A MECHANICAL MODEL OF FATIGUE CRACK PROPAGATION.

REPORT 1

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The cyclic strength of structural members is often determined to a large extent by the crack propagation process. The life of the structural members in the crack propagation stage is estimated on the basis of integration of various equations linking the rate of fatigue crack propagation with the stress intensity factor for maximum and minimum loads [1-3]. However, the use of these equations for the evaluation of the service life of real structures, primarily of welded members in these structures, is associated with considerable problems because the welding stresses greatly alter the nature of deformation at the crack tip. For example, in welded tee joints subjected to cyclic service loading with an asymmetry factor (stress ratio) equal to zero, the asymmetry factor of loading the material at the crack tip K_{min}/K_{max} changes from 0.8 to 0 in the course of crack propagation [4]. Consequently, the life of welded structures can be evaluated only on the basis of the equations which take into account the loading asymmetry which varies along the trajectory of crack propagation. In most cases, these equations are empirical and contain a large number of interconnected parameters which can be determined only by means of experiments with statistical processing of the data. This greatly complicates the determination and application of these dependences.

The proposed model of fatigue crack propagation is based on analysis of deformation of the material at the crack tip and takes into account the effect of the triaxial nature of the stress state, the strain criterion of low-cycle fracture, and the principle of linear damage summation. The model can be used to determine the effect of the loading asymmetry varying along the crack trajectory, on the rate of fatigue crack growth.

Solution of the Cyclic Elastoplastic Problem of the Stress–Strain State at the Crack Tip

The determination of the stress–strain at the crack tip in cyclic loading consists of two stages. In the first stage, attention is given to loading from the minimum to maximum load or (in terms of the stress intensity factor) from K_{min} to K_{max}, whereas in the second stage, unloading or loading from K_{max} to K_{min} is examined. Direct loading, similar to simple loading, is realized in both stages, if the process is examined in the coordinate system connected with the instant of the start of unloading, i.e., deformation of the material in each half-cycle of loading starts with unloading. In this case, it is justified to use the strain theory of plasticity [5] for both stages. To determine the intensity of the stresses and strains within the framework of the strain theory of plasticity, we may use the dependence proposed in [6]

$$\sigma_i = \left(\frac{E_s}{E}\right)^{1/2}\sigma_i^e; \qquad \varepsilon_i = \left(\frac{E}{E_s}\right)^{1/2}\varepsilon_i^e, \tag{1}$$

where σ_i and ε_i are respectively the intensity of the stresses and strains at the crack tip in solving the elastoplastic problem; σ_i^e and ε_i^e are the same in solving the elastic problem; E is the modulus of normal elasticity; E_s is the secant modulus.

The ratio of the stresses acting in the direction perpendicular (σ_1) and parallel (σ_2) to the crack trajectory in the elastoplastic region is assumed to be constant and independent of the degree of deformation of the material, i.e., $q = \sigma_2/\sigma_1 = \text{const}$ [7]. The deformation diagram of the material which links the stresses and strains in elastoplastic loading is represented by the generalized diagram of cyclic deformation which is independent of the number of the load half-cycle, i.e., cyclically stable materials are examined. In this case, hardening of the material in the stage of plastic deformation is approximated by a linear

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law; this is fully acceptable for the high strains reached at the crack tip. Taking into account these assumptions, cyclic deformation of the material may be described by a kinematic model based on the mechanism of translation hardening [8].

We shall examine the stress—strain state at the crack tip in loading to specific value K in the zero half-cycle. We shall analyze deformation of the material at the point on the crack trajectory which satisfies the criterion of maximum tensile stresses [9]. The distribution of the main stresses in the vicinity of the crack tip in elastic loading in the plane strain conditions is described using the relationships [1]

$$\left.\begin{aligned} \sigma_1 &= \frac{K}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\right);\\ \sigma_2 &= \frac{K}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\right);\\ \sigma_3 &= 2\mu\frac{K}{\sqrt{2\pi r}}\cos\frac{\theta}{2}, \end{aligned}\right\} \tag{2}$$

where σ_1, σ_2, σ_3 are the components of the main stresses; K is the stress intensity factor; r, Θ are the polar coordinates of the examined point; μ is Poisson's factor in the elastic loading range.

In the present case, the stress intensity in plane strain along the crack line is determined by the relationship

$$\sigma_i^e = \frac{1-2\mu}{\sqrt{2\pi r}}K. \tag{2'}$$

The stress and strain intensity at the crack tip in elastoplastic deformation may be calculated using Eqs. (1). The diagram of deformation of the material in the loading stage in linear approximation of hardening is expressed by the relationships

$$\sigma_i = E_u\left(\varepsilon_i - \frac{\sigma_T}{E} + \sigma_T\right) \text{ for } \sigma_i > \sigma_T; \quad \sigma_i = E\varepsilon_i \text{ for } \sigma_i \leqslant \sigma_T, \tag{3}$$

where E_u is the tangential modulus of hardening; σ_T is the yield stress.

Expressing the secant modulus E_s by means of the parameters of the deformation diagram included in Eqs. (3), in the form $E_s = \sigma_i / \left(\frac{\sigma_i - \sigma_T}{E_u} + \frac{\sigma_T}{E}\right)$, and substituting Eq. (2') into Eq. (1), we obtain

$$\sigma_i = \frac{\sigma_T\left(1-\frac{E_u}{E}\right) + \sqrt{\sigma_T^2\left(1-\frac{E_u}{E}\right)^2 + 4\frac{E_u}{E}\frac{(1-2\mu)^2}{2\pi r}K^2}}{2}. \tag{4}$$

The intensity of the elastoplastic strains at the crack tip is determined by the following equation, taking into account Eqs. (3)

$$\varepsilon_i = \frac{\sigma_i - \sigma_T}{E_u} + \frac{\sigma_T}{E}. \tag{5}$$

The components of the stresses σ_1, σ_2, σ_3 at the crack tip depend on the severity of the stress state and Poisson's factor. Assuming to a first approximation that Poisson's factor in the elastoplastic range is equal to 0.5, and using the equation for determining σ_i by means of the stress components observing the plane strain conditions, we obtain

$$\sigma_1 = \sigma_i / D, \tag{6}$$

where D is a coefficient which takes into account the increase of the first main stress as a result of the effect of the triaxial stress state; for the above-mentioned conditions, this coefficient is calculated from the equation

$$D = \frac{\sqrt{3}}{2}(1-q). \tag{7}$$

The degree of increase of the first main stress caused by the effect of the triaxial stress state may be determined by comparing the stress state in elastic and elastoplastic deformation. For a sufficiently localized zone of plastic deformation at the crack tip, the distribution of the stresses outside this zone in elastic and elastoplastic deformation will be approximately identical. Consequently, to satisfy the conditions of equilibrium in the elastoplastic region, the following relationship must be satisfied

$$\frac{1}{D}\int_0^{r_p} \sigma_i dr = \int_0^{r_p} \frac{K}{\sqrt{2\pi r}} dr, \tag{8}$$

where r_p is the size of the elastoplastic zone,

$$r_p = \frac{(1-2\mu)^2}{2\pi}\left(\frac{K}{\sigma_T}\right)^2. \tag{8'}$$

Integrating the right- and left-hand parts of Eq. (8) and using Eq. (4), we obtain the following equation for calculating the value of coefficient D by means of the parameters of the deformation diagram

$$D = (1-2\mu)\left[\left(1-\frac{E_u}{E}\right)\cdot\frac{1}{4} + 2M\right], \tag{9}$$

where M is the parameter calculated from the equation

$$M = \frac{1}{4\left(\frac{E}{E_u}-1\right)} \ln\frac{E}{E_u} + \frac{1}{8}\left(1+\frac{E_u}{E}\right). \tag{10}$$

Knowing the coefficient which takes into account the increase of the first main stress caused by the effect of the triaxial stress state, using Eq. (7) we can easily determine the relationship between the main stresses σ_1 and σ_2 in the elastoplastic region in plane strain

$$q = 1 - \frac{2}{\sqrt{3}} D. \tag{11}$$

To determine more accurately the stress—strain state at the crack tip, the proposed calculation method must take into account the dependence of Poisson's factor on the degree of deformation of the material. In this case, calculations of the stress—strain state at the crack tip in loading should be carried out in accordance with the following sequence.

1. Determination of the stress and strain intensity at the crack tip in the elastoplastic region from Eqs. (4), (5).

2. Calculation of the running value of Poisson ratio ν_r from the equation proposed in [7]

$$\nu_r = 0.5 - (0.5-\mu)\frac{\sigma_i}{E\varepsilon_i}. \tag{12}$$

3. Determination of coefficient D and of the relationship between the main stresses σ_1 and σ_2 using Eqs. (9)-(11).

4. Calculation of the main stresses at the crack tip

$$\left.\begin{aligned} &\sigma_1 = \sigma_i\left[\frac{1}{\sqrt{2}}\sqrt{(1-q)^2 + (q-\nu_r(1+q))^2 + (1-(1+q)\nu_r)^2}\right]^{-1};\\ &\sigma_2 = q\sigma_1;\\ &\sigma_3 = (1+q)\nu_r\sigma_1. \end{aligned}\right\} \tag{13}$$

5. Calculation of the total strains at the crack tip in plane strain

$$\left.\begin{aligned} &\varepsilon_1 = \frac{1}{E_s}[\sigma_1 - \nu(\sigma_2+\sigma_3)];\\ &\varepsilon_2 = \frac{1}{E_s}[\sigma_2 - \nu(\sigma_3+\sigma_1)];\\ &\varepsilon_3 = 0. \end{aligned}\right\} \tag{14}$$

6. Calculation of elastic strains

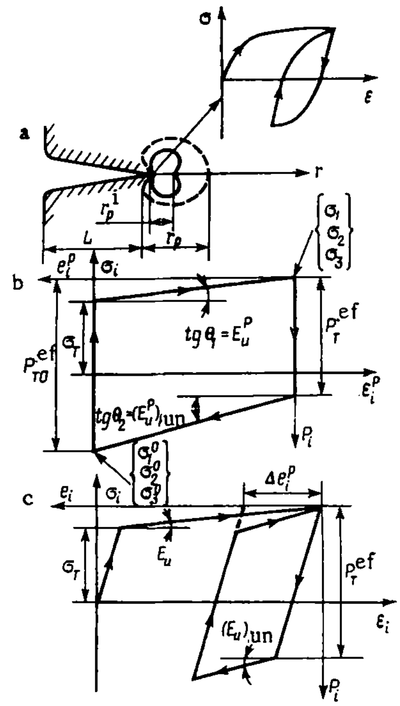

Fig. 1. Diagram of deformation of the material in the elastoplastic zone at the crack tip (a) and determination of the effective yield stress in unloading (b) and the range of plastic strain intensity (c).

$$\left.\begin{aligned}\varepsilon_1^e &= \frac{1}{E}[\sigma_1 - \mu(\sigma_2 + \sigma_3)];\\ \varepsilon_2^e &= \frac{1}{E}[\sigma_2 - \mu(\sigma_3 + \sigma_1)];\\ \varepsilon_3^e &= \frac{1}{E}[\sigma_3 - \mu(\sigma_1 + \sigma_2)].\end{aligned}\right\} \tag{15}$$

7. Calculation of plastic strains

$$\left.\begin{aligned}\varepsilon_1^p &= \varepsilon_1 - \varepsilon_1^e;\\ \varepsilon_2^p &= \varepsilon_2 - \varepsilon_2^e;\\ \varepsilon_3^p &= \varepsilon_3 - \varepsilon_3^e.\end{aligned}\right\} \tag{16}$$

It may be seen that Eqs. (13)-(16) characterize completely the stress–strain state at the crack tip at the instant in which the given value of stress intensity factor K is reached in loading.

In the stress–strain state at the crack tip at the start of unloading we examined the process of loading from K_{max} to K_{min} in the P_i–e_i coordinate system connected with the start of loading (Fig. 1). To analyze the stress–strain state, it is necessary to determine the vector of the stress increment at which plastic deformation in inverse loading is renewed, i.e., the conditions for the start of yielding are fulfilled. In this case, the most important condition is that it is necessary to take into account the fact that the ratio of the stress components at the start of unloading differs from the ratio of the stress increment components which lead to restoration of plastic deformation. As a result of taking into account this fact, it becomes evident that the effective yield stress in unloading P_T^{ef} differs from the identical parameter in uniaxial loading S_T ($S_T = 2\sigma_T$).

Since the required stress increment vector {Δp}, which leads to restoration of plastic deformation in inverse loading, is determined, in accordance with the unloading theorem [8], in the condition of elastic deformation, the vector can be expressed by means of the increment of the components of the main stresses at the crack tip taking into account Eq. (2)

$$\{\Delta p\} = \begin{Bmatrix} \Delta p_1 \\ \Delta p_2 \\ \Delta p_3 \end{Bmatrix} = \begin{Bmatrix} -\frac{\Delta K}{\sqrt{2\pi r}} \\ -\frac{\Delta K}{\sqrt{2\pi r}} \\ -2\mu\cdot\frac{\Delta K}{\sqrt{2\pi r}} \end{Bmatrix} = \begin{Bmatrix} \Delta p \\ \Delta p \\ 2\mu\Delta p \end{Bmatrix}, \tag{17}$$

where Δp_1, Δp_2, Δp_3 are the components of the increment of the main stresses at the crack tip in loading in the P_i-e_i coordinates in the stage of unloading from the stress state corresponding to K_{max}.

In the present case, the components of the stress deviator at the start of plastic deformation in unloading are determined taking into account Bauschinger's effect described by the model of translation hardening [8]

$$\left.\begin{aligned} P_1^* &= S_1 + (1-2\mu)\frac{\Delta p}{3}; \\ P_2^* &= S_2 + (1-2\mu)\frac{\Delta p}{3}; \\ P_3^* &= S_3 - (1-2\mu)\frac{2\cdot\Delta p}{3}, \end{aligned}\right\} \tag{18}$$

where P_1^*, P_2^*, P_3^* are the components of the stress deviator corresponding to the start of yielding in unloading; S_1, S_2, S_3 are the components of the stress deviator corresponding to the start of unloading, i.e., to the conditions of maximum loading in the zero half-cycle.

The values of S_1, S_2, and S_3 are determined by means of the components of the main stresses σ_1, σ_2, σ_3 and the components of the deviator of microstresses $\{\rho\}$ from the equations

$$\left.\begin{aligned} S_1 &= \sigma_1 - \sigma - \rho_1; \\ S_2 &= \sigma_2 - \sigma - \rho_2; \\ S_3 &= \sigma_3 - \sigma - \rho_3, \end{aligned}\right\} \tag{19}$$

where σ_1, σ_2, σ_3 are the stress components determined using Eq. (13); σ is the component of the spherical stress tensor, $\sigma = \frac{\sigma_1+\sigma_2+\sigma_3}{3}$; ρ_1, ρ_2, ρ_3 are the components of the deviator of the microstresses which depend on the plastic strains in loading in the zero half-cycle

$$\left.\begin{aligned} \rho_1 &= C\varepsilon_1^p; \\ \rho_2 &= C\varepsilon_2^p; \\ \rho_3 &= C\varepsilon_3^p. \end{aligned}\right\} \tag{20}$$

The components of the plastic strains in this case are determined from Eqs. (16) and constant C depends only on hardening of the material [8]

$$C = \frac{2}{3}\,\frac{E_u}{(1 - E_u/E)}.$$

The condition for the start of plastic deformation in unloading may be written in the form [8]

$$P_1^*P_2^* + P_2^*P_3^* + P_3^*P_1^* + \frac{\sigma_т^2}{3} = 0. \tag{21}$$

Substituting into Eq. (21) Eqs. (18)-(20), and solving the former in relation to the vector of the stress increment Δp, we obtain

$$\Delta p = 0 \text{ and} \Delta p = \frac{3S_3}{1-2\mu}. \tag{22}$$

The first root ($\Delta p = 0$) corresponds to the start of unloading and is not interesting, whereas the second root corresponds to the start of plastic deformation in inverse loading. Examining deformation in inverse loading in the coordinate system, linked with the start of unloading, the effective yield stress for the given triaxial stress state may be determined as the intensity of the stress increments

$$P_т^{ef} = \frac{1}{\sqrt{2}}\sqrt{(\Delta p_1 - \Delta p_2)^2 + (\Delta p_2 - \Delta p_3)^2 + (\Delta p_3 - \Delta p_1)^2} = (1-2\mu)\,|\,\Delta p\,| = 3S_3. \tag{23}$$

Thus, the effective yield stress in unloading is equal to $P_T{}^{ef} = 3S_3$, where S_3 is determined from Eqs. (13), (19), and (20) in relation to the ratio of the components of the main stresses at the moment of maximum loading in the zero half-cycle.

Since the effective yield stress in unloading depends on the triaxial stress state at the crack tip, in the general case in inverse deformation, the tangential modulus of hardening may differ from the identical parameter of the zero half-cycle.

To determine the hardening modulus in unloading and subsequent elastoplastic deformation, we shall examine deformation of the material in the P_i—$e_i{}^p$ coordinate system linked with the start of loading (Fig. 1b). We shall calculate the plastic modulus of hardening $(E_u{}^p)_{un}$ characterizing the stress increment in the elastoplastic region, in relation to the plastic strain increment. For this purpose, it is necessary to know the stress state for at least two points of the deformation path in unloading. One of these points may be represented by the point of the start of plastic yielding which is characterized by the value of the effective yield stress in unloading $P_T{}^{ef}$, and the second point may be the point at which strain in inverse loading $e_i{}^p$ is equal to the plastic strain in direct loading, i.e., in the case $\varepsilon_i{}^p = 0$ in the σ_i—$\varepsilon_i{}^p$ coordinates (Fig. 1b).

Let it be that at a certain point at the crack tip, the material with the effective yield stress $P_T{}^{ef}$ is deformed during unloading. The stress state at this point corresponds to the yielding conditions along the entire deformation path because plastic yielding takes place. At $e_i{}^p = \varepsilon_i{}^p$ the condition $\varepsilon_i{}^p = 0$ is fulfilled and, consequently, also $\varepsilon_1{}^p = \varepsilon_2{}^p = \varepsilon_3{}^p = 0$. The microstress vector $\{\rho\}$ will also be equal to zero. Consequently, the condition $\sigma_i = \sigma_T$ should be fulfilled at this point (in the σ_i—$\varepsilon_i{}^p$ coordinates) because it is assumed that the yield stresses of the material in the initial condition in tension and compression are identical. At the same time, the stress state may be determined using Eqs. (2) and (2') because the plastic strain vector is equal to $\{\varepsilon^p\} = 0$. In this case, we may write

$$\sigma_i = (1-2\mu)\,|\sigma_1|. \tag{24}$$

Consequently, taking into account Eq. (24), and also the compressive nature of the stresses in the direction perpendicular to the crack trajectory, the stress state at the crack tip in the stage of inverse loading in the σ_i—$\varepsilon_i{}^p$ coordinates at the examined deformation point at $\varepsilon_i{}^p = 0$ may be characterized by the following vector

$$\begin{Bmatrix} \sigma_1^0 \\ \sigma_2^0 \\ \sigma_3^0 \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\sigma_T}{1-2\mu} \\ -\dfrac{\sigma_T}{1-2\mu} \\ -\dfrac{2\mu\sigma_T}{1-2\mu} \end{Bmatrix}. \tag{25}$$

In the P_i—$e_i{}^p$ coordinate system the stress state at the examined deformation point is characterized by the vector $\{p^\circ\}$ which is determined from

$$\begin{Bmatrix} p_1^0 \\ p_2^0 \\ p_3^0 \end{Bmatrix} = -\left(\begin{Bmatrix} \sigma_1^0 \\ \sigma_2^0 \\ \sigma_3^0 \end{Bmatrix} - \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{Bmatrix}\right), \tag{26}$$

where $\{\sigma\}$ is the stress vector characterizing the stress state of the material at the crack tip at the instant of the start of loading and is determined using Eqs. (13).

The intensity of the stresses $p^{ef}{}_{T_0}$ in the P_i—$e^p{}_i$ coordinate system at the deformation point $e^p{}_i = \varepsilon^p{}_i$ may be calculated from the equation

$$P_{T_0}^{ef} = \frac{1}{\sqrt{2}}\sqrt{(p_1^0-p_2^0)^2+(p_2^0-p_3^0)^2+(p_3^0-p_1^0)}. \tag{27}$$

Thus, in the case of inverse loading, the variation of plastic strain by the value $e^p{}_i = \varepsilon^p{}_i$ is accompanied by the change of the stress intensity by the value $(p^{ef}{}_{T_0} - p^{ef}{}_T)$. Consequently, the plastic modulus of hardening $(E^p{}_u)_{un}$ can be calculated, in accordance with its definition, from the relationship

$$(E_u^p)_u = \frac{P_{T_0}^{ef} - P_T^{ef}}{\varepsilon_i^p}. \tag{28}$$

Correspondingly, the modulus of hardening in inverse loading in the P_i—e_i coordinate system, characterizing the deformation diagram with linear hardening in the form of Eqs. (3), is determined using the equation [8]

$$(E_u)_{un} = \frac{(E_u^p)_{un}}{[1 + (E_u^p)_{un}/E]} . \qquad (29)$$

Consequently, the effective yield stress in unloading P^{ef}_T determined using Eqs. (13), (19), (20), and (23) and the hardening modulus $(E_u)_{un}$ determined using Eqs. (25)-(29) make it possible, taking into account the previously derived dependences, to characterize fully the stress—strain state at the crack tip in cyclic loading in the unloading stage or in inverse loading from K_{max} to K_{min}.

Since it was assumed to a first approximation that the material is cyclically stable, it may be assumed that a closed deformation loop forms in its cyclic deformation. In this case, within the framework of the strain theory of plasticity, the range of the plastic strain intensity at the crack tip which determines the degree of damage of the material in the region of elastoplastic deformation is equal to the plastic strain intensity determined in accordance with the relationships derived for the unloading stage (Fig. 1c). The range of the plastic strain intensity at the crack tip may be calculated from the following equation

$$\Delta e_i^p = \frac{p_i^0 - p_T^{ef}}{(E_u)_{un}} + \frac{p_T^{ef} - p_i^0}{E} , \qquad (30)$$

where $\Delta e_i{}^p$ is the plastic strain intensity range at the crack tip; $p_i{}^0$ is the stress intensity calcualted from Eq. (4) with the values of σ_T, E_u, and K replaced by $P_T{}^{ef}$, $(E_u)_{un}$, and $\Delta K = K_{max} - K_{min}$, respectively.

In the general case, expressing the dependence of the plastic strain intensity range on the main parameters, we may write the following functional dependence

$$\Delta e_i^p = \psi (K_{max}, \Delta K, (E_u)_{un}, P_T^{ef}, r), \qquad (31)$$

where the relationship between $\Delta e_i{}^p$ with K_{max} follows from the corresponding functional relationship of the latter quantity with $(E_u)_{un}$ and $P_T{}^{ef}$. Consequently, in calculating the stress—strain state at the crack tip, it is possible to take into account the changes in the loading conditions and, correspondingly, asymmetry both in respect of the parameter K_{max} and ΔK.

Thus, the proposed solution of the cyclic elastoplastic problem of the stress—strain state at the crack tip can be used to determine all the necessary parameters taking into account the effect of the triaxial stress state in calculating the fatigue crack propagation rate on the basis of the proposed strain criteria of fracture.

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