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On the primitive irreducible representations of finitely generated nilpotent groups

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We develop some tecniques whish allow us to apply the methods of commutative algebra for studing the representations of nilpotent groups. Using these methods, in particular, we show that any irreducible representation of a finitely generated nilpotent group G over a finitely generated field of characteristic zero is induced from a primitive representation of some subgroup of G.

Keywords: *nilpotent groups, group rings, primitive representations.*

If a group G has a finite series such that each of its factors is either cyclic or quasicyclic, then G is said to be minimax. If, in such series, all infinite factors are cyclic, then the group G is said to be polycyclic. By Theorem 2.13 in [1], any finitely generated nilpotent group is polycyclic.

Let G be a group, let H be a subgroup of the group G, and let U be a right RH-module. Since the group ring RG can be considered as a left RH-module, we can define the tensor product $U \otimes_{RH} RG$, which is a right RG-module named the RG-module induced from the RH-module U. If M is an RG-module and $U \leqslant M$, then $M = U \otimes_{RH} RG$, iff $M = \oplus_{t\in T} Ut$, where T is a right transversal to the subgroup H in G.

If a kG-module M of some representation φ of a group G over a field k is induced from some kH-module U, where H is a subgroup of the group G, then we say that the representation φ is induced from a representation ϕ of the subgroup H, where U is the module of the representation ϕ. If the module M is not induced from any kH-submodule for any subgroup $H < G$, then we say that the module M and the representation φ are primitive. Recall that the representation φ is said to be faithful. If $\mathrm{Ker}\varphi = 1$, it is equal to $C_G(M) = 1$.

Certainly, the primitive irreducible modules are a basic subject for investigations, when we are dealing with irreducible representations. In [2], Harper solved a problem posed by Zaleskii and proved that any not Abelian-by-finite finitely generated nilpotent group has an irreducible

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primitive representation over a not locally finite field. In [3], we proved that if a minimax nilpotent group G of class 2 has a faithful irreducible primitive representation over a finitely generated field of characteristic zero, then the group G is finitely generated. In [4], Harper studied polycyclic groups which have faithful irreducible representations. It is well known that any polycyclic group is finitely generated soluble, has finite rank, and meets the maximal condition for subgroups (in particular, for normal subgroups). In [5], we showed that, in the class of soluble groups of finite rank with the maximal condition for normal subgroups, only polycyclic groups may have faithful irreducible primitive representations over a field of characteristic zero.

It is well known that any finitely dimensional irreducible representation of a group G is induced from a primitive representation of some subgroup of G. Our main result shows that this property may hold for infinitely dimensional representations.

Theorem 1. *Let F be a finitely generated field of characteristic zero, and let G be a finitely generated nilpotent group. Let M be an irreducible FG-module. Then there are an element $0 \neq a \in M$ and a subgroup $L \leqslant G$ such that $aFG = aFL \otimes_{FL} FG$ and aFL is a primitive FL-module.*

So, the following questions naturally arise.

Question 1. *Is it true that any irreducible representation of a finitely generated nilpotent group G is induced from a primitive representation of some subgroup of G?*

Question 2. *Is it true that any irreducible representation of a minimax nilpotent group G is induced from a primitive representation of some subgroup of G?*

Our methods of investigations are based on the following techniques introduced by Brookes in [6]. A commutative ring R is a Hilbert one, if every prime ideal in R is an intersection of some maximal ideals of R. A ring R is an absolutely Hilbert one, if it is commutative, Noetherian, Hilbert, and all field images of R are locally finite.

Let G be a group, and let K be a normal subgroup of G. Let R be a ring, and let I be a G-invariant ideal of the group ring RK. Then $I^{\dagger} = G \bigcap (I+1)$ is a G-invariant subgroup of G. We say that a G-invariant ideal I is G-large, if $R/(R \bigcap I) = k$ is a field and $\left| K/I^{\dagger} \right| < \infty$.

Let R be an absolutely Hilbert ring, let V be an RK-module, and let U be a submodule of V. We say that a G-large ideal I of RK culls U in V, if $VI < V$ and $VI + U = V$.

Let $\pi \subseteq \mathbb{N}$ be a set of prime integers. We will say that a G-large ideal I is (π,G)-large, if $\mathrm{char}(\mathrm{R}/(\mathrm{R} \bigcap \mathrm{I})) \notin \pi$. We will say that an RK-module V is (π, G)-large critical, if, for any nonzero submodule U of V, there is a (π, G)-large ideal I of RK which culls the submodule U in V.

Proposition 1. *Let K be a normal subgroup of a finitely generated nilpotent group G. Let R be an absolutely Hilbert domain of characteristic zero. Let $0 \neq W$ be a finitely generated RK-module which is R-torsion-free. Then there is a cyclic uniform RK-submodule $0 \neq V \leqslant W$ which is (π, G)-large critical for any finite set of prime integers $\pi \subseteq \mathbb{N}$.*

Let P be a prime ideal of a commutative ring S, and let S_P be the localization of S at the ideal P. Let M be an S-module, and let the support $\mathrm{Supp}_S M$ of the module M consist of all prime ideals P of S such that $M_P = M \otimes_S S_P \neq 0$ (see. [7] Chap. II, § 4.4). By [7, Chap. IV, § 1.4, Theorem 2], if S and M are Noetherian, then the set $\mu_S(M)$ of minimal elements of $\mathrm{Supp}_S M$ is finite and coincides with $\mu_S(\mathrm{Ann}_S M)$. Here, $\mu_S(\mathrm{Ann}_S M)$ is the set of prime ideals of S which are minimal over $\mathrm{Ann}_S M$. Since $\mathrm{Ann}_S(S/\mathrm{Ann}_S M) = \mathrm{Ann}_S M$, we can conclude that

$$\mu_S(M) = \mu_S(\mathrm{Ann}_S M) = \mu_S(S/\mathrm{Ann}_S M)$$

where $\mu_S(S/\mathrm{Ann}_S M)$ is the set of minimal elements of the support $\mathrm{Supp}_S(S/\mathrm{Ann}_S M)$ of the quotient module $S/\mathrm{Ann}_S M$.

Let G be a group, and let K be a normal subgroup of G. Let L be a subgroup of G such that $K \leqslant L \leqslant G$ and the quotient group L/K is finitely generated free Abelian. Let R be a ring. If I is a G-large ideal of the group ring RK, then the derived subgroup of the quotient group $L/I^{\dagger}$ is finite, and, hence, $L/I^{\dagger}$ has a characteristic central subgroup A of finite index. Proposition 1 shows that there are plenty of G-large ideals of the group ring RK. Since the quotient group L/K is finitely generated free Abelian, the subgroup A may be chosen finitely generated free Abelian. Then, by [1, Theorem 3.7], the group ring kA is a Noetherian.

Let W be a finitely generated RL-module. Then W/WI is a finitely generated kA-module, and, hence, W/WI is a Noetherian kA-module. So, the finite set

$$\mu_{kA}(W/WI) = \mu_{kA}(\mathrm{Ann}_{kA}(W/WI)) \tag{1}$$

is defined. Some basic properties of the set $\mu_{kA}(W/WI)$ are gathered in the following Lemma.

Lemma 1. *Let G be a finitely generated nilpotent group, and let K be a normal subgroup of G such that the quotient group G/K is free Abelian. Let R be an absolutely Hilbert domain, and let W be a finitely generated RG-module. Let I be a G-large ideal of RK. Let A be a central free Abelian subgroup of finite index in $G/I^{\dagger}$, $k = R/(R \cap I)$. Then, for any RG-submodule $W_1 \leqslant W$:*

(i) W_1/W_1I has a finite series, and each of whose quotients is isomorphic to some section of W/WI;

(ii) $\mathrm{Supp}_{kA}\ W_1/W_1I \subseteq \mathrm{Supp}_{kA}\ W/WI$;

(iii) for any $P \in \mu_{kA}(W_1/W_1I)$, there is $Q \in \mu_{kA}(W/WI)$ such that $Q \leqslant P$;

(iv) $\mu_{kA}(W_2/W_2I) \cap \mu_{kA}(W/WI) \subseteq \mu_{kA}(W_1/W_1I) \cap \mu_{kA}(W/WI)$ for any RG-submodule W_2 of W such that $W_2 \leqslant W_1$.

Equality (1) relates properties of the module W with properties of ideals in the commutative domain kA. This approach allows us to apply the methods of commutative algebra for studying the module W. The next proposition shows how this relationship may be established.

Proposition 2. *Let G be a finitely generated nilpotent group, and let H be a subgroup of* G. *Let R be an absolutely Hilbert domain, and let $W \neq 0$ be an RG-module. Let K be a normal subgroup of G such that $K \leqslant H$. Let I be a G-large ideal of RK, let $k = R/(I \cap R)$, let A be a free Abelian central subgroup of finite index in $G/I^{\dagger}$, and let $B = A \cap H/I^{\dagger}$. Let $0 \neq d \in W$ such that $(dRG)/(dRG)I \neq 0$, let w be the image of d in $(dRG)/(dRG)I$, and let $J = \mathrm{Ann}_{kA}(w)$. Suppose that $dRG = dRH \otimes_{RH}$. Then:*

(i) $J = (J \cap kB)kA$;

(ii) $\mu_{kA}((dRG)/(dRG)I) = \mu_{kA}(J)$;

(iii) if $(dRG)/(dRG)I \neq 0$ is kA-torsion, then $\mu_{kA}(J)$ consists of proper ideals of kA.

The following proposition plays the key role in the proof of Theorem 1.

Proposition 3. *Let G be a finitely generated nilpotent group, and let K be a normal subgroup of G such that the quotient group G/K is free Abelian. Let R be an absolutely Hilbert domain, let F be the field of fractions of R, and let M be an irreducible FG-module. Let I be a G-large ideal of RK, let A be a central free Abelian subgroup of finite index in $G/I^{\dagger}$, and let $k = R/(R \cap I)$.*

Then, for any non-zero elements $a, b \in M$:

(i) *aRK contains a submodule which is isomorphic to bRK;*

(ii) $\mu_{kA}((aRG)/(aRG)I) = \mu_{kA}((bRG)/(bRG)I)$;

(iii) *if* $(akG)/(akG)I \neq 0$, *then* $(bkG)/(bkG)I \neq 0$;

(iv) *if* $(akG)/(akG)I$ *is* kA *-torsion, then* $(bkG)/(bkG)I$ *is* kA *-torsion.*

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ПРО ПРИМІТИВНІ НЕЗВІДНІ ЗОБРАЖЕННЯ
СКІНЧЕННО ПОРОДЖЕНИХ НІЛЬПОТЕНТНИХ ГРУП

Розвивається підхід, що дає змогу застосовувати методи комутативної алгебри до вивчення зображень нільпотентних груп. Із застосуванням цього підходу, зокрема, показано, що усяке незвідне зображення скінченно породженої нільпотентної групи *G* над скінченно породженим полем нульової характеристики є індукованим з примітивного зображення деякої підгрупи групи *G*.

***Ключові слова**: нільпотентні групи, групові кільця, примітивні зображення.*