INCREASING THE LOAD-CARRYING CAPACITY OF THREE-LAYER PLATES AND SHALLOW SHELLS

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This article examines three-layer plates rectangular in plan and shallow shells of symmetrical construction subjected to bending by a transverse distributed load. The load-carrying capacity of the plates and shells is evaluated by means of a generalized strength criterion. We solve the problem of finding the optimum law of variation in the thickness of the external layers of these structure. The resulting thickness distribution keeps the structures as far as possible from the limiting stress state.

Formulation of the Problem. As the theoretical model we take a three-layer packet of symmetrical construction with a lightweight filler. We assume that the broken line hypothesis is valid for the entire packet.

Let $\omega = (\omega_i)(i=\overline{1, 5})$ be a vector function, the components of which represent displacements of points of the middle surface of the bearing (external and internal) layers of the shell (Fig. 1). Here $\omega_1 = u_1$, $\omega_2 = v_1$, $\omega_3 = u_2$, $\omega_4 = v_2$. We assume that the deflection in the direction of the axis is the same for all layers and $\omega_5 = w$. In the case of the plates, considering the symmetrical construction of the packet, we take $\omega_1 = -\omega_3$ and $\omega_2 = -\omega_4$ [1].

We will examine a symmetrical bilinear form:

$$a_t(\omega', \omega'') = \int_\Omega \Big\{ \frac{E_1 t}{1-\nu_1\nu_2}\Big[\varepsilon_1'\varepsilon_1'' + \frac{\nu_2}{\nu_1}\varepsilon_2'\varepsilon_2'' + \nu_2(\varepsilon_1'\varepsilon_2'' + \varepsilon_1''\varepsilon_2') + \varepsilon_3'\varepsilon_3'' + \frac{\nu_2}{\nu_1}\varepsilon_4'\varepsilon_4'' + \nu_2(\varepsilon_3'\varepsilon_4'' + \varepsilon_3''\varepsilon_4')\Big] + G_{12}t(\varepsilon_5'\varepsilon_5'' + \varepsilon_6'\varepsilon_6'') + \frac{E_1 t^3}{6(1-\nu_1\nu_2)}\Big[\varepsilon_7'\varepsilon_7'' + \frac{\nu_2}{\nu_1}\varepsilon_8'\varepsilon_8'' + \nu_2(\varepsilon_7'\varepsilon_8'' + \varepsilon_7''\varepsilon_8')\Big] + G_{12}t^3\varepsilon_9'\varepsilon_9'' + (G_{31}\varepsilon_{10}'\varepsilon_{10}'' + G_{32}\varepsilon_{11}'\varepsilon_{11}'')h\Big\} d\Omega.$$

Here, ε_i' and ε_i'' are strain components generated by the vectors ω' and ω'' [1]; $t(x_1, x_2)$, $h(x_1, x_2)$ are functions of point $(x_1, x_2) \in \Omega$; $\Omega = [a_1, b_1] \times [a_2, b_2]$ is a projection of the middle surface of the shell; E_1 and E_2 are Young's moduli; G_{12} is the shear modulus; ν_1 and ν_2 are Poisson's ratios of the orthotropic bearing layers ($E_1\nu_2 = E_2\nu_1$) G_{31} and G_{32} are the shear moduli of the filler. Here $\frac{1}{2}t(x_1, x_2) + h(x_1, x_2) = \text{const}$.

We form the energy space $H\Omega$ [2], closing in the norm $\|\omega\|_{H(\Omega)} = a_t(\omega, \omega)^{\frac{1}{2}}$ a set of smooth functions ω satisfying conditions of shell support such that $\omega = 0$ follows from the equation $a_t(\omega, \omega) = 0$.

As in [2], we will call the generalized solution of the problem of the stress–strain state of a three-layer shell of variable thickness the function $\omega(x_1, x_2) \in H(\Omega)$, for which

$$a_t(\omega, \omega') = \int_\Omega g\omega' d\Omega, \quad \forall\omega' \in H(\Omega), \tag{1}$$

where g is the intensity of the external load.

Considering the function $t(x_1, x_2)$ as a control, we determine the permissible set of controls:

$$U_d = \Big\{t \mid t \in W_p^1(\Omega), \quad \|t\|_{W^1_{p(\Omega)}} \leqslant C_1, \quad p \geqslant 2, \quad t_1 \leqslant t \leqslant t_2, \quad \varphi(t) = \int_\Omega t d\Omega \leqslant C_2, \quad \Phi_i(t, \omega(t)) \leqslant 1, \ i = 1, \ldots, m\Big\}, \tag{2}$$

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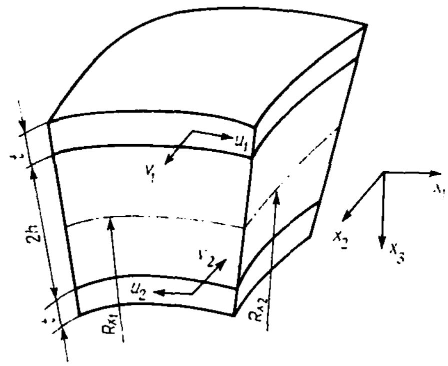

Fig. 1. Element of a three-layer shell.

where C_1, C_2, t_1, and t_2 are positive constants; $W_p^1(\Omega)$ is a Sobolev space; the value of the functional $\varphi(t)$ is equal to the volume of the material of the bearing part of the shell; $\Phi_i(t, \omega(t))$ is a functional representing the chosen strength criterion and characterizing the stress state of the structure in the neighborhood of the point $(x_j(i))$ $(j = \overline{1, 3})$. It is determined through the solution $\omega(t)$ of problem (1). Here, we assume that the set U_d is not empty.

We introduce the object function

$$\Phi(t) = \sup_i \Phi_i(t, \omega(t)).$$

The problem of finding a structure of maximum strength is formulated in accordance with the formulation of an infinite problem of optimum control: find the function $t^*(x_1, x_2)$ for which

$$t^* \in U_d, \quad \Phi(t^*) = \inf_{t \in U_d} \Phi(t). \tag{3}$$

The physical meaning of problem (3) consists of selecting the optimum law of change in the thickness of the bearing layers of the structure in question. With such a law, the stress state at the point liable to the greatest stress (accounted for in the form of the safety factor) will be as mild as possible. In other words, the resulting thickness distribution will correspond to a shell (plate) which is maximally unloaded in accordance with the chosen strength criteria. This in turn makes it possible to additionally load the structure and thereby increase its load-carrying capacity.

As was done in [3], the existence of a solution to problem (3) can be proven.

Numerical Realization of the Direct and Optimum Problems. Let the following subdivisions be prescribed on the segments $[a_k, b_k]$ $(k = 1, 2)$

$$\Delta_k = a_k = x_k^0 < x_k^1 <, \ldots, < x_k^{N_k} = b_k.$$

We will designate $h_k^i = x_k^i - x_k^{i-1}$, where $1 \leqslant i \leqslant N_k$; $h_k = \max_i h_k^i$; $\alpha_k^i = \sum_{l=1}^{i} h_k^l$. Assuming $\alpha_k^0 = 0$, for $0 \leqslant i \leqslant N_k$ we note the following obvious relation:

$$x_k^i = a_k + \alpha_k^i. \tag{4}$$

We introduce into the region $\Omega = [a_1, b_1] \times [a_2, b_2]$ a grid $\Delta = \Delta_1 \times \Delta_2$ dividing Ω into rectangular cells, from among which we pick out

$$\Omega_{ij} = \{(x_1, x_2) \mid x_1 \in [a_1, b_1] \cap [x_1^{i-1}, x_1^{i+1}],$$
$$i = 0, \ldots, N_1;$$
$$x_2 \in [a_2, b_2] \cap [x_2^{j-1}, x_2^{j+1}], \quad j = 0, \ldots, N_2\}.$$

In the region Ω, considering (4), we assign the fundamental splines:

$$T_{ij} = \begin{cases} \left(1 - \dfrac{x_1 - x_1^i}{\delta_1 h_1^i + \delta_2 h_1^{i+1}}\right)\left(1 - \dfrac{x_2 - x_2^j}{\delta_1 h_2^j + \delta_2 h_2^{j+1}}\right) \\ \text{at} \quad (x_1, x_2) \in \Omega_{ij}; \\ 0 \ \text{at} \quad (x_1, x_2) \notin \Omega_{ij}, \end{cases} \tag{5}$$

where $\delta_1 = \left[\frac{x_k - x_k^i}{h_k}\right]$; $\delta_2 = \delta_1 + 1$; $[a]$ is the integral part of the number a.

We use functions (5) to form an N-dimensional space of controls $H_N(\Omega) \subset W_2^1(\Omega)$:

$$H_N(\Omega) = \left\{ t_N(x_1, x_2) \,|\, t_N(x_1, x_2) = \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} t_{ij} T_{ij}(x_1, x_2), \right. \qquad (6)$$

$$\left. N = (N_1 + 1)(N_2 + 1) \right\}.$$

The geometric sense of the coefficients t_{ij} is given by the relation

$$t_{ij} = t_N(x_1^i, x_2^j). \qquad (7)$$

Regarding $H_N(\Omega)$ as a space of controls, we can approximate ∞-dimensional optimization problem (3) by a sequence of finite-dimensional problems: for each fixed number N we find the function

$$t_N^* \in U_d^N, \quad \Phi(t_N^*) = \inf_{t_N \in U_d^N} \Phi(t_N). \qquad (8)$$

Here, the allowable set is represented in the form

$$U_d^N = \left\{ t_N(x_1, x_2) \,|\, t_N \in H_N(\Omega), \ \|t_N\|_{W_p^1(\Omega)} \leqslant C_1, \ p \geqslant 2, \right.$$
$$t_1 \leqslant t_{ij} \leqslant t_2, \quad \Phi_k(t_N, \omega^N(t_N)) \leqslant 1,$$
$$\left. \varphi(t_N) = \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} t_{ij} \int_\Omega T_{ij}(x_1, x_2) d\Omega \leqslant C_2, \ k = 1, \ldots, m \right\}, \qquad (9)$$

where $\omega^N(t_N)$ is the approximate solution of problem (1) found by the Ritz method, and it can in turn be approximated by means of bicubic polynomial splines [4].

It should be noted that the functions $t_N(x_1, x_2)$ from (6) are ambiguously determined by specification of the coefficients t_{ij}, which can be considered components of an N-dimensional vector. This fact allowed us to examine problem (8) as a nonlinear programming problem, which was solved by the methods of penalty functions and coordinate descent [5].

In the problems examined below, the load-carrying capacity of the plates was evaluated by means of a generalized strength criterion [6] which can be represented in the form of a functional

$$\Phi_k(t_N) = A_1\sigma_{11} + A_2\sigma_{22} + \frac{1}{2}[A_3\sigma_{11}^2 + A_4\sigma_{22}^2 + A_5\sigma_{11}\sigma_{22} + A_6\sigma_{12}^2]^{\frac{1}{2}}, \qquad (10)$$

where A_i $(i = \overline{1, 6})$ are constants expressed through the physical constants of the material and characterizing its strength in compression, tension, and shear.

Considering the nonlinearity of the stresses σ_{pq} in (10) with respect to t, it can be shown that the greatest value of $\Phi_k(t)$ is reached on one of the surfaces bounding the bearing layers of the plate: $x_3 = -t-h$, $x_3 = -h$, $x_3 = h$, $x_3 = h + t$.

In approximating the thickness of the plate $t(x_1, x_2)$ by means of the functions $T_{ij}(x_1, x_2)$ from (5) the segments $[a_k, b_k]$ $(k = 1, 2)$ were subdivided into an odd number of intervals N_k. Here, the nodes of the subdivision (4) were chosen so that the lengths of the corresponding intervals satisfied the relations

$$h_k^{2i} = h_{k,0}, \quad h_k^{2i-1} \gg h_{k,0}, \quad i = 1, \ldots, n_k; \qquad (11)$$
$$n_k = \frac{N_k + 1}{2}.$$

In the calculations we assumed that the constant $h_{k,0}$ is equal to

$$h_{k,0} = \frac{1}{20} \min_{1 \leqslant i \leqslant n_k} h_k^{2i-1}.$$

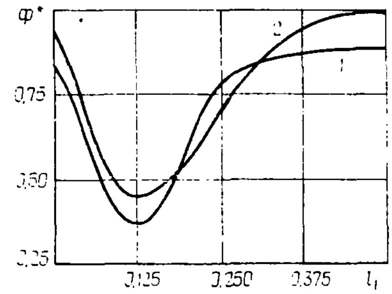

Fig. 2. Comparison of the load-carrying capacity of an optimum plate (1) and a plate of constant thickness (2).

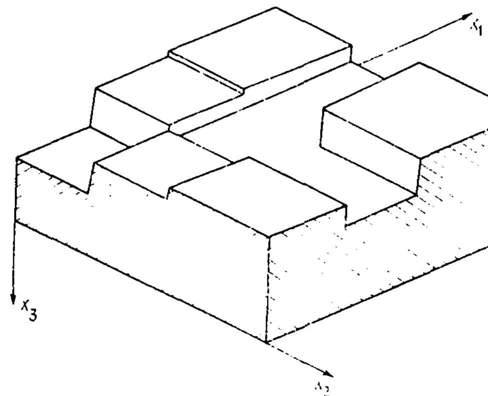

Fig. 3. Form of the optimum plate.

Also, the coefficients t_{ij} (7) were subject to the following conditions:

$$t_{ij} = t_{i+1j} = t_{ij+1} = t_{i+1j+1} = \theta_{ij},$$
$$i = 0,\ 2, \ldots,\ N_1 - 1; \qquad j = 0,\ 2, \ldots,\ N_2 - 1, \tag{12}$$

where Θ_{ij} are positive constants.

Equations (11) and (12) correspond to the fact that controls $t_N(x_1,\ x_2)$ from the allowable set U_d^N (9) are continuous functions which are close to being piecewise-constant. The proposed approximation for the thickness of the plate makes it possible to diminish the dimensionality of finite-dimensional problem (8) (to reduce the number of optimization parameters t_{ij}) and to thereby reduce computer operating time.

As an example, we solved problem (8) on finding the optimum law of change in the thickness of a plate which is square in plan (with a side $c = b_k - a_k$, $k = 1, 2$). The plate is freely supported at its contour and is bent by a uniformly distributed pressure of intensity g_0. Also, for comparison we solved auxiliary problem (1) on the stress state of a plate of constant thickness for the same boundary conditions, geometric dimensions, and external load. Here, the intensity of the pressure was chosen so that the value of functional (10) at the most heavily stressed point in a plate of constant thickness reached the limiting value, i.e., $\sup\limits_{1\leqslant k\leqslant m} \Phi_k = 1$. Solving problem (8), in accordance with (9) we assumed $\varphi(t_N) = C_2 = V$, where V is the volume of the material of the bearing part of the constant-thickness plate. This is evidence of the fact that the optimum plate was found from among a set of plates of equal weight in the case of constant thickness, with the stress state reaching the limiting value in the plate. Here, the weight of the filler was not considered.

We introduce the dimensionless coordinate $l_i = \frac{x_i}{c}\ (i = 1, 2)$. In these coordinates, a rectangle Ω — the plan of the plate being examined — is given by the inequalities $0 \leqslant l_i \leqslant 1$. As indicated above, with fixed values of l_1 and l_2 the greatest value of the functional (10) is reached at one of four points $x_3 = -h - t,\ -h;\ h;\ h + t$ (Fig. 1). Calculations showed that, in accordance with the chosen strength criterion, the most heavily stressed zone in the plate of constant thickness is the section $l_2 = 0.5$.

Figure 2 shows distribution curves of dimensionless values of the function $\Phi^*(l_1) = \max_{x_3} \Phi(l_1, 0.5, x_3)$, where the functional Φ is determined by Eq. (10). Due to the symmetry of the geometric dimensions (square plate), the boundary conditions, and the external load, the results are shown for one-fourth of the plate, i.e., for $0 \leqslant l_1 \leqslant 0.5$. It is apparent from the graphs that the maximum value of $\Phi^*(l_1)$ is reduced by 15% in the optimum plate compared to the plate of constant thickness.

The accuracy of the solution of problem (1) on the stress state of a plate was checked by comparing the solution for different numbers of coordinate functions. Thus, the graphs shown in Fig. 2 were obtained from the solution $\omega = (u, v, w)$ approximated by 104 coordinate functions. Here, we took 20 functions for tangential displacements u, v and 64 functions for the deflection w. The stress state found from the refined solution differs no more than 3% from the above solution with a twofold increase in the number of dimensions.

Figure 3 shows the form of 1/8 of the resulting optimum plate. The law of thickness change shown in the figure was obtained by varying nine parameters Θ_{ij} (see (12)).

In conclusion, we note that the solution of problem (1) is linearly dependent on the intensity of the load g, and the maximum value of $\Phi^*(l_1)$ for the optimum plate found is 15% lower than for the constant-thickness plate. It follows from the foregoing that the optimum plate, having the same weight as the constant-thickness plate, has a load-carrying capacity 15% greater than the constant-thickness plate.

Taking into account the multiple extreme character of problem (8) and increasing the number of optimization parameters, it would be possible to find an optimum plate design with an even greater load-carrying capacity.

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