & 1 & 0 \\ \tau & 0 & 1 \end{bmatrix}.$$

As a result, the components of stress tensor have the form

$$\sigma_{12} = \sigma_{21} = \sigma_{13} = \sigma_{31} = 2C_1\left(1+\tau\right)^{-10/3}\tau; \ \sigma_{32} = \sigma_{23} = 0;$$

$$\sigma_{11} = (8/3)C_1\left(1+\tau\right)^{-10/3}\left(\tau - 1\right)\tau + 2D_1\tau\left(\tau+2\right); \qquad (15)$$

$$\sigma_{22} = \sigma_{33} = -(4/3)C_1\left(1+\tau\right)^{-10/3}\left(1+2\tau\right)\tau + 2D_1\tau\left(\tau+2\right).$$

The formulas (15) show that the Poynting effect (when the values of shear angle increase from the sufficiently small values to the moderate ones, then the shear stress depends nonlinearly on the shear angle) is described by the Neo-Hookean model, because equation (15) demonstrates just this nonlinear dependence for the moderate values of shear angle. The Figure 7 shows the dependence of the shear stress on the shear angle $\sigma_{12} \sim \tau$ for the silicon rubber (in all the plots, a stress corresponds to 1 MPa).

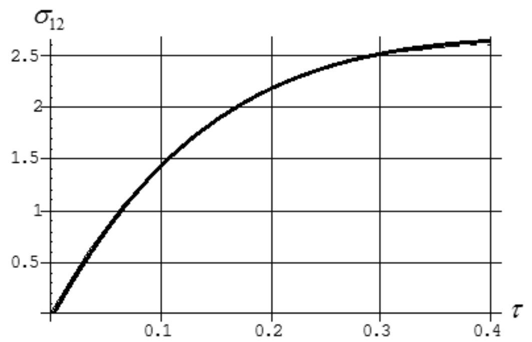

Fig. 7

Description by model 2. The expressions for displacement gradient **F** and components of tensor **B** are the same as for the Neo-Hookean model. As a result, the expressions from formula (12) are simplified

$$\lambda_1 = 1, \lambda_2 = \lambda_3 = 1 + \tau; \quad J = (1+\tau)^2; \quad I_1 = 1 + 2\tau^2; \quad I_2 = (1+\tau)^2 \left[2 + (1+\tau)^2\right]$$

and components of the stress tensor have the form

$$\sigma_{12} = \sigma_{21} = 2C_{10}(1+\tau)^{-10/3}\tau - 2C_{01}(1+\tau)^{-14/3}(1+4\tau)\tau; \tag{16}$$

$$\sigma_{23} = \sigma_{32} = -2C_{01}(1+\tau)^{-14/3}\tau^2; \tag{17}$$

$$\begin{aligned}\sigma_{11} = {} & 2C_{10}(1+\tau)^{-10/3}(4/3)\left(1+\tau+2\tau^2\right) + 2D_1\tau(1+2\tau) + \\ & + 2C_{01}(1+\tau)^{-14/3}(4/3)\left(3+5\tau+5\tau^2+4\tau^3-2\tau^4\right);\end{aligned} \tag{18}$$

$$\sigma_{22} = \sigma_{33} = 2C_1(1+\tau)^{-2}\left[(1+\tau)^{-4/3} + \left(1+2\tau^2\right)\left(1-(1+\tau)^{-4/3}\right) - 1\right] + 2D_1\tau(1+2\tau). \tag{19}$$

Thus, the Mooney-Rivlin model (that is more complicate as compared with the Neo-Hookean model) describes the more complicate stress state, which is characterized by six components of stress tensor. This model describes well-known nonlinear effects. The Poynting effect follows from representation of the shear stresses by formula (16). The Kelvin effect follows from formulas (18), (19).

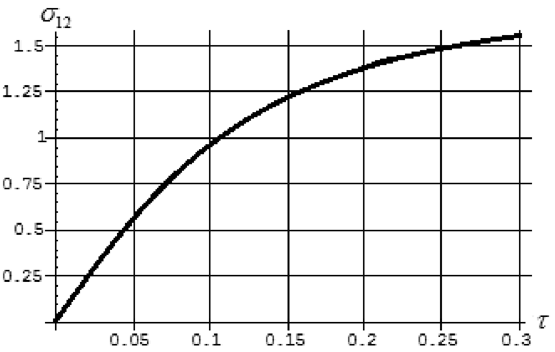

Fig. 8

Also, formula (17) describes one more nonlinear effect: an initiation of shear stresses $\sigma_{23} = \sigma_{32}$. The Figure 8 shows the nonlinear dependence of shear stress σ_{12} on the shear strain τ, that is built for the silicon rubber. Comparison with Figure 7, which corresponds to the Neo-Hookean model, shows that the Mooney-Rivlin model describes the more essential deviation from the linear Hookean description of simple shear.

4.3. Description by model 3. The Cauchy-Green strain tensor is characterized by three components

$$\varepsilon_{22} = (1/2)\left(u_{2,2} + u_{2,2} + u_{k,2}u_{k,2}\right) = (1/2)\tau^2; \tag{20}$$

$$\varepsilon_{12} = \varepsilon_{21} = (1/2)\left(u_{1,2} + u_{2,1} + u_{k,1}u_{k,2}\right) = (1/2)\tau. \tag{21}$$

To calculate the stresses, it is necessary to write the potential (14) with respect to the formulas (20), (21)

$$W(\varepsilon_{ik}) = (1/2)\lambda(\varepsilon_{22})^2 + \mu\left[(\varepsilon_{22})^2 + (\varepsilon_{12})^2 + (\varepsilon_{21})^2\right] + \tag{22}$$

$$+(1/3)A\left[\varepsilon_{22}\left(\varepsilon_{12}\varepsilon_{12} + \varepsilon_{21}\varepsilon_{21} + \varepsilon_{12}\varepsilon_{21}\right) + (\varepsilon_{22})^3\right] + B\left[(\varepsilon_{22})^2 + (\varepsilon_{12})^2 + (\varepsilon_{21})^2\right]\varepsilon_{22} + (1/3)C(\varepsilon_{22})^3;$$

$$W(\tau) = (1/2)\mu\tau^2 + (1/8)\left[(\lambda+2\mu) + A + B\right]\tau^4 + (1/24)\left[A + 3B + C\right]\tau^6. \tag{23}$$

The Lagrange stress tensor is determined by the formula $\sigma_{ik}\left(x_n, t\right) = \partial W / \partial \varepsilon_{ik}$ and has two nonlinear components

$$\sigma_{22} = \left(\lambda + 2\mu\right)\varepsilon_{22} + A\left[\left(\varepsilon_{22}\right)^2 + \left(1/3\right)\left(\varepsilon_{12}\varepsilon_{12} + \varepsilon_{21}\varepsilon_{21} + \varepsilon_{12}\varepsilon_{21}\right)\right] +$$

$$+ B\left[3\left(\varepsilon_{22}\right)^2 + \left(\varepsilon_{12}\right)^2 + \left(\varepsilon_{21}\right)^2\right] + C(\varepsilon_{22})^2 = \quad (24)$$

$$= \left(1/4\right)\left[2\left(\lambda + 2\mu\right) + \left(A + 2B\right)\right]\tau^2 + \left(1/4\right)\left(A + 3B + C\right)\tau^4;$$

$$\sigma_{12} = 2\mu\varepsilon_{12} + \left(\left(1/3\right)A\left(\varepsilon_{12} + \varepsilon_{21}\right) + 2B\varepsilon_{12}\right)\varepsilon_{22}; \quad (25)$$

$$\sigma_{12} = \sigma_{21} = \mu\tau + \left(1/6\right)\left(A + 3B\right)\tau^3. \quad (26)$$

The shear stress contains the linear and nonlinear summands and describes the simple shear. The normal stress describes the change of volume under deformation and testifies the break of the state of simple shear under nonlinear description of deformation. To build the plots of dependence (24) choose two nonstandard for the Murnaghan mo del materials – foam and polystyrene – which can experience not only the small by values strains, but also the moderate ones.

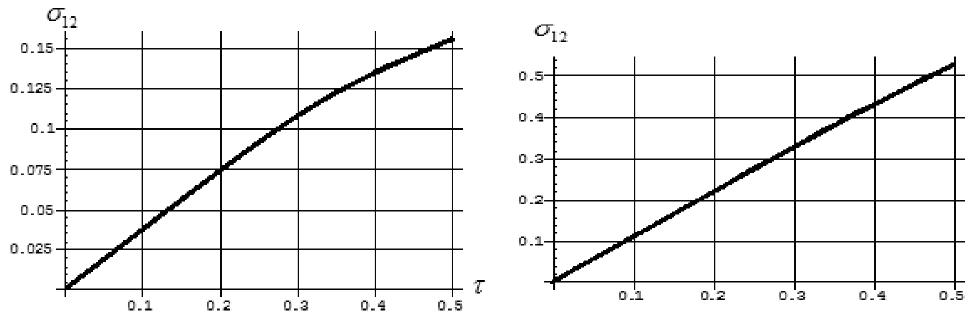

Fig. 9

The Figure 9 shows a dependence of the shear stress on the shear angle.

Note to dependence $\sigma_{12} \sim \tau$ for models 1 – 3. These models describe the nonlinear Poynting effect. At the same time many scientists working with auxetic materials report the experimental dependences that coincide quantitatively with the shown here theoretical dependences (for example, [4, 19]).

Conclusion to dependence $\sigma_{12} \sim \tau$ for models 1 – 3. The developed in mechanics of materials nonlinear models of deformation of elastic materials can be recommended for description of auxetic materials.

5. Uniaxial tension.

5.1. Description by model 1. The formulas $\lambda_2 = \lambda_3$, $J = \lambda_1\lambda_2^2$, $I_1 = \lambda_1^2 + 2\lambda_2^2$ are valid and the normal stresses are given by the formulas

$$\sigma_{11} = \left(2/3\right)\mu\left(\lambda_1\lambda_2^2\right)^{-5/3}\left(\lambda_1^2 - \lambda_2^2\right) + k\left(\lambda_1\lambda_2^2 - 1\right); \quad (27)$$

$$\sigma_{22} = \sigma_{33} = -\left(1/3\right)\mu\left(\lambda_1\lambda_2^2\right)^{-5/3}\left(\lambda_1^2 - \lambda_2^2\right) + k\left(\lambda_1\lambda_2^2 - 1\right). \quad (28)$$

If to assume that the normal stresses on outside of the sample are absent

$$-\left(1/3\right)\mu\left(\lambda_1\lambda_2^2\right)^{-5/3}\left(\lambda_1^2 - \lambda_2^2\right) + k\left(\lambda_1\lambda_2^2 - 1\right) = 0; \quad (29)$$

$$\text{then } \sigma_{11} = 3k\left(\lambda_1\lambda_2^2 - 1\right). \quad (30)$$

It follows from Eq. (30) that the Poynting-type effect (when the principal extensions increase from the sufficiently small values to the moderate ones, then the normal stress in the direction of tension depends nonlinearly on these extensions) is described by the Neo-Hookean model.

It should be noted that the stretching in the longitudinal direction stress depends in the model 1 on two principal extensions – longitudinal and transverse.

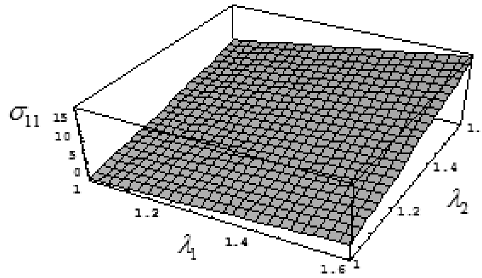

Fig. 10

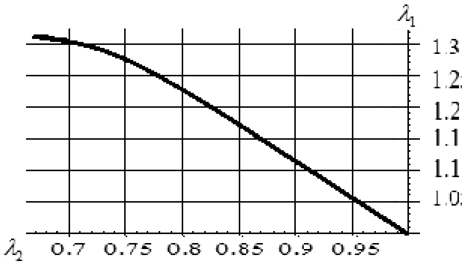

Fig. 11

The Figure 10 shows a dependence of the longitudinal stress on principal extensions and is built for the rubber with allowance for that the value $(\mu/3k) = 0.00334$ is very small compared to the unit. Then Eq. (29) is simplified to the form

$$\varepsilon_{22} = (1/2)(1+2\varepsilon_{11})^{-2} - 1/2 . \tag{31}$$

The Figure 11 corresponds to formula (29) and shows a dependence of the longitudinal principal extension on the transverse principal extension. Note that the silicon rubber is characterized by the big difference between values of shear and bulk moduli that can reach hundred times. Therefore, the new material is chosen further for the numerical analysis – the foam, which values of elastic constants is characterized by about equal by the order.

It looks in this case to be illogical to neglect the first summand in Eq. (29). Note that the ratio (λ_2 / λ_1) corresponds in the linear theory to the Poisson's ratio.

The Figure 6 shows that with an increase of extension λ_1 the increase of extension λ_2 slows.

5.2. Description by model 2. The uniaxial tension in direction of abscissa axis is characterized by parameters: $\lambda_2 = \lambda_3$, $J = \lambda_1\lambda_2^2$, $I_1 = \lambda_1^2 + 2\lambda_2^2$, $I_2 = \lambda_2^4 + 2\lambda_1^2\lambda_2^2$, $B_{11} = \lambda_1^2$, $(BB)_{11} = \lambda_1^4$. The normal stresses are given by the formulas

$$\sigma_{11} = 2C_{10}(2/3)(\lambda_1\lambda_2^2)^{-5/3}(\lambda_1^2 - \lambda_2^2) +$$

$$+ 2C_{01}(\lambda_1\lambda_2^2)^{-7/3}(\lambda_1^4 + (2/3)\lambda_1^2\lambda_2^2 - (5/3)\lambda_2^4) + 2D_1(\lambda_1\lambda_2^2 - 1); \tag{32}$$

$$\sigma_{22} = \sigma_{33} = (2/3)C_{10}(\lambda_1\lambda_2^2)^{-5/3}(\lambda_2^2 - \lambda_1^2) +$$

$$+ 2C_{01}(\lambda_1\lambda_2^2)^{-7/3}(1/3)\lambda_2^2(\lambda_2^2 - \lambda_1^2) + 2D_1(\lambda_1\lambda_2^2 - 1). \tag{33}$$

Assume that the normal stresses over the sample outer surface are absent. Then equation (33) is simplified to the form

$$\sigma_{11} = 2C_{01}(\lambda_1\lambda_2^2)^{-7/3}(\lambda_1^4 - \lambda_2^4) + 6D_1(\lambda_1\lambda_2^2 - 1). \tag{34}$$

The last formula testifies: the Mooney-Rivlin model describes the Poynting-type effect. Two elastic constants are presented in Eq. (34) in contrast to the Neo-Hookean model, where the shear modulus was absent.

It should be noted that in both models – Neo-Hookean and Mooney-Rivlin – the tension in the longitudinal direction stress σ_{11} depends already on two principal extensions.

The Figure 12 shows a dependence of the longitudinal stress on principal extensions is built for the silicon rubber. It coincides practically with Figure 10 (Neo-Hookean model) and shows that the constant C_{01} of the Mooney-Rivlin model effects not essentially on the stress σ_{11} and the dependence (32) rests the weakly nonlinear within the accepted restrictions.

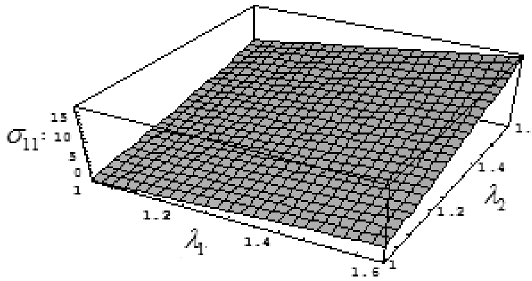

Fig. 12

The equation (33) can be transformed to the form

$$\lambda_1^6 - \lambda_1^3/\sigma^2 + \left[\left(2C_{10}/6D_1\right)\sqrt[-3]{\sigma^4} + \left(2C_{01}/6D_1\right)\sqrt[-3]{\sigma^2}\right]\sigma^{-4}(\sigma^2-1) = 0; \quad (35)$$

$$\sigma = \left(\lambda_2/\lambda_1\right); \quad \lambda_1^3 = 1/2\sigma^2 \pm 1/2\sigma^2\sqrt{1 - \begin{bmatrix}\left(2C_{10}/6D_1\right)\sqrt[-3]{\sigma^4} + \\ + \left(2C_{01}/6D_1\right)\sqrt[-3]{\sigma^2}\end{bmatrix}\sigma^{-4}(\sigma^2-1)}.$$

The corresponding to the model 1 plot from Figure 11 is practically identical with the plot corresponding to the model 2.

5.3. Description by model 3. The uniaxial tension is characterized by three nonzero components of the strain tensor ε_{kk} and one non-zero component of the stress tensor σ_{11}. Then the constitutive equations are somewhat simplified and have the form

$$\begin{gathered}\sigma_{11} = \lambda\left[\left(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}\right) = I_1\right] + 2\mu\varepsilon_{11} + A\left(\varepsilon_{11}\right)^2 + \\ + B\left\{\left[\left(\varepsilon_{11}\right)^2 + \left(\varepsilon_{22}\right)^2 + \left(\varepsilon_{33}\right)^2 = \mathrm{E}\right] + 2\varepsilon_{11}I_1\right\} + C\left(\mathrm{E} + 2\varepsilon_{22}\varepsilon_{11} + 2\varepsilon_{33}\varepsilon_{11}\right).\end{gathered} \quad (36)$$

$$0 = \lambda I_1 + 2\mu\varepsilon_{22} + A\left(\varepsilon_{22}\right)^2 + B\left(\mathrm{E} + 2\varepsilon_{22}I_1\right) + C\left(\mathrm{E} + 2\varepsilon_{22}\varepsilon_{33} + 2\varepsilon_{22}\varepsilon_{11}\right); \quad (37)$$

$$0 = \lambda I_1 + 2\mu\varepsilon_{33} + A\left(\varepsilon_{33}\right)^2 + B\left(\mathrm{E} + 2\varepsilon_{33}I_1\right) + C\left(\mathrm{E} + 2\varepsilon_{22}\varepsilon_{11} + 2\varepsilon_{22}\varepsilon_{33}\right). \quad (38)$$

Let us remind that in the linear theory of elasticity, corresponding to the Hookean model, the constitutive equations are significantly simpler

$$\sigma_{11} = \lambda I_1 + 2\mu\varepsilon_{11}; \quad 0 = \lambda I_1 + 2\mu\varepsilon_{22}; \quad 0 = \lambda I_1 + 2\mu\varepsilon_{33}. \quad (39)$$

Apply further to the nonlinear equations (36) – (38) the procedure of analysis of the state of uniaxial tension that is used in the linear theory of elasticity as applied to equations (39). Sub traction of equation (38) from equation (37) gives the formula

$$0 = 2\mu\left(\varepsilon_{22}-\varepsilon_{33}\right) + A\left(\left(\varepsilon_{22}\right)^2 - \left(\varepsilon_{33}\right)^2\right) + 2B\left(\varepsilon_{22}-\varepsilon_{33}\right)\left(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}\right),$$

from which follows the equality of components of transverse strains $\varepsilon_{22} = \varepsilon_{33}$.

Addition of formulas (36)-(38) results in the following formula

$$\sigma_{11}/(3\lambda+2\mu) - \left[(A+3B+C)/3\lambda+2\mu\right]\left[\left(\varepsilon_{11}\right)^2 + 2\left(\varepsilon_{22}\right)^2\right] - \left[2B/(3\lambda+2\mu)\right]\left(\varepsilon_{11}+2\varepsilon_{22}\right)^2 -$$

$$-\left[4C/(3\lambda+2\mu)\right]\left[\left(\varepsilon_{22}\right)^2 + 2\varepsilon_{22}\varepsilon_{11}\right] = \varepsilon_{11} + 2\varepsilon_{22}. \quad (40)$$

Substitution of formula (40) into the relation (36) gives new relation

$$\sigma_{11} = E\varepsilon_{11} + \left(A + \frac{2\lambda + 3\mu}{\lambda + \mu} B + C \right) (\varepsilon_{11})^2 - \tag{41}$$

$$- \frac{\lambda}{\lambda+\mu} \left(A + \frac{4\lambda - 2\mu}{\lambda} B - \frac{2\mu}{\lambda} C \right) (\varepsilon_{22})^2 + \frac{2(\lambda+2\mu)}{\lambda + \mu} (B + C) \varepsilon_{11}\varepsilon_{22}.$$

The last relation shows that the model 3, like the models 1 and 2, describes the Poyntingtype effect.

The Figure 13 shows a dependence among the longitudinal stress σ_{11} and strains $\varepsilon_{11}, \varepsilon_{22}$ $\sigma_{11} = \sigma_{11}(\varepsilon_{11}, \varepsilon_{22})$ for the foam and polystyrene and the moderate values of strains. Both plots demonstrate an essential nonlinearity under moderate strains.

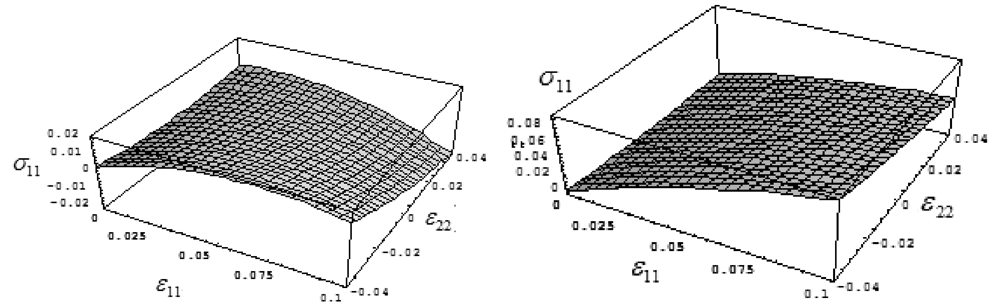

Fig. 13

Write now the constitutive equation (36) with allowance for equality $\varepsilon_{22} = \varepsilon_{33}$ and transform it to the form of quadratic equation relative to the ratio $\varepsilon_{22} / \varepsilon_{11}$

$$\left(\frac{\varepsilon_{22}}{\varepsilon_{11}} \right)^2 + 2 \frac{\left[\left((\lambda+\mu)/\varepsilon_{11} \right) + (B+C) \right]}{(A + 6B + 4C)} \frac{\varepsilon_{22}}{\varepsilon_{11}} + \frac{(\lambda/\varepsilon_{11}) + (B+C)}{(A + 6B + 4C)} = 0 .$$

The solution of this equation has the form

$$(\varepsilon_{22}/\varepsilon_{11}) = -\left\{ \left[(\lambda + \mu)/\varepsilon_{11} + (B + C) \right] / (A + 6B + 4C) \right\} \times$$

$$\times \left[1 \pm \sqrt{1 - \frac{(A + 6B + 4C)\left[\lambda / \varepsilon_{11} + (B + C) \right]}{\left[(\lambda + \mu) / \varepsilon_{11} + (B + C) \right]^2}} \right]. \tag{42}$$

Thus, equation (42) shows that for the ratio $(-\varepsilon_{22} / \varepsilon_{11})$ is not constant in the Murnaghan nonlinear model.

The plots in Fig. 14 shows a dependence of the ratio $(-\varepsilon_{22} / \varepsilon_{11})$ on the strain ε_{11} .and are built for the foam and polystyrene for the moderate strains. The plot main feature: the ratio $(-\varepsilon_{22} / \varepsilon_{11})$ is decreased essentially from the initial value, which corresponds to the Poisson ratio for small strain, to the negative values under the moderate values of longitudinal strain. So, the ratio, that is treated as the Poisson's ratio for small strain, in the case of moderate strain becomes the characteristics of transition of the material from the category of conventional materials into the category of nonconventional materials. This can be considered as the new revealed theoretically nonlinear effect.

Thus, an analysis of universal deformation of uniaxial tension for the model 3 revealed the new property: the material with conventional properties under small strains is transformed under moderate strains into the nonconventional (auxetic) material. Uncommonness of this observation consists in that usually the material is considered either the conventional or the nonconventional during all the process of deformation.

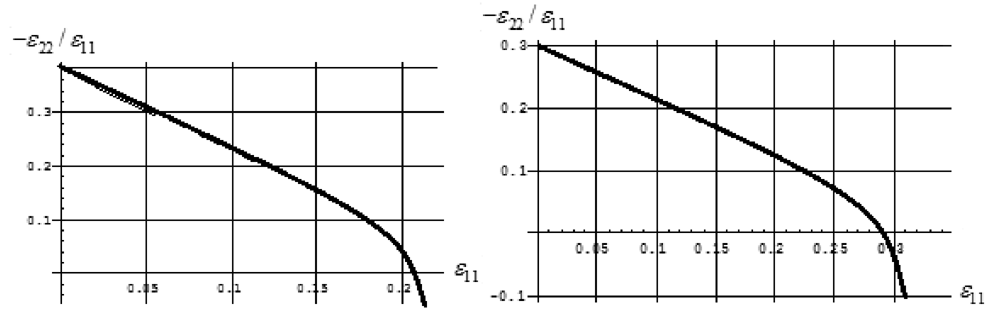

Fig. 14

Let us compare the plots from Fig. 14 with the experimental data from [4, Fig.4] shown here as Fig. 15 (dependence of the ratio $(-\varepsilon_{22} / \varepsilon_{11})$ on the strain ε_{11}), where the deformation of the foams was studied for the finite strans with increasing the longitudinal strain ε_{11} from 0,1 to 1,4. Note that the theoretical plots are constructed for the range from $\varepsilon_{11}= 0$ to the moderate values 0.23 (foam) and 0,33 (polystyrene). This comparison shows that $\left(\varepsilon_{22} / \varepsilon_{11}\right)$ increases within the range $\varepsilon_{11} \in (0,0; 0,3)$. Thus, the model 3 describes some experimental observation of the foam.

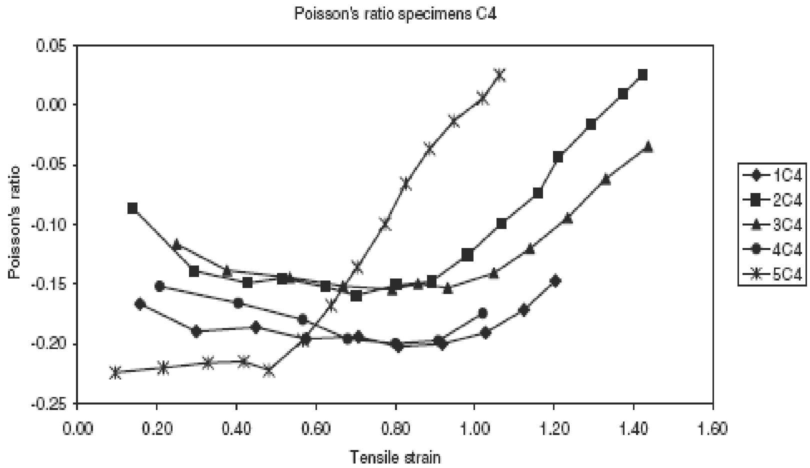

Fig. 15

The plots in Figure 16 show dependence of longitudinal and transverse strains. Three stages can be marked out: 1. A decrease of transverse strain becomes slower under transition to the moderate strains. 2.The strain ε_{22} reaches the local minimum and further increases.

3. When the strain ε_{11} continues to increase, the strain ε_{22} possesses zero value and further increases possessing already the positive values.

The shown feature confirms once again the new mechanical effect – a transition of the material under its deformation to the level of moderate values of the longitudinal stretching from the class of conventional materials into the class of the auxetic materials. In other words, the standard sample in conditions of universal deformation of uniaxial tension is deformed for small strains as if it is made of the conventional material (its cross-section is decreased) and with increasing the values of longitudinal stretching to the moderate values the sample cross-section starts to increase, what is the characteristic just for auxetic materials.

The plots from Figure 11 can be compared with the plot, obtained experimentally in [41].

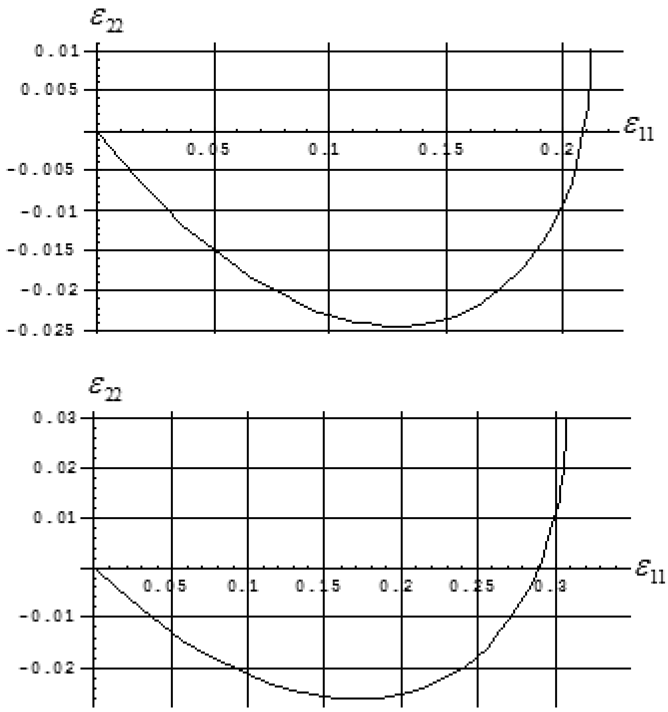

Fig. 16

This article reports that the new metamaterials were created from the soft silicon rubber. The samples were deformed in conditions of uniaxial compression up to moderate values of longitudinal strain 0,35. The shown in Figure 17 plot corresponds to Figure 2a in [41] and shows a dependence of longitudinal and transverse strains. Comparison of plots from Figure16 (uniaxial stretching) and Figure 17 (uniaxial compression) demonstrates the common property of forming the hump in the area of negative values of transverse strain, which is transformed with the increasing values of longitudinal strain roughly into the straight line in the area of positive values of transverse strain.

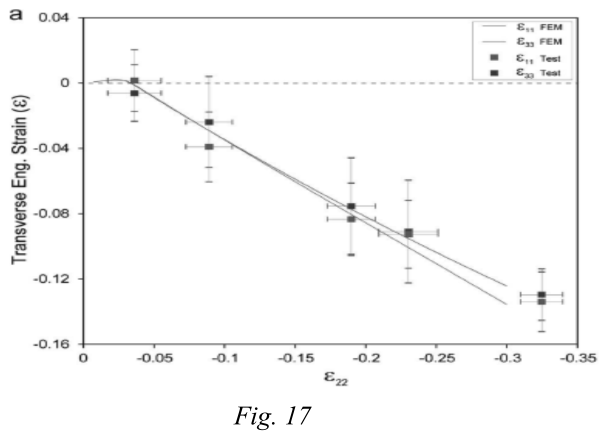

Fig. 17

Main conclusion to subsection 6. The nonlinear Murnaghan model describes within conditions of uniaxial tension some nonlinear phenomena of deformation, which can be linked with the properties of deformation of auxetic materials. Note that the shown feature is clearly visible only within the framework of the Murnaghan model, but the Neo-Hookean and Mooney -Rivlin models also describe the hump formation, as it can be seen in Fig. 11.

6. Multiaxial tension.

6.1. Description by model 1. In this case $\lambda_1 = \lambda_2 = \lambda_3$, $J = \lambda_1^3, I_1 = 3\lambda_1^2$ and the normal stress is equal

$$\sigma_{11} = 2D_1(\lambda_1^3 - 1). \qquad (43)$$

The formula (43) describes the Poyntingtype effect relative to the bulk modulus (the dependence of σ_{11} on the extension λ_1 is evidently nonlinear). The Figure 18 shows a dependence of longitudinal stress on the longitudinal principal extension and is built for the silicon rubber. The plot testifies that the model 1 describes the nonlinear change of the sample volume, while being subjected the universal deformation of uniform compression-tension.

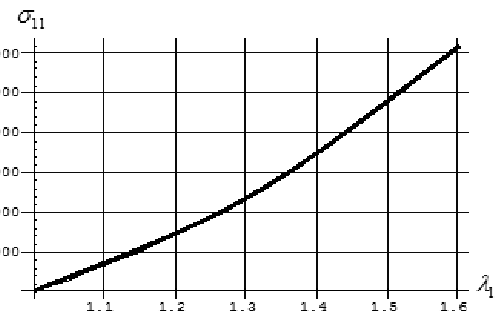

Fig. 18

6.2. Description by model 2. In this case $\lambda_1 = \lambda_2 = \lambda_3$, $J = \lambda_1^3, I_1 = 3\lambda_1^2$, $I_2 = 3\lambda_1^4$ and they are true for any nonlinear model. The formula for normal stress coincides with the analogous formula for the model 1 (43) and verifies the nonlinear dependence of tension stresses on the principal extension.

6.3. Description by model 3. The components of displacement gradients and Cauchy-Green strain tensor are as follows

$$u_{1,1} = u_{2,2} = u_{3,3} = \varepsilon > 0; \quad u_{1,1} + u_{2,2} + u_{3,3} = 3\varepsilon = e; \quad u_{k,m} = \left(\partial u_k / \partial x_m\right) = 0 \ \left(k \neq m\right);$$

$$\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = \varepsilon + \left(1/2\right)\varepsilon^2; \ \varepsilon_{ik} = 0 \quad \left(i \neq k\right). \qquad (44)$$

The corresponding algebraic invariants of the Cauchy-Green tensor are written in the form

$$I_1 = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = e; \quad I_2 = \left(\varepsilon_{11}\right)^2 + \left(\varepsilon_{22}\right)^2 + \left(\varepsilon_{33}\right)^2 = \left(1/3\right)e^2;$$

$$I_3 = \left(\varepsilon_{11}\right)^3 + \left(\varepsilon_{22}\right)^3 + \left(\varepsilon_{33}\right)^3 = \left(1/9\right)e^3. \qquad (45)$$

The formulas for invariants (45) allow to write the potential in the simpler form

$$W(\varepsilon) = \left(3/2\right)\left(3\lambda + 2\mu\right)\varepsilon^2 + \left(\left(9/2\right)\lambda + 3\mu + A + 9B + 9C\right)\varepsilon^3 +$$
$$+ \left(3/2\right)\left(4\left(3\lambda + 2\mu\right) + \left(A + 9B + 9C\right)\right)\varepsilon^4 +$$
$$+ \left(3/4\right)\left(A + 9B + 9C\right)\varepsilon^5 + \left(1/8\right)\left(A + 9B + 9C\right)\varepsilon^6. \qquad (46)$$

The stresses are evaluated by the formulas

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = \left(3\lambda + 2\mu\right)\varepsilon + \left(\left(3/2\right)\left(3\lambda + 2\mu\right) + \left(A + 9B + 7C\right)\right)\varepsilon^2 +$$
$$+ \left(A + 9B + 7C\right)\left(\varepsilon^3 + (1/4)\,\varepsilon^4\right); \ \sigma_{12} = \sigma_{23} = \sigma_{31} = 0.$$

Thus, the normal stresses only are nonzero and they contain the linear and nonlinear summands.

The interdependence between the first invariant of the stress tensor σ_{kk} and the parameter of the multiaxial tension e has the form

$$\sigma_{kk} = (3\lambda + 2\mu)e + \big[(1/2)(3\lambda + 2\mu) +$$

$$+ (1/3)(A + 9B + 7C)\big]e^2 + (A + 9B + 7C)\big[(1/9)e^3 + (1/108)e^4\big]. \quad (47)$$

The plots in Figure 19 shows a dependence $\sigma_{kk}(e)$ for the foam and polystyrene and are evaluated by formula (47). It follows from them that they are similar to the parabola with a vertex in a positive half of the plane $\sigma_{kk}Oe$. The parabola right branch of then passes into the negative half of the plane. Both plots have "the hump" in the positive branch of the plane.

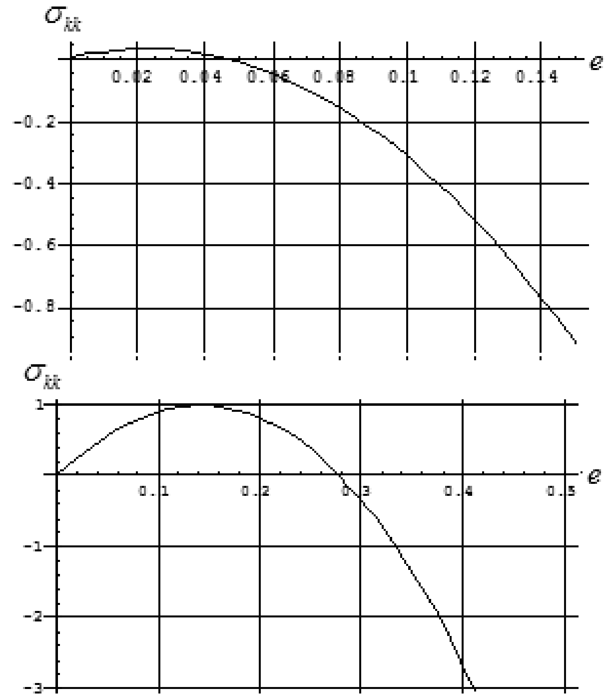

Fig. 19

A presence of "the hump" testifies that the nonlinear Murnaghan model describes the transition of the material of the sample-cube from the class of conventional materials into the class of auxetic materials. The fact is that the sample is compressed for the small values of uniform tension and in the following increase of the tension strain the sample swells. But this phenomenon is characteristic for only auxetic materials.

Conclusions.

Conclusion 1. Three nonlinear models are used in the analysis. They describe the nonlinear Poynting-type effects in conditions of three used above universal deformations and the moderate strains. This agrees quantitatively with experimental observations of nonlinear dependences $\sigma \sim \varepsilon$ in auxetic materials for the moderate strains.

Conclusion 2. In the case of uniaxial and omniaxial tension, the nonlinear Murnaghan model describes a transition of the material from the class of conventional materials into the class of the auxetic materials. This occurs when the material is deformed to the level of moderate values of the longitudinal stretching. In other words, the shown experiments and proposed theoretical analysis testify that the standard sample in conditions of the mentioned universal deformation of uniaxial tension is deformed for small strains as if it is made of the conventional material (its cross-section is decreased) and with increasing the values of longitudinal stretching to the moderate values the sample cross-section starts to increase, what is the characteristic just for auxetic materials.

Р Е З Ю М Е. Запропоновано теоретичну спробу пояснення ауксетичності матеріалів за допомогою нелінійних моделей пружного деформування у широкому діапазоні значень деформацій від малого до помірного рівня. Отримано аналітичні вирази, які відповідають трьом видам універсальної деформації (простий зсув, односторонній розтяг, всесторонній розтяг) і рамках трьох відомих в класичній теорії нелінійної пружності моделей – двоконстантної неогукової моделі, триконстантної моделі Муні-Рівліна, п'ятиконстантної моделі Мурнагана. Найбільш цікавий новий результат полягає у тому, що як в показаних експериментах, так і в запропонованих теоретичних розрахунках зразок з пружного матеріалу деформується як традиційний матеріал для малих значень деформацій, тоді як при збільшенні деформацій до помірних він деформується як нетрадиційний (ауксетичний) матеріал.

References.

1. *Alderson A., Alderson K.L.* Auxetic materials // I. Mech. E.: J. Aerosp. Eng. – 2007. **– 221,** N 4. – P. 565 – 575.
2. *An introduction* to auxetic materials: an interview with Professor Andrew Alderson, AZoMaterials, August 29, 2015.
3. *Anurag C., Anvesh C.K., Katam S.* Auxetic materials // Int. J. for Research in Appl. Science and Eng. Technology. – 2015. – **3**, N 4, P. 1176 – 1183.
4. *Babaee S., Shim J., Weaver J.C., Chen E.R., Patel N., Bertoldi K.* 3D Soft Metamaterials with Negative Poisson's Ratio // Advanced Materials. – 2013. – 25(36). – P. 5044-5049.
5. *Cabras L., Brun M.* Auxetic two-dimensional lattices with Poisson's ratio arbitrarily close to –1 // Proc. Roy. Soc. London A. – 2014. – **470**, 20140538. – P. 1 – 23.
6. *Carneiro V.H., Meireles J., Puga H.* Auxetic Materials – A Review // Materials Science – Poland. – 2013**. – 31**, N 4. – P. 561 – 571.
7. *Cattani C., Rushchitsky J.J.* Wavelet and Wave Analysis as applied to Materials with Micro or Nano-Structures. - Singapore-London: World Scientific Publishing, 2007. – 466 p.
8. *Christensen R.M.* Mechanics of Composite Materials. – New York: J. Wiley and Sons, 1979. – 334 p.
9. *Dagdelen J., Montoya J., de Jong M., Persson K.* Computational prediction of new auxetic materials // Nature. Communications. - 2017. – **323**. – P. 1 – 8.
10. *Dudek K.K., Attard D., Caruana-Gauci R., Wojciechowski K.W., Grima J.N.* Unimode metamaterials exhibiting negative linear compressibility and negative thermal expansion // Smart Materials and Structures. – 2016**. – 25**, N 2, – 025009.
11. *Encyclopedia of Smart Materials*. In 2 vols. – New York: John Wiley and Sons, 2002. – 1193 p.
12. *Evans K.E.* Auxetic polymers: a new range of materials // Endeavour. – 1991. – **15**. – P. 170 – 174.
13. *Flügge S.* Encyclopedia of Physics. Vol. VIa/I. Mechanics of Solids. – Berlin: Springer-Verlag, 1973. – 468 p.
14. *Hauk V.* (ed) Structural and Residual Stress Analysis. – Amsterdam: Elsevier Science B.V., 1997 (evariant 2006). – 640 p.
15. *Hetnarski R.B., Ignaczak J.* The Mathematical Theory of Elasticity. – Boca Raton: CRC Press, 2011. – 804 p.
16. *Holzapfel G.A.* Nonlinear Solid Mechanics. A Continuum Approach for Engineering, – Chichester: Wiley, 2006. – 470 p.
17. *Gibson L.J., Ashby M.F., Schayer G.S., Robertson C.I.* The Mechanics of Two-Dimensional Cellular Materials // Proc. Roy. Soc. Lond. A. – 1982. – **382**. – P. 25 – 42.
18. *Gibson L.J., Ashby M.F.* The Mechanics of Three-Dimensional Cellular Materials // Proc. Roy. Soc. Lond. A 1982. **– 382**. P. 43 – 59.
19. *Greaves G.N.* Poisson's ratio over two centuries: challenging hypotheses // Notes and Records of Roy. Soc. – 2013. – **67**, N 1. – P. 37 – 58.
20. *Grima J.N.* Auxetic metamaterials // European Summer Campus, Strasbourg, France, 2010. – P. 1 – 13. http://www.auxetic.info
21. *Guz I.A., Rodger A.A., Rushchitsky J.J., Guz A.N.* Developing the mechanical models for nanomaterials // Phil. Trans. Roy. Soc. A: Math., Phys. and Eng. Sci. – 2008. – **365** (**1860**). – P. 3233 – 3239.
22. *Guz A.N.* Elastic Waves in Bodies with Initial (Residual) Stresses. In 2 vols. V.1. General Problems. V.2. Regularities of Propagation. – Kyiv: Naukova Dumka, 1986. – 376 p., 536 p. (In Russian).
23. *Karnaukhov V.G., Kirichok I.F., Kozlov V.I.* Thermomechanics of Inelastic Thin-Walled Structural Members with Piezoelectric Sensors and Actuators Under Harmonic Loading (Review) // Int. Appl. Mech. – 2017. – **53**, N1. – P. 6 – 58.
24. *Lakes R.S.* Foam Structures with a Negative Poisson's Ratio // Science. – 1987. – **235**. – P. 1038 – 1040.

25. *Liu Y.P., Hu H.* A review on auxetic structures and polymeric materials // Scientific Research and Essays. – 2010. **– 5**, N 10. – P. 1052 – 1063. http://www.auxetic.info.
26. *Lim T.C.* Auxetic Materials and Structures. – Berlin: Springer, 2015. – 564 p.
27. *Love A.E.H.* The Mathematical Theory of Elasticity. 4th edition. – New York: Dover Publications, 1944. – 643 p.
28. *Lurie A.I.* Theory of Elasticity, Series: Foundations of Engineering Mechanics. – Berlin: Springer, 2005. – 1050 p.
29. *Materials.* Special issue "Auxetics 2017 – 2018".
30. *Metamaterials Handbook* – Two Volume Slipcase Set. Capolino F. (ed). – Boca Raton: CRC Press, 2009. – 1736 p.
31. *Mooney M.* A theory of large elastic deformations // J. Appl. Phys. – 1940. – **11**, N9. – P. 582 – 592.
32. *Murnaghan F.D.* Finite Deformation in an Elastic Solid. – New York: John Wiley, 1951(1967). – 218 p.
33. *Ogden R.W.* Nonlinear elastic deformations. – New York: Dover, 1997. – 544 p.
34. *Pravoto Y.* Seeing auxetic materials from the mechanics point of view: A structural review on the negative Poisson's ratio // Computational Materials Sci. – 2012. – **58**. P. 140 **–** 153.
35. *Rivlin R.S.* Large elastic deformations of isotropic materials. IV. Further development of general theory // Phil. Trans. Roy. Sci. London, Ser. A. Math. and Phys. Sci. – 1948. – **241** (**835**). – P. 379 – 397.
36. *Rushchitsky J.J.* On Universal Deformations in an Analysis of the Nonlinear Signorini Theory for Hyperelastic Medium // Int. Appl. Mech. – 2007. – **43**, N 12. – P. 1347 – 1350.
37. *Rushchitsky J.J.* Nonlinear elastic waves in materials, Series: Foundations of engineering mechanics. – Heidelberg: Springer, 2014. – 454 p.
38. *Rushchitsky J.J.* On the Constraints for Displacement Gradients in Elastic Materials // Int. Appl. Mech. – 2016. – **52**, N 3. – P. 339 – 352.
39. *Saccomandi G.*, *Ciambella J.* A continuum hyperelastic model for auxetic materials // Proc. Roy. Soc. A. – 2014. – **470**. – P.1 – 14.
40. *Scarpa F.* Auxetics: From Foams to Composites and Beyond (presentation to Sheffield May 2011) http://www.bris.ac.uk/composites.
41. *Scarpa F., Pastorino P., Garelli A., Patsias S, Ruzzene M.* Auxetic compliant flexible PU foams: static and dynamic properties // Physica Status Solidi B – 2005. – **242**, N 3. – P. 681 – 694.
42. *Truesdell C.* A first course in rational continuum mechanics. Baltimore: The John Hopkins University, 1972; 2nd edition, New York: Academic Press, 1991. – 352 p.
43. *Uzun M.* Mechanical properties of auxetic and conventional polyprophylene random short fibre rein – forced composites // Fibres and Textiles in Eastern Europe. – 2012. – **20,** N 5(94). – P. 70 – 74.
44. *Wojciechowski K.W.* Constant thermodynamic tension Monte – Carlo studies of elastic properties of a two – dimensional system of hard cyclic hexameters // Molecular Physics. – 1987**. – 61**. – P. 1247 – 1258.
45. *Yao Y.T., Uzun M., Patel I.* Working of auxetic nanomaterials // J. Achievements in Materials and Manufacturing Engineering. – 2011. – **49**, N 2. – P. 585 – 594.
46. *Zhuk Ya.A., Guz I.A.* Active damping of the forced vibration of a hinged beam with piezoelectric layer, geometrical and physical nonlinearities taken into account // Int. App. Mech. – 2009. – **45**, N 1 – P. 94 – 108.

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