a) the "SPRINT" program complex gives stable results for different grid densities and can be used for three-dimensional calculation of the stress–strain state of ribbed GTE casings;

b) to obtain results accurate to within about 10%, the minimum grid density should be about 1/3 of the circumference in the annular direction and about 1/10 of the diameter of the shell in the axial direction;

c) use of the method developed significantly reduces the amount of work necessary at the design stage and produces results with the required degree of accuracy.

USE OF THE FINITE ELEMENTS METHOD TO CALCULATE THE STRENGTH OF CONICAL SHELLS WITH NOTCHES

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There are serious mathematical difficulties in determining the stress–strain state of conical shells with notches, but these obstacles can be surmounted by using numerical methods. Mainly shells with small notches in the case of uniform loading have been examined in the well-known solutions of this problem. The stress-strain state has been assumed momentless when the notches are large.

The present study determines the moment stress–strain state of conical shells with notches of arbitrary size and form under complex loading. The problem is solved by the finite elements method in displacements. As the finite element we chose a curvilinear arbitrary tetragonal element of natural curvature (Fig. 1) with 24 degrees of freedom.

We are examining a conical shell of length L and thickness h with a cone angle γ and radius R_0 at $x = x_0$. The shell is weakened by a notch with a radius $r = r(\varphi)$ and is loaded by a system of surface loads $q_i(x, y)$, local forces P_{lI}, and moments $M_{lI}(y)$, as well as lineal contour forces $P_{kI}(y)$ and moments $M_{kI}(y)$. The indices I = 1, 2, 3 correspond to the directions of the axes x, y, z (Fig. 1).

We subdivide the shell into n parts along the boundary of the notch and into m parts along a line connecting the contour of the hole with the outer contour of the shell (Fig. 1). The shell is thus replaced by a set of $m \times n$ arbitrary tetragonal curvilinear elements of natural curvature.

We approximate the displacements of points of a finite element with the polynomials

$$\left.\begin{aligned}
u &= \alpha_1\xi\eta + \alpha_2\xi + \alpha_3\eta + \alpha_4;\\
v &= \alpha_5\xi\eta + \alpha_6\xi + \alpha_7\eta + \alpha_8;\\
w &= \alpha_9\xi^3\eta^3 + \alpha_{10}\xi^3\eta^2 + \alpha_{11}\xi^3\eta + \alpha_{12}\xi^3 +\\
&+ \alpha_{13}\xi^2\eta^3 + \alpha_{14}\xi^2\eta^2 + \alpha_{15}\xi^2\eta + \alpha_{16}\xi^2 +\\
&+ \alpha_{17}\xi\eta^3 + \alpha_{18}\xi\eta^2 + \alpha_{19}\xi\eta + \alpha_{20}\xi + \alpha_{21}\eta^3 +\\
&+ \alpha_{22}\eta^2 + \alpha_{23}\eta + \alpha_{24}.
\end{aligned}\right\} \tag{1}$$

We write (1) as follows in matrix form:

$$\mathbf{u} = \mathbf{P}\alpha, \tag{2}$$

where $\mathbf{P}$ is a coupling matrix of order 3×24; $\mathbf{u} = u, v, w$ is the column vector of the displacements of points of the element; $\alpha = \{\alpha_1, \ldots, \alpha_{24}\}$ is the vector of the unknown coefficients of the polynomials.

Using the solutions in [1], for the displacements, strains, and forces of a finite element, we obtain the expressions

$$\mathbf{u} = \mathbf{P}_1\bar{\mathbf{u}}; \quad \mathbf{P}_1 = \mathbf{P}\mathbf{B}^{-1}; \quad \varepsilon = \mathbf{A}\mathbf{u}; \quad \mathbf{T} = \mathbf{D}\varepsilon, \tag{3}$$

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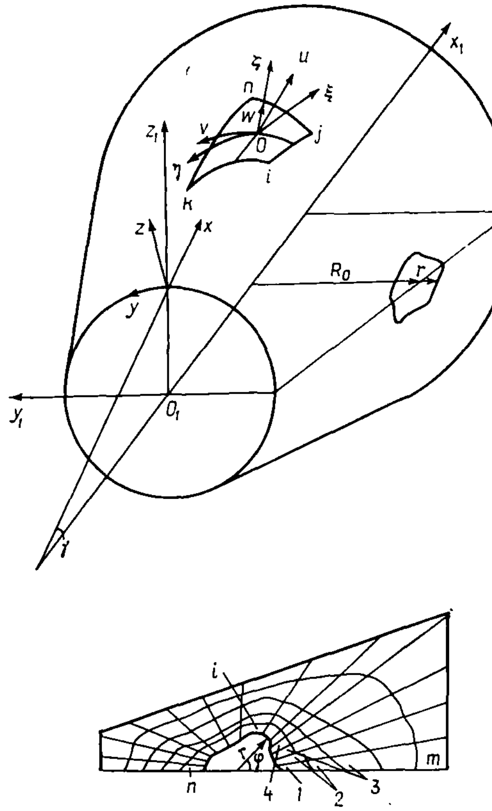

Fig. 1. Conical shell and its finite element.

where

$$\boldsymbol{\varepsilon} = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \chi_1, \chi_2, \chi_3\},$$
$$\mathbf{T} = \{T_1, T_2, T_3, M_1, M_2 M_3\}$$

are the vectors of the strains and forces of the finite element;

$$\bar{\mathbf{u}} = \{u_\mathrm{I}, v_\mathrm{I}, w_\mathrm{I}, w_{\xi\mathrm{I}}, w_{\eta\mathrm{I}}, w_{\xi\eta\mathrm{I}}, u_j, \ldots, w_{\xi\eta j}, \quad u_k, \ldots, w_{\xi\eta k}, \ldots, w_{\xi\eta n}\}$$

are the vectors of the nodal displacements of the finite element; **B** is a matrix of order 24 × 24, the elements of which have the form

$$\left.\begin{aligned}
& b_{1j} = p_{1j}, \ b_{2j} = p_{2j}, \quad b_{3j} = p_{3j}, \quad b_{4j} = (p_{3j})_\xi, \\
& b_{5j} = (p_{3j})_\eta, \quad b_{6j} = (p_{3j})_{\xi\eta} \quad \text{at} \quad \xi = -1; \\
& \eta = -1, \quad b_{7j} = p_{1j}, \quad b_{8j} = p_{2j}, \quad b_{9j} = p_{3j}, \\
& b_{10j} = (p_{3j})_\xi, \quad b_{11j} = (p_{3j})_\eta, \quad b_{12j} = (p_{3j})_{\xi\eta} \\
& \ \text{at } \xi = 1, \quad \eta = -1; \quad b_{13j} = p_{1j}, \quad b_{14j} = p_{2j}, \\
& b_{15j} = p_{3j}, \quad b_{16j} = (p_{3j})_\xi, \quad b_{17j} = (p_{3j})_\eta, \\
& b_{18j} = (p_{3j})_{\xi\eta} \quad \text{at} \quad \xi = -1, \quad \eta = 1; \\
& b_{19j} = (p_{1j}), \quad b_{20j} = p_{2j}, \quad b_{21j} = p_{3j}, \\
& b_{22j} = (p_{3j})_\xi, \quad b_{23j} = (p_{3j})_\eta, \quad b_{24j} = (p_{3j})_{\xi\eta}, \\
& \ \text{at } \xi = 1, \quad \eta = 1
\end{aligned}\right\} \qquad (4)$$

(p_{Ij} are elements of the matrix **P**; ξ and η in the indices denote differentiation).

In contrast to the matrix presented in [1], the matrix **A** has the form

$$
\mathbf{A}=\begin{vmatrix}
(\ )_x & 0 & 0 \\
\frac{1}{\bar{x}}(\ ) & (\ )_y & k_2(\ ) \\
(\ )_y & (\ )_x-\frac{1}{\bar{x}}(\ ) & 0 \\
0 & 0 & -(\ )_{xx} \\
0 & k_2(\ )_y & -\frac{1}{\bar{x}}(\ )_x-(\ )_{yy} \\
0 & 2k_2(\ )_x-2\frac{k_2}{\bar{x}}(\ ) & -2(\ )_{xy}+\frac{2}{\bar{x}}(\ )_y
\end{vmatrix},
$$

$$
k_2=\frac{1}{\bar{x}\,\mathrm{tg}\gamma}, \tag{5}
$$

where $\bar{x}$ is the coordinate of a point of the element from the pole of the cone; x and y in the indices denote differentiation.

We write the relation between the x, y coordinates and the curvilinear coordinates ξ and η in the following form (Fig. 1):

$$
x=x(\xi,\eta);\qquad y=y(\xi,\eta), \tag{6}
$$

where in accordance with the data in [2]

$$
\left.\begin{aligned}
&x(\xi,\eta)=f_1(\xi,\eta)\,x_1+f_2(\xi,\eta)\,x_2+f_3(\xi,\eta)\,x_3+\\
&+f_4(\xi,\eta)\,x_4;\qquad y(\xi,\eta)=f_1(\xi,\eta)\,y_1+\\
&+f_2(\xi,\eta)\,y_2+f_3(\xi,\eta)\,y_3+f_4(\xi,\eta)\,y_4;\\
&f_1(\xi,\eta)=\frac{1}{4}(1-\xi)(1-\eta);\\
&f_2(\xi,\eta)=\frac{1}{4}(1+\xi)(1-\eta);\\
&f_3(\xi,\eta)=\frac{1}{4}(1-\xi)(1+\eta);\\
&f_4(\xi,\eta)=\frac{1}{4}(1+\xi)(1+\eta)
\end{aligned}\right\} \tag{7}
$$

(x_I and y_I are nodal coordinates).

We represent arbitrary functions $f(\xi,\eta)$ with respect to the coordinates ξ and η as follows

$$
\mathbf{f}'=\mathbf{G}\mathbf{F}';\qquad \mathbf{f}''=\mathbf{H}\mathbf{F}'', \tag{8}
$$

where

$$
\left.\begin{aligned}
&\mathbf{f}'=\{f_\xi,f_\eta\};\qquad \mathbf{f}''=\{f_{\xi\xi},f_{\xi\eta},f_{\eta\eta}\};\\
&\mathbf{F}'=\{f_x,f_y\},\qquad \mathbf{F}''=\{f_{xx},f_{xy},f_{yy}\};\\
&\mathbf{G}=\begin{vmatrix} x_\xi y_\xi \\ x_\eta y_\xi \end{vmatrix};\ \mathbf{H}=\begin{vmatrix} x_\xi^2 & 2x_\xi y_\xi & y_\xi^2 \\ x_\xi x_\eta & x_\xi y_\eta+y_\xi x_\eta & y_\xi y_\eta \\ x_\eta^2 & 2x_\eta y_\eta & y_\eta^2 \end{vmatrix}.
\end{aligned}\right\} \tag{9}
$$

From (8) we find

$$
\mathbf{F}'=\mathbf{G}^{-1}\mathbf{f}';\qquad \mathbf{F}''=\mathbf{H}^{-1}\mathbf{f}''. \tag{10}
$$

The elements of the matrices $\mathbf{G}^{-1}$ and $\mathbf{H}^{-1}$ are equal to

$$
\left.\begin{aligned}
&g_{11}=y_\eta/\Delta;\quad g_{12}=-y_\xi/\Delta;\quad g_{21}=-x_\eta/\Delta;\\
&g_{22}=x_\xi/\Delta;\ \ h_{11}=y_\eta^2/\Delta_1;\ \ h_{12}=-2y_\xi y_\eta/\Delta_1;\\
&h_{13}=y_\xi^2/\Delta_1;\ \ h_{23}=-x_\xi y_\xi/\Delta_1;\ \ h_{31}=x_\eta^2/\Delta_1;\\
&h_{32}=-2x_\xi x_\eta/\Delta_1;\quad h_{33}=-x_\xi^2/\Delta_1;\quad h_{21}=\\
&=-y_\eta x_\eta/\Delta_1;\ \ h_{22}=(x_\xi y_\eta+y_\xi x_\eta)/\Delta_1;\ \ \Delta_1=\Delta^2;\ \ \Delta=x_\xi y_\eta-x_\eta y_\xi.
\end{aligned}\right\} \tag{11}
$$

We obtain the deviatives x_ξ and y_ξ from (6) with allowance for (7)

$$\left.\begin{aligned} x_\xi &= a_1x_1 + b_1x_2 + c_1x_3 + d_1x_4; \\ y_\xi &= a_1y_1 + b_1y_2 + c_1y_3 + d_1y_4; \\ x_\eta &= a_2x_1 + b_2x_2 + c_2x_3 + d_2x_4; \\ y_\eta &= a_2y_1 + b_2y_2 + c_2y_3 + d_2y_4, \end{aligned}\right\} \quad (12)$$

where

$$\left.\begin{aligned} a_1 &= -\frac{1}{4}(1-\eta); \quad b_1 = \frac{1}{4}(1-\eta); \\ c_1 &= -\frac{1}{4}(1+\eta); \quad d_1 = \frac{1}{4}(1+\eta); \\ a_2 &= -\frac{1}{4}(1-\xi); \quad b_2 = -\frac{1}{4}(1+\xi); \\ c_2 &= \frac{1}{4}(1-\xi); \quad d_2 = \frac{1}{4}(1+\xi). \end{aligned}\right\} \quad (13)$$

Using (10), we obtain the nontrivial elements of the matrix **A** in the form:

$$\left.\begin{aligned} & a_{11} = g_{11}(\ )_\xi + g_{12}(\ )_\eta; \quad a_{21} = \frac{1}{x}(\ ); \\ & a_{22} = g_{21}(\ )_\xi + g_{22}(\ )_\eta; \quad a_{23} = k_2(\ ); \\ & a_{31} = g_{21}(\ )_\xi + g_{22}(\ )_\eta; \quad a_{33} = g_{11}(\ )_\xi + \\ & + g_{12}(\ )_\eta - \frac{1}{x}(\ ); \quad a_{43} = -(h_{11}(\ )_{\xi\xi} + \\ & + h_{12}(\ )_{\xi\eta} + h_{13}(\ )_{\eta\eta}); \quad a_{52} = k_2(g_{21}(\ )_\xi + \\ & + g_{22}(\ )_\eta); \quad a_{53} = -(h_{31}(\ )_{\xi\xi} + h_{32}(\ )_{\xi\eta} + \\ & + h_{33}(\ )_{\eta\eta}) - \frac{1}{x}(g_{11}(\ )_\xi + g_{12}(\ )_\eta); \quad a_{63} = \\ & = -2(h_{21}(\ )_{\xi\xi} + h_{22}(\ )_{\xi\eta} + h_{23}(\ )_{\eta\eta} + \\ & + \frac{2}{x}(g_{21}(\ )_\xi + g_{22}(\ )_\eta); \quad a_{62} = 2k_2\Big(g_{11}(\ )_\xi + \\ & + g_{12}(\ )_\eta - \frac{1}{x}(\ )\Big). \end{aligned}\right\} \quad (14)$$

We write the expressions of strain energy and the work of the external forces for the finite element:

$$W = \frac{1}{2}\iint_s \mathbf{T}^T \boldsymbol{\varepsilon} ds; \qquad U = \iint_s \mathbf{q}^T \mathbf{u} ds + \int_l \bar{\mathbf{R}}_k^T \mathbf{u}_k dl + \bar{\mathbf{R}}_l^T \bar{\mathbf{u}}_l, \quad (15)$$

where $\mathbf{q} = \{q_1, q_2, q_3\}$ is the vector of the external surface load; $\bar{\mathbf{R}}_k = \{P_{k1}, P_{k2}, P_{k3}, M_{k1}, M_{k2}, M_{k3},\}$, $\bar{\mathbf{R}}_l = \{P_{l1}, P_{l2}, P_{l3}, M_{l1}, M_{l2}, M_{l3}\}$ are vectors of the contour and local forces acting on the finite element; $\mathbf{u}_k = \{u, v, w, w_\xi, w_\eta, w_{\xi\eta}\}$; $\bar{\mathbf{u}}_l = \{u_l, v_l, w_l, w_{\xi l}, w_{\eta l}, w_{\xi\eta l}\}$.

Introducing the new variables

$$ds = |\det \mathbf{G}|\, d\xi d\eta; \qquad dl = |\det \mathbf{G}|\, dl, \quad (16)$$

we obtain an expression for the total potential energy of the finite element

$$\Pi_i = W - U = \frac{1}{2}\iint_s \mathbf{T}^T \boldsymbol{\varepsilon} |\det \mathbf{G}|\, d\xi d\eta - \iint_s \mathbf{q}^T \mathbf{u} |\det \mathbf{G}|\, d\xi d\eta - \int_l \bar{\mathbf{R}}_k^T \mathbf{u}_k |\det \mathbf{G}|\, dl - \bar{\mathbf{R}}_l^T \bar{\mathbf{u}}_l. \quad (17)$$

With allowance for (3), we find

$$\Pi_i = \frac{1}{2}\bar{\mathbf{u}}^T (\mathbf{B}^{-1})^T \iint_s \mathbf{P}^T \mathbf{A}^T \mathbf{D} \mathbf{A} \mathbf{P} |\det[G]|\, d\xi d\eta \mathbf{B}^{-1} \bar{\mathbf{u}} - \iint_s \mathbf{q}^T \mathbf{P} |\det \mathbf{G}|\, d\xi d\eta \mathbf{B}^{-1} \bar{\mathbf{u}} - \int_l \bar{\mathbf{R}}_k^T \mathbf{P}_k |\det \mathbf{G}|\, dl \mathbf{B}^{-1} \bar{\mathbf{u}} -$$

$$- \bar{\mathbf{R}}_l^T \bar{\mathbf{u}}_l = \frac{1}{2}\bar{\mathbf{u}}^T \mathbf{K} \bar{\mathbf{u}} - \mathbf{Q}\bar{\mathbf{u}} - \mathbf{Q}_k \bar{\mathbf{u}} - \mathbf{Q}_l \bar{\mathbf{u}}, \quad (18)$$

where

$$
\left.\begin{array}{l}
\mathbf{K} = (B^{-1})^T \iint_s \mathbf{P}^T\mathbf{A}^T\mathbf{DAP}\,|\det\mathbf{G}|\,d\xi d\eta\mathbf{B}^{-1};\\
\mathbf{Q} = \iint_s \mathbf{q}^T\mathbf{P}\,|\det\mathbf{G}|\,d\xi d\eta\mathbf{B}^{-1};\quad \mathbf{Q}_k = \int_l \overline{\mathbf{R}}_k^T\mathbf{P}_k \times\\
\times\,|\det\mathbf{G}|\,dl\mathbf{B}^{-1};\quad (p_k)_{1j} = p_{1j};\quad (p_k)_{2j} = p_{2j};\\
(p_k)_{3j} = p_{3j};\quad (p_k)_{4j} = [g_{11}(\ )_\xi + g_{12}(\ )_\eta]\,p_{3j};\\
(p_k)_{5j} = [g_{21}(\ )_\xi + g_{22}(\ )_\eta]\,p_{3j};\qquad (p_k)_{6j} =\\
= [h_{21}(\ )_{\xi\xi} + h_{22}(\ )_{\xi\eta} + h_{23}(\ )_{\eta\eta}]\,p_{3j},\\
\qquad\qquad j = 1, \ldots, 24
\end{array}\right\} \tag{19}
$$

($(p_k)_{Ij}$ are elements of the matrix $\mathbf{P}_k$).

The nodal displacements $\tilde{\mathbf{u}}$ in the system of coordinates ξ and η are connected with the nodal displacements $\bar{\mathbf{u}}$ in the system of coordinates x and y by the relation

$$\tilde{\mathbf{u}} = \mathbf{C}\bar{\mathbf{u}}, \tag{20}$$

where

$$
\mathbf{C} = \begin{vmatrix} \lambda_I & 0 & 0 & 0\\ 0 & \lambda_j & 0 & 0\\ 0 & 0 & \lambda_k & 0\\ 0 & 0 & 0 & \lambda_n \end{vmatrix};\quad \lambda_I = \begin{vmatrix} 1 & 0 & 0 & & & 0\\ 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \\ & & & x_\xi^I y_\xi^I & & \\ & & & x_\eta^I y_\eta^I & & \\ 0 & & & & & x_\xi^I y_\eta^I \end{vmatrix}
$$

(x_ξ^I, y_ξ^I are derivatives of x and y with respect to ξ at point I of the finite element).

By means of (20) we obtain

$$\Pi_l = \frac{1}{2}\bar{\mathbf{u}}^I\mathbf{K}^*\bar{\mathbf{u}} - \mathbf{Q}^*\bar{\mathbf{u}}\,\mathbf{Q}_k^*\bar{\mathbf{u}} - \mathbf{Q}_l^*\bar{\mathbf{u}}, \tag{21}$$

where

$$\mathbf{K}^* = \mathbf{C}^T\mathbf{KC};\quad \mathbf{Q}^* = \mathbf{QC};\quad \mathbf{Q}_k^* = \mathbf{Q}_k\mathbf{C};\quad \mathbf{Q}_l^* = \mathbf{Q}_l\mathbf{C}. \tag{22}$$

Summing the potential energies of individual elements, we determine the potential energy of the shell. By varying the potential energy of the shell with respect to the nodal displacements, in accordance with the principle of possible displacements ($\delta\Pi = 0$), as in [1] we obtain a system of linear algebraic equations of equilibrium to determine the displacements:

$$\overline{\mathbf{K}}\mathbf{u}' = \overline{\mathbf{Q}}, \tag{23}$$

where $\overline{\mathbf{K}}$ is the elastic stiffness matrix of the shell; $\mathbf{u}'$ and $\overline{\mathbf{Q}}$ are vectors of the nodal displacements and nodal forces of the shell.

The matrix $\overline{\mathbf{K}}$ has a band structure and can be obtained by summing the elements of matrices $\mathbf{K}^*$ with the use of the index matrix in [3]. We obtain the vector $\overline{\mathbf{Q}}$ by summation of the vectors $\mathbf{Q}' = \mathbf{Q}^* + \mathbf{Q}_k^* + \mathbf{Q}_l^*$. System (23) is solved by the method of expanding the matrix into two triangular matrices. Having solved it, we find the vector $\mathbf{u}'$ and we use Eq. (3) to find all of the components of the stress–strain state of the shell.

Example. To check the effectiveness of the finite element, we studied the stress–strain state of a circular conical shell (length L = 20 cm, radius R_0 = 13.4 cm, thickness h = 0.2 cm, cone angle $\gamma/2$ = 0.647) weakened by a circular notch with a radius r = 2 cm and loaded by uniform longitudinal tensile forces N.

Figure 2 shows the change in the stress concentration factors $k_1 = \sigma_{\Theta C}/\sigma_p$, $k_2 = \sigma_{\Theta B}/\sigma_p$ ($\sigma_{\Theta C}$, $\sigma_{\Theta B}$ are the maximum stresses in the middle surface of the shell and the bending stresses at the contour of the hole on the outside surface of the shell, $\sigma_p = N/2\pi R_0 h$) on the contour of the hole. The dashed line shows the results of the calculation in [4]. It can be seen

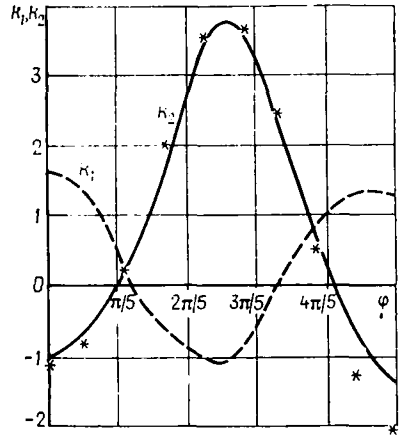

Fig. 2. Bending and membrane stress concentration factors at the contour of the hole.

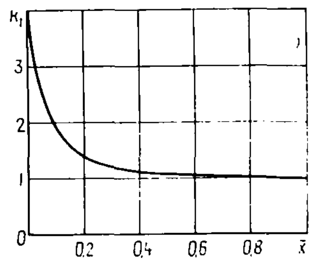

Fig. 3

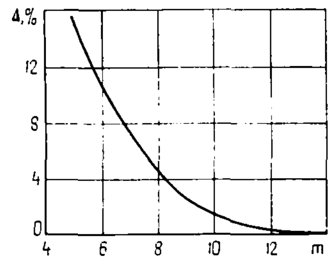

Fig. 4

Fig. 3. Dependence of the membrane stress concentration factor on the coordinate $\bar{x}$.

Fig. 4. Dependence of the error of the stress calculation on the number m.

that there is satisfactory agreement. The concentration of the bending stresses is comparable to the stress concentration in the middle surface of the shell.

Figure 3 shows the decay of the maximum stress concentration in the middle surface of the shell from the contour of the hole ($\bar{x} = y/5r$, with the origin of the coordinate y located at the hole contour).

Figure 4 shows the dependence of the error of the stress calculation on the number of finite elements m for $\gamma/2 = 0.12$. There is sufficiently good convergence with respect to the number m.

The time of solution of the problem on a BÉSM-6 computer was about 40 min for m = 13, n = 14 with a number of unknowns N = 1260 and a half-width of the shell stiffness matrix B = 96. Thus, the algorithm developed makes it possible to efficiently determine the stress–strain state of conical shells with notches.

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