UDC 517.983

ON CONTINUOUS SPECTRUM OF TRANSPORT OPERATOR

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Abstract. It is proved that the point $\zeta = 0$ in difference of other points of continious spectrum is point of branchement of logarithmic type of the resolvent of transport operator.

Introduction

We consider partial case of so-called "equation of transmission". There is much literature concerning (during many years) different problems in this direction. One of such problems, namely the problem of neutron transport, leads to the operator

$$Lf(x,\mu) = -i\mu\frac{\partial f}{\partial x}(x,\mu) + c(x)\int\limits_{-1}^{1} f(x,\mu')d\mu' \tag{1}$$

in the space $L^2(D)$, where $D = R \times [-1, 1]$. In [1] in the case

$$c(x) = \begin{cases} c, & |x| < a \\ 0, & |x| > a, c = \text{const} \end{cases}$$

it was obtained that continuous spectrum of the operator L coincides with real axis R and that the set of eingen-values is finite. In [2] in the case $c \in L^\infty(R),\ \operatorname{supp} c \subset [-a, a],\ c(x) \geq 0$ well-known functional model is applied.

1. Statement of the problem

Among other publications we mention only several of them, which are the closest to our problem. In [3] the authors use Friedrichs' model to study the operator L. In the case of exponentialy decreasing potential the sufficient condition of finiteness of point spectrum was obtained. The methods of this work were used in [4] in more general case of the operator

$$Lf(x,\mu) = -i\mu\frac{\partial f}{\partial x}(x,\mu) + a(x)\int\limits_{-1}^{1} b(\mu')f(x,\mu')d\mu'. \tag{2}$$

As it was proved in [4] the value $\zeta = 0$ only can be the point of accumulation of point spectrum of the operator L if the following conditions hold:

a) the function $a(x)$ is locally integrable and satisfies the estimate

$$|a(x)| \leq Me^{-|x|},\ x \in R, \tag{3}$$

where $\varepsilon > 0,\ M > 0$ are some constants;

b) the function $b(\mu),\ \mu \in (-1, 1)$ admits analytic prolongation $b(z)$ into the circle $|z| < 1 + \varepsilon$.

In that work it was proved that resolvent has analytic prolongation over the semi-axis $(-\infty;0)$ and $(0,\infty)$. But in this work the point of spectrum $\zeta = 0$ remains to be unstudied.

Our aim is to prove that the point $\zeta = 0$ is the point of branchment of the resolvent.Apropos in a similar situation in the work [3] it was proved that the point $\zeta = 0$ was spectral singularity of considered operator. Like [3-4] we use unitary equivalence of the operator L to the operator of Friedrichs' model.

2. Preliminary

Here we give some notations and results from [4].

Let H be Hilbert space of the functions on two variables $\varphi(s,\mu), (s,\mu) \in D$ with norm

$$||\varphi||_H^2 = \int\limits_R \int\limits_{-1}^{1} |\varphi(s,\mu)|^2 \frac{1}{|\mu|} ds d\mu$$

and let $G = L^2(R)$. We denote by $(\cdot,\cdot)$, $(\cdot,\cdot)_H$ scalar product in the spaces G and H respectively. We denote by $S : H \to H$ the operator of multiplication by independent variable $(S\varphi)(\tau,\mu) \equiv \tau\varphi(\tau,\mu)$, $\tau \in R$ with maximal domain of definition. Using Fourier transformation it was proved in [4] that the operator $L : L^2(D) \to L^2(D)$ is unitary equivalent to the operator $T = S + A^*B : H \to H$ (Friedrich's model) with bounded operators $A^* : G \to H$, $B : H \to G$ under the form

$$A^*c(s,\mu) = \frac{1}{2\pi} \int\limits_R a_1(y)c(y)e^{-iy\frac{s}{\mu}} dy, \tag{4}$$

and

$$B\varphi(x) = a_2(x) \int\limits_R e^{ix\tau} \left(\int\limits_{-1}^{1} b(\mu')\varphi(\tau\mu',\mu')d\mu' \right) d\tau. \tag{5}$$

We use the traditional form of perturbation A^*B, that's why we don't need the operator $A : H \to G$ itself. The representations (4)-(5) contain the factors $a_{1,2}(x)$ of arbitrary factorization such that

$$a(x) = \overline{a_1(x)}a_2(x), \quad |a_1(x)| = |a_2(x)|.$$

The relation between the resolvents $T_\zeta = (T-\zeta)^{-1}$ and $S_\zeta = (S-\zeta)^{-1}$ of the operators T and S is the following $T_\zeta = S_\zeta - S_\zeta A^* K(\zeta)^{-1} B S_\zeta$, where $K(\zeta) = 1 + BS_\zeta A^*$.

3. Estimate of the operator $K(\zeta), \zeta \to 0$

It is shown in [3] that

$$((K(\zeta)-1)c)(x) = \int\limits_R k(x,y,\zeta)c(y)dy, \quad \text{where} \quad k(x,y,\zeta) = \frac{1}{2\pi} a_2(x)\overline{a_1(y)} I(x-y,\zeta) \tag{6}$$

and

$$I(u,\zeta)=\int\limits_{R} l(\tau,\zeta)e^{iu\tau}d\tau,\ u=x-y,\quad \text{where}\quad l(\tau,\zeta)=\int\limits_{-1}^{1}\frac{b(\mu')}{\tau\mu'-\zeta}d\mu',\ \text{Im}(\zeta)\neq 0. \quad (7)$$

Let δ be arbitrary value such that $0<\delta<\varepsilon$ (see (3)) and $\Omega_{\pm}(\delta)=\{\zeta:|\zeta|<\delta,\ \pm\text{Im}\ \zeta>0\}$. By $\ln\zeta$ we denote the branch of logarithmic function which is continuous in the domain $\zeta\notin[0,\infty)$ and such that $\ln(-1)=\pi i$.

If $b(\mu)\equiv 1$ then we denote by $I_0(u,\zeta)$ the expression $I(u,\zeta)$ (see[3])

$$I_0(u,\zeta)=\gamma(\zeta)+R_0(u,\zeta),\ \zeta\in\Omega_{\pm}(\delta), \quad (8)$$

where

$$\gamma(\zeta)=-\pi i\ \text{sign}\ \nu\cdot\ln\zeta,\quad \nu=\text{Im}\zeta \quad (9)$$

and the term $R_0(u,\zeta)$ admits the estimate

$$|R_0(u,\zeta)|\leq M\left[\frac{1}{|u|^{\frac{1}{e}}}+|u|\right],\ p>1,\ \zeta\in\Omega_{\pm}(\delta),\ M=\text{const}, \quad (10)$$

which is independent of ζ. Underline that $\gamma(\zeta)\to\infty,\zeta\to 0$ and the decomposition like (8) is not unique. Let us introduce the following notation

$$||b||_{C^1}=\sup_{|z|<1+\varepsilon}|b(z)|+\sup_{|z|<1+\varepsilon}|b'(z)|\quad\text{and}\quad N_0(\delta)=\sup_{\zeta\in\Omega_{\pm}(\delta)}\left(\int\limits_{\varepsilon}^{\infty}\frac{dt}{|t-\zeta|^q}\right)^{\frac{1}{q}}. \quad (11)$$

Lemma 1. *The function $I(u,\zeta)$, defined in the relations (6)-(7), can be represented in the form*

$$I(u,\zeta)=b(0)\gamma(\zeta)+R(u,\zeta), \quad (12)$$

where

$$|R(u,\zeta)|\leq N_1(\delta)\left[\frac{1}{|u|^{\frac{1}{e}}}+|u|\right],\ p>1,\ \zeta\in\Omega_{\pm}(\delta), \quad (13)$$

where $N_1(\delta)=CN_0(\delta)\,||b||_{C^1}$ and C denote some constant, which is independent of δ, ζ and also of the function $b(\mu)$.

Proof. Let us denote $b_1(z)\equiv b(z)-b(0)$. We substitute in (7) the decomposition $b(z)=b_1(z)+b(0)$, separating $I_0(u,\zeta)$ (what correspond to with $b(z)\equiv 1$) in the right part of (7) and taking into account the decomposition $I_0(u,\zeta)$ itself (8), we obtain for $\zeta\in\Omega_{\pm}(\delta)$

$$I(u,\zeta)=\int\limits_0^{\infty}\frac{1}{t-\zeta}f_{-\omega,1}(t\,|u|)dt-\int\limits_0^{\infty}\frac{1}{t+\zeta}f_{\omega,1}(t\,|u|)dt+b(0)\gamma(\zeta)+b(0)R_0(u,\zeta), \quad (14)$$

where

$$f_{-\omega,1}(\tau)=\int\limits_{\tau}^{\infty}\frac{1}{y}\left[b_1\left(\frac{\tau}{y}\right)e^{i\omega y}+b_1\left(-\frac{\tau}{y}\right)e^{-i\omega y}\right]dy. \quad (15)$$

Integrating by parts, we get the estimate

$$\left|f_{-\omega,1}(\tau)\right| \le \begin{cases} 2\left\|b\right\|_{C^1}, & \tau\in(0,1) \\ 4\left\|b\right\|_{C^1}/\tau, & \tau\in(1,\infty). \end{cases} \tag{16}$$

It follows from here that the interval of integrations $(0,\infty)$ in (14) can be changed by the interval $(0,\varepsilon)$ (the value of ε see in (3)) and the difference between the integrals will have the estimate like (13). In the integral (15) we put $\tau = t\left|u\right|$ and make the change of variable $\frac{y}{|u|}=\theta$, then in view that $u = \text{sign}\, u\cdot\left|u\right| = \omega\left|u\right|$ we have:

$$f_{-\omega,1}(t\left|u\right|) = \int_t^\infty \frac{1}{\theta}\left[b_1\left(\frac{t}{\theta}\right)e^{iu\theta} + b_1\left(-\frac{t}{\theta}\right)e^{-iu\theta}\right]d\theta. \tag{17}$$

According to (14) we need the value $t<\varepsilon$. It's easy to verify that in (17) the interval of integrating (t,∞) can be changed by (t,ε) and therefore we can consider the integrals

$$g_\pm(t,u) = \int_0^\varepsilon \frac{1}{\theta}\cdot b_1\left(\pm\frac{t}{\theta}\right)e^{\pm iu\theta}d\theta. \tag{18}$$

In the right part of (14) it remains to consider the sum $I_+(u,\zeta)+I_-(u,\zeta)$, where

$$I_\pm(u,\zeta) = \int_0^\varepsilon \frac{g_\pm(t,u)}{t-\zeta}dt - \int_0^\varepsilon \frac{g_\pm(t,-u)}{t+\zeta}dt. \tag{19}$$

□

Theorem 1. *Let $\delta<\varepsilon$, then*

$$K(\zeta)-1 = \frac{b(0)}{2\pi}\gamma(\zeta)(\bullet,a_1)a_2 + Q(\zeta),\quad \zeta\in\Omega_\pm(\delta), \tag{20}$$

where the elements a_1, a_2 are defined by the factorization $a(x)=\overline{a_1(x)}a_2(x)$, $\left|a_1(x)\right|=\left|a_2(x)\right|$ and the operator $Q(\zeta): L^2(R)\to L^2(R)$ is compact with the norm bounded uniformly with respect to ζ, namely

$$\left\|Q(\zeta)\right\| \le M\left\|a\right\|_\delta,\quad \left\|a\right\|_\delta^2 \equiv \int_R \left|a(x)\right|^2 e^{2\delta|x|}dx,\quad \zeta\in\Omega_\pm(\delta). \tag{21}$$

Proof. According to (6) and (12), we have

$$k(x,y,\zeta) = \frac{b(0)}{2\pi}\gamma(\zeta)a_2(x)\overline{a_1(y)} + \frac{1}{2\pi}a_2(x)\overline{a_1(y)}R(x-y,\zeta),$$

what proves the decomposition (20). Further

$$\left\|Q(\zeta)\right\|^2 \le \frac{1}{4\pi^2}\int_R\int_R \left|a_2(x)\right|^2\left|a_1(y)\right|^2\left|R(x-y,\zeta)\right|^2dxdy.$$

Due to the relations $|a_2(x)|^2 = |a(x)|$, $|a_1(y)|^2 = |a(y)|$, we obtain

$$\|Q(\zeta)\|^2 \le \frac{1}{4\pi^2}\int\limits_R\int\limits_R |a(x)|\,|a(y)|\,e^{\delta|x|}e^{\delta|y|}\left[e^{-\delta(|x|+|y|)}\,|R(x-y,\zeta)|^2\right]dxdy \le$$

$$\le M_0^2\left(\int\limits_R\int\limits_R |a(x)|^2\,|a(y)|^2\,e^{2\delta|x|}e^{2\delta|y|}dxdy\right)^{\frac{1}{2}} =$$

$$= M_0^2\left(\left(\int\limits_R |a(x)|^2\,e^{2\delta|x|}dx\right)^2\right)^{\frac{1}{2}} = M_0^2\,\|a\|_\delta^2\,, \qquad (22)$$

where due to the estimate (13) under the condition $p > 2$ the value

$$M_0^2 = \left(\int\limits_R\int\limits_R e^{-2\delta(|x|+|y|)}\,|R(x-y,\zeta)|^2\,dxdy\right)^{\frac{1}{2}}$$

is finite. Theorem is proved. □

We substitute (18) in (19), change the order of integrating and we make the change of variable $\frac{t}{\theta} = \tau$, then in the case of sign „+“

$$I_+(u,\zeta) = \int\limits_0^1 b_1(\tau)\left(\int\limits_{-\varepsilon}^{\varepsilon}\frac{e^{iu\theta}}{\theta t-\zeta}d\theta\right)d\tau.$$

By integrating by parts, we have the decomposition

$$\int\limits_{-\varepsilon}^{\varepsilon}\frac{e^{iu\theta}}{\theta\tau-\zeta}d\theta = \frac{1}{\tau}\left[e^{iu\varepsilon}\ln(\varepsilon\tau-\zeta) - e^{-iu\varepsilon}\ln(-\varepsilon\tau-\zeta) - iu\int\limits_{-\varepsilon}^{\varepsilon}e^{iu\theta}\ln(\theta\tau-\zeta)d\theta\right], \qquad (22)$$

which leads us to the estimate $|I_+(u,\zeta)| \le C\,\|b\|_{C^1}\,[|u|+1]\,,\quad \zeta\in\Omega_\pm(\delta),\quad C = \text{const}.$ The value $I_-(u,\zeta)$ has analogic estimate. Really (let us consider $b_1(\tau)\equiv 1$), the value $\int\limits_0^1 \ln(\varepsilon\tau-\zeta)d\tau$ is bounded for $\zeta\in\Omega_\pm(\delta)$ if the integral

$$G(\zeta) \equiv \int\limits_0^1[\ln(\varepsilon\tau-\zeta) - \ln\varepsilon\tau]d\tau = \int\limits_0^1 \ln(1-\frac{\zeta}{\varepsilon\tau})d\tau$$

is bounded too. If $\tau = |\zeta|\,s$, then

$$G(\zeta) = |\zeta|\int\limits_0^{\frac{1}{|\zeta|}}\ln\left(1-\frac{\zeta}{\varepsilon\,|\zeta|}\cdot\frac{1}{s}\right)ds$$

and using the inequality $\left|\ln\left((1-\frac{\zeta}{\varepsilon|\zeta|}\cdot\frac{1}{s})\right)\right| \le \frac{M}{s}$, $s > 1$, $M = \text{const}$, we obtain $|G(\zeta)| \le C\,|\zeta|\ln\frac{1}{|\zeta|}$ or $|G(\zeta)| \le C$, $\zeta \in \Omega_{\pm}(\delta)$. Lemma is proved.

Now we consider $\zeta = 0$ as the point of the spectrum of the operator L (or the operator T).

Statement 1. *The value $\zeta = 0$ is not eigen-value of the operator L.*

Proof. If (see(2))

$$-i\mu\frac{\partial f}{\partial x}(x,\mu) + a(x)\int\limits_{-1}^{1} b(\mu')f(x,\mu')d\mu' \equiv 0,\ f \in L^2(D).$$

then integrating from 0 to x gives

$$-i\mu(f(x,\mu)) - f(\nu,\mu)) = B(x),$$

where the function

$$B(x) = -\int\limits_0^x a(t)\left(\int\limits_{-1}^{1} b(\mu')f(t,\mu')d\mu'\right)dt$$

has not limit value $\lim\limits_{x\to+\infty} B(x) = B_\infty$. Then $i\mu f(0,\mu) = B_\infty$ and

$$-i\mu f(x,\mu) + B_\infty = B(x). \text{ But } f(x,\mu) = (B(x) - B_\infty)/(-i\mu) \notin L^2(D),$$

what proves Statement. □

Statement 2. *The value $\zeta = 0$ is point of branchment of linear form of the resolvent $(T_\zeta\varphi,\psi)$, where φ,ψ are smooth elements.*

Proof. We consider the functions $\varphi(\tau)$, $\psi(\tau)$, which admit analytic prolongation in the band $|\mathrm{Im}\zeta| < \varepsilon$. According to (20)

$$|K_+(\tau)| = c\,|\ln|\tau|| + O(1),\ \tau \to 0.$$

So, the function $K_+(\zeta)$ is not bounded if $\zeta \to 0$. By the same way $\zeta = 0$ is not pole of the function $K_+(\zeta)$ what proves the statement. □

Conclusion

As a result in this work it was obtained: the point $\zeta = 0$ in difference of other points of continious spectrum is point of branchement of logarithmic type of the resolvent of transport operator. The operators which are more general than (1) are interesting in different applications so the same problem will be actual for such operators.

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Статья поступила в редакцию 07.10.2010