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NONPOSITIVE SOLUTIONS TO A CERTAIN FUNCTIONAL DIFFERENTIAL INEQUALITY

НЕДОДАТНІ РОЗВ'ЯЗКИ ДЕЯКИХ ФУНКЦІОНАЛЬНО-ДИФЕРЕНЦІАЛЬНИХ НЕРІВНОСТЕЙ

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In the paper, efficient conditions are found guaranteeing that every solution to the problem

$$u'(t) \geq \ell(u)(t), \quad u(a) \geq h(u)$$

is nonpositive, where $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ *and* $h : C([a,b];\mathbb{R}) \to \mathbb{R}$ *are linear bounded operators. The results obtained are very useful for the investigation of the question on solvability and unique solvability of the nonlocal boundary-value problems for the first order functional differential equations in both linear and nonlinear cases.*

Знайдено ефективні умови для того, щоб кожен розв'язок задачі

$$u'(t) \geq \ell(u)(t), \quad u(a) \geq h(u),$$

де $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ $h : C([a,b];\mathbb{R}) \to \mathbb{R}$ *— лінійні обмежені оператори, був недодатним. Отримані результати є корисними для вивчення задачі розв'язності та існування единого розв'язку нелокальних граничних задач для функціонально-диференціальних рівнянь першого порядку як в лінійному, так і в нелінійному випадках.*

1. Introduction and notation. On the interval $[a,b]$, we consider the functional differential inequality

$$u'(t) \geq \ell(u)(t), \tag{1.1}$$

where $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ is a linear bounded operator. By a solution to inequality (1.1) we understand an absolutely continuous function $u : [a,b] \to \mathbb{R}$ satisfying inequality (1.1) almost everywhere on the interval $[a,b]$.

Theorems on differential inequalities play a very important role in the theory of differential equations. For example, well-known Gronwall's inequality is also a corollary of a certain theorem on differential inequalities. Various types of differential inequalities are studied in the literature (see, e.g., [1, 3–6, 8, 9, 11, 13, 15–17]). In the present paper, effective sufficient conditions are found guaranteeing that every solution to inequality (1.1) satisfying the condition

$$u(a) \geq h(u) \tag{1.2}$$

with a linear bounded functional $h : C([a,b];\mathbb{R}) \to \mathbb{R}$ is nonpositive on the interval $[a,b]$. Statements obtained here can be used in the investigation of the question on solvability and unique solvability of the nonlocal boundary-value problems for functional differential equations in both linear and nonlinear cases.

In order to simplify the formulation of the main results we introduce the following definition.

Definition 1.1. *Let $h \in F_{ab}$. An operator $\ell \in \mathcal{L}_{ab}$ is said to belong to the set $\widetilde{V}_{ab}^-(h)$ (resp. $\widetilde{V}_{ab}^+(h)$) if every solution to the problem (1.1), (1.2) is nonpositive (resp. nonnegative).*

As it was mentioned above, the aim of the paper is to find conditions guaranteeing the inclusions $\ell \in \widetilde{V}_{ab}^-(h)$ and $\ell \in \widetilde{V}_{ab}^+(h)$ to hold. In the case where the functional h is given by the formula

$$h(v) \stackrel{\text{df}}{=} \lambda v(b) \quad \text{for} \quad v \in C([a,b];\mathbb{R})$$

with $\lambda \geq 0$, the sets $\widetilde{V}_{ab}^+(h)$ and $\widetilde{V}_{ab}^-(h)$ are described in detail (see [7, 8]). In [14], the case where $h \in PF_{ab}$ is considered. However, a general case of h has not been studied yet.

We shall suppose throughout the paper that the functional $h \in F_{ab}$ is defined by the formula

$$h(v) \stackrel{\text{df}}{=} \lambda v(b) + h_0(v) - h_1(v) \quad \text{for} \quad v \in C([a,b];\mathbb{R}), \tag{1.3}$$

where $\lambda > 0$ and $h_0,\, h_1 \in PF_{ab}$. There is no lost of generality in assuming this, because an arbitrary linear bounded functional can be represented in this form.

The following notation is used in the sequel:

(1) $\mathbb{N}$ is the set of all natural numbers, $\mathbb{R}$ is the set of all real numbers, $\mathbb{R}_+ = [0,+\infty[$. If $x \in \mathbb{R}$ then we put

$$[x]_+ = \frac{|x| + x}{2}, \quad [x]_- = \frac{|x| - x}{2}.$$

(2) $C([a,b];\mathbb{R})$ is the Banach space of continuous functions $v : [a,b] \to \mathbb{R}$ endowed with the norm $\|v\|_C = \max\{|v(t)| : t \in [a,b]\}$.

(3) $\widetilde{C}([a,b];D)$, where $D \subseteq \mathbb{R}$, is the set of absolutely continuous functions $v : [a,b] \to D$.

(4) $L([a,b];\mathbb{R})$ is the Banach space of Lebesgue integrable functions $p : [a,b] \to \mathbb{R}$ endowed with the norm $\|p\|_L = \int_a^b |p(s)|ds$.

(5) $L([a,b];D) = \{p \in L([a,b];\mathbb{R}) : p : [a,b] \to D\}$, where $D \subseteq \mathbb{R}$.

(6) $C([a,b];D) = \{v \in C([a,b];\mathbb{R}) : v : [a,b] \to D\}$, where $D \subseteq \mathbb{R}$.

(7) $\mathcal{L}_{ab}$ is the set of linear bounded operators $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$, P_{ab} is the set of operators $\ell \in \mathcal{L}_{ab}$ mapping the set $C([a,b];\mathbb{R}_+)$ into the set $L([a,b];\mathbb{R}_+)$.

(8) F_{ab} is the set of linear bounded functionals $h : C([a,b];\mathbb{R}) \to \mathbb{R}$, PF_{ab} is the set of functionals $h \in F_{ab}$ mapping the set $C([a,b];\mathbb{R}_+)$ into the set $\mathbb{R}_+$.

Definition 1.2. *Let $t_0 \in [a,b]$. We say that $\ell \in \mathcal{L}_{ab}$ is a t_0 Volterra operator if, for arbitrary $a_1 \in [a,t_0]$, $b_1 \in [t_0,b]$, $a_1 \neq b_1$, and $v \in C([a,b];\mathbb{R})$ with the property*

$$v(t) = 0 \quad \textit{for} \quad t \in [a_1,b_1],$$

the relation

$$\ell(v)(t) = 0 \quad \textit{for a.e.} \quad t \in [a_1,b_1]$$

holds.

2. Preliminary remarks. Recall that we suppose $\ell \in \mathcal{L}_{ab}$ and $h \in F_{ab}$. The following two assumptions are natural:

(A) If $h(1) = 1$ then the operator ℓ is supposed to be nontrivial in the sense that the condition $\ell(1) \not\equiv 0$ holds.

(B) $\widetilde{h} \not\equiv 0$, where the functional $\widetilde{h}$ is defined by the formula

$$\widetilde{h}(v) = h(v) - v(a) \quad \text{for} \quad v \in C([a,b];\mathbb{R}).$$

Remark 2.1. It follows from Definition 1.1 that if $\ell \in \widetilde{V}^-_{ab}(h)$ (resp. $\ell \in \widetilde{V}^+_{ab}(h)$) then the homogeneous problem

$$u'(t) = \ell(u)(t), \quad u(a) = h(u) \tag{2.1}$$

has only the trivial solution. Therefore, the inclusion $\ell \in \widetilde{V}^-_{ab}(h)$ (resp. $\ell \in \widetilde{V}^+_{ab}(h)$) guarantees the unique solvability of the problem

$$u'(t) = \ell(u)(t) + q(t), \quad u(a) = h(u) + c \tag{2.2}$$

for every $q \in L([a,b];\mathbb{R})$ and $c \in \mathbb{R}$. This fact follows from the Fredholm property of problem (2.2) (see, e.g., [2, 10]; in the case, where the operator ℓ is strongly bounded, see also [1, 12, 18]). Moreover, under the condition $\ell \in \widetilde{V}^-_{ab}(h)$ (resp. $\ell \in \widetilde{V}^+_{ab}(h)$), the unique solution to the problem (2.2) is nonpositive (resp. nonnegative) whenever $q \in L([a,b];\mathbb{R}_+)$ and $c \in \mathbb{R}_+$.

Remark 2.2. It is easy to verify that the condition $(-P_{ab}) \cap \widetilde{V}^-_{ab}(h) \neq \varnothing$ implies

$$h(1) > 1. \tag{2.3}$$

Indeed, if $\ell \in (-P_{ab}) \cap \widetilde{V}^-_{ab}(h)$ and $h(1) \leq 1$, then the function $u \equiv 1$ is a positive solution to the problem (1.1), (1.2), which contradicts the inclusion $\ell \in \widetilde{V}^-_{ab}(h)$.

On the other hand if, together with (2.3), the inequality $h_0(1) \leq 1$ holds then the zero operator belongs to the set $\widetilde{V}^-_{ab}(h)$. Indeed, let $u \in \widetilde{C}([a,b];\mathbb{R})$ satisfy (1.2) and

$$u'(t) \geq 0 \quad \text{for a.e.} \quad t \in [a,b].$$

Then it is clear that

$$u(a) \leq u(t) \leq u(b) \quad \text{for} \quad t \in [a,b]. \tag{2.4}$$

By virtue of condition (2.4) and the assumptions $h_0, h_1 \in PF_{ab}$, it follows from (1.2) that

$$u(a) \geq \lambda u(b) + h_0(u) - h_1(u) \geq u(a)h_0(1) + (\lambda - h_1(1))\, u(b).$$

Taking now condition (2.4) and the assumption $h_0(1) \leq 1$ into account, we get

$$(\lambda - h_1(1))\, u(b) \leq (1 - h_0(1))\, u(a) \leq (1 - h_0(1))\, u(b),$$

and thus

$$(h(1) - 1)\, u(b) \leq 0.$$

The last inequality and (2.3) result in $u(b) \leq 0$. Hence, condition (2.4) guarantees $u(t) \leq 0$ for $t \in [a, b]$, and thus $0 \in \widetilde{V}_{ab}^{-}(h)$.

We have shown that condition (2.3) is necessary for the validity of the relation $(-P_{ab}) \cap \cap \widetilde{V}_{ab}^{-}(h) \neq \varnothing$ and conditions (2.3) and $h_0(1) \leq 1$ are sufficient for the inclusion $0 \in \widetilde{V}_{ab}^{-}(h)$ to hold.

Definition 2.1. *An operator $\ell \in \mathcal{L}_{ab}$ is said to belong to the set $S_{ab}(a)$ (resp. $S_{ab}(b)$) if every solution u to inequality (1.1), which satisfies $u(a) \geq 0$ (resp. $u(b) \leq 0$), is nonnegative (resp. nonpositive).*

Remark 2.3. The sets $S_{ab}(a)$ and $S_{ab}(b)$ are investigated in [6].

3. Auxiliary statements. In this section, auxiliary statements are given. More precisely, properties of the sets U_{ab}^{-} and $\widetilde{U}_{ab}^{+}(h)$ are studied that are very useful in the investigation of the validity of the desired inclusion $\ell \in \widetilde{V}_{ab}^{-}(h)$.

3.1. Formulation of results. We first formulate all the results, the proofs are given in the next section.

Definition 3.1. *Let $h \in F_{ab}$. An operator $\ell \in \mathcal{L}_{ab}$ is said to belong to the set U_{ab}^{-}, if the problem (1.1), (1.2) has no nontrivial nonnegative solution.*

Remark 3.1. It follows immediately from Definitions 1.1 and 3.1 that $\widetilde{V}_{ab}^{-}(h) \subseteq U_{ab}^{-}(h)$.

Since the set $U_{ab}^{-}(h)$ is wider than $\widetilde{V}_{ab}^{-}(h)$, conditions for the inclusion $\ell \in U_{ab}^{-}$ can be derived relatively easy. In Theorem 3.1 (Theorem 3.2), the case $\ell \in P_{ab}$ ($-\ell \in P_{ab}$) is considered, whereas Theorems 3.3 and 3.4 concern the case where $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in P_{ab}$.

Theorem 3.1. *Let $\ell \in P_{ab}$ and*

$$h_1(1) < \lambda. \tag{3.1}$$

Let, moreover, there exist a function $\gamma \in \widetilde{C}([a, b]; \mathbb{R}_+)$ satisfying

$$\gamma'(t) \leq \ell(\gamma)(t) \quad \textit{for a.e.} \quad t \in [a, b], \tag{3.2}$$

$$\gamma(a) < h(\gamma). \tag{3.3}$$

Then $\ell \in U_{ab}^{-}(h)$.

Remark 3.2. If $\ell \in P_{ab}$, $h(1) \geq 1$, and $h_1(1) < \lambda$ then the operator ℓ belongs to the set $U_{ab}^{-}(h)$ without any additional assumptions. Indeed, since the operator ℓ is supposed to be nontrivial in the case where $h(1) = 1$, the function

$$\gamma(t) = 1 + \int_a^t \ell(1)(s)ds \quad \text{for} \quad t \in [a, b]$$

satisfies the conditions (3.2) and (3.3).

Theorem 3.2. *Let* $-\ell \in P_{ab}$ *and*

$$h(1) > 1, \quad h_0(1) \le 1. \tag{3.4}$$

Then $\ell \in U_{ab}^-(h)$ *if and only if there exists a function* $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ *satisfying the conditions (3.2) and (3.3).*

Theorem 3.3. *Let* $\ell = \ell_0 - \ell_1$, *where* $\ell_0, \ell_1 \in P_{ab}$, *and*

$$h(1) \le 1, \quad h_1(1) < \lambda. \tag{3.5}$$

If, moreover,

$$\int_a^b \ell_1(1)(s)\,ds < (\lambda - h_1(1)) \min\left\{1, \frac{1}{\lambda}\right\} \tag{3.6}$$

and

$$\int_a^b \ell_0(1)(s)\,ds > \frac{(1-h_0(1))\min\left\{1,\frac{1}{\lambda}\right\}}{(\lambda - h_1(1))\min\left\{1,\frac{1}{\lambda}\right\} - \int_a^b \ell_1(1)(s)\,ds} - 1, \tag{3.7}$$

then $\ell \in U_{ab}^-(h)$.

Theorem 3.4. *Let* $\ell = \ell_0 - \ell_1$, *where* $\ell_0, \ell_1 \in P_{ab}$, *and*

$$h(1) > 1, \quad h_1(1) < \lambda. \tag{3.8}$$

Let, moreover, the inequality (3.6) hold and

$$\int_a^b \ell_0(1)(s)\,ds > \omega\left(\int_a^b \ell_1(1)(s)\,ds\right), \tag{3.9}$$

where

$$\omega(y) = \begin{cases} \dfrac{(y + h_1(1))\left(1 - \frac{1}{\lambda}h_1(1)\right)}{1 - \frac{1}{\lambda}h_1(1) - y} - (h_0(1) + \lambda - 1) \\ \qquad \text{if} \quad \lambda \ge 1, \quad y < \dfrac{(h(1)-1)\left(1-\frac{1}{\lambda}h_1(1)\right)}{\lambda - 1 + h_0(1)}, \\ \dfrac{\left(y + \frac{1}{\lambda}h_1(1)\right)\left(1 - \frac{1}{\lambda}h_1(1)\right)}{1 - \frac{1}{\lambda}h_1(1) - y} - \left(\dfrac{1}{\lambda}h_0(1) + \dfrac{\lambda-1}{\lambda}\right) \\ \qquad \text{if} \quad \lambda \ge 1, \quad y \ge \dfrac{(h(1)-1)\left(1-\frac{1}{\lambda}h_1(1)\right)}{\lambda - 1 + h_0(1)}, \\ \dfrac{\left(y + \frac{1-\lambda}{\lambda} + \frac{1}{\lambda}h_1(1)\right)(\lambda - h_1(1))}{\lambda - h_1(1) - y} - \dfrac{1}{\lambda}h_0(1) \\ \qquad \text{if} \quad \lambda < 1, \quad y < \dfrac{(h(1)-1)(\lambda - h_1(1))}{h_0(1)}, \\ \dfrac{(y + 1 - \lambda + h_1(1))(\lambda - h_1(1))}{\lambda - h_1(1) - y} - h_0(1) \\ \qquad \text{if} \quad \lambda < 1, \quad y \ge \dfrac{(h(1)-1)(\lambda - h_1(1))}{h_0(1)}. \end{cases} \tag{3.10}$$

Then $\ell \in U_{ab}^{-}(h)$.

Now we introduce the following definition.

Definition 3.2. *Let* $h \in F_{ab}$. *An operator* $\ell \in \mathcal{L}_{ab}$ *is said to belong to the set* $\widetilde{U}_{ab}^{+}(h)$ *if there is no nonpositive solution* u *to inequality (1.1) satisfying the condition*

$$u(a) > h(u). \tag{3.11}$$

Remark 3.3. It is clear that $\widetilde{U}_{ab}^{+}(0) = \mathcal{L}_{ab}$ and $\widetilde{V}_{ab}^{+}(h) \subseteq \widetilde{U}_{ab}^{+}(h)$.

Theorem 1. *Let* $\ell \in P_{ab}$ *and* $h \in PF_{ab}$ *be such that* $h(1) \leq 1$. *If there exists a function* $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ *satisfying the conditions*

$$\gamma'(t) \geq \ell(\gamma)(t) \quad \text{for a.e.} \quad t \in [a,b], \tag{3.12}$$

$$\gamma(a) \geq h(\gamma), \tag{3.13}$$

then $\ell \in \widetilde{U}_{ab}^{+}(h)$.

3.2. Proofs. We first recall a result established in [6].

Lemma 3.1 ([6], Theorem 1.1). *Let* $\ell \in P_{ab}$. *Then* $\ell \in \mathcal{S}_{ab}(a)$ *if and only if there exists a function* $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ *satisfying condition (3.12).*

Proof of Theorem 3.1. Let u be a nonnegative solution to the problem (1.1), (1.2). We shall show that $u \equiv 0$. Since $\ell \in P_{ab}$ and u is a nonnegative function, it follows from (1.1) that

$$0 \leq u(a) \leq u(t) \leq u(b) \quad \text{for} \quad t \in [a,b]. \tag{3.14}$$

Suppose that $u(b) > 0$. Then condition (1.2), in view of (3.1), (3.14), and the assumptions $h_0, h_1 \in PF_{ab}$, results in

$$u(a) \geq \lambda u(b) + h_0(u) - h_1(u) \geq (\lambda - h_1(1))\, u(b) > 0.$$

Consequently, the relation (3.14) implies

$$u(t) > 0 \quad \text{for} \quad t \in [a,b]. \tag{3.15}$$

Put

$$v(t) = ru(t) - \gamma(t) \quad \text{for} \quad t \in [a,b],$$

where

$$r = \max\left\{\frac{\gamma(t)}{u(t)} : \ t \in [a,b]\right\}.$$

According to (3.3), (3.15), and the assumption $\gamma \in \widetilde{C}([a,b];\mathbb{R}_+)$, we get

$$r > 0. \tag{3.16}$$

It is obvious that

$$v(t) \geq 0 \quad \text{for} \quad t \in [a, b] \tag{3.17}$$

and there exists $t_0 \in [a, b]$ such that

$$v(t_0) = 0. \tag{3.18}$$

Taking now (1.1), (3.2), (3.16), (3.17), and the assumption $\ell \in P_{ab}$ into account, we obtain

$$v'(t) \geq \ell(v)(t) \geq 0 \quad \text{for a.e.} \quad t \in [a, b]. \tag{3.19}$$

Therefore, relation (3.19), on account of (3.17) and (3.18), yields

$$0 = v(a) \leq v(t) \leq v(b) \quad \text{for} \quad t \in [a, b]. \tag{3.20}$$

However, using (1.2), (3.1), (3.3), (3.16), (3.20), and the assumptions $h_0, h_1 \in PF_{ab}$, we get

$$0 = v(a) > \lambda v(b) + h_0(v) - h_1(v) \geq (\lambda - h_1(1))\, v(b) \geq 0,$$

which is a contradiction.

The contradiction obtained proves that $u(b) \leq 0$. However, relation (3.14) then implies $u \equiv 0$, and thus $\ell \in U_{ab}^{-}(h)$.

The theorem is proved.

Proof of Theorem 3.2. First suppose that there exists a function $\gamma \in \widetilde{C}([a, b];]0 + \infty[)$ satisfying relations (3.2) and (3.3). Let u be a nonnegative solution to the problem (1.1), (1.2). We shall show that $u \equiv 0$. Suppose that, on the contrary, there exists $t^* \in [a, b]$ such that

$$u(t^*) > 0. \tag{3.21}$$

Put

$$v(t) = r\gamma(t) - u(t) \quad \text{for} \quad t \in [a, b],$$

where

$$r = \max\left\{ \frac{u(t)}{\gamma(t)} : t \in [a, b] \right\}.$$

According to (3.21), inequality (3.16) holds. It is clear that condition (3.17) is satisfied and there exists $t_0 \in [a, b]$ such that (3.18) is true. Taking now (1.1), (3.2), (3.16), (3.17), and the assumption $-\ell \in P_{ab}$ into account, we obtain

$$v'(t) \leq \ell(v)(t) \leq 0 \quad \text{for a.e.} \quad t \in [a, b]. \tag{3.22}$$

Therefore, on account of (3.17) and (3.18), the relation (3.22) yields

$$0 = v(b) \leq v(t) \leq v(a) \quad \text{for} \quad t \in [a, b]. \tag{3.23}$$

However, using (1.2), (3.3), (3.4), (3.16), (3.23), and the assumptions $h_0, h_1 \in PF_{ab}$, we get

$$0 = \lambda v(b) = r\lambda\gamma(b) - \lambda u(b) > v(a) - h_0(v) + h_1(v) \geq v(a)\,(1 - h_0(1)) \geq 0,$$

a contradiction. The contradiction obtained proves that $u \equiv 0$, and thus $\ell \in U_{ab}^-(h)$.

Now suppose that $\ell \in U_{ab}^-(h)$. We first show that the homogeneous problem (2.1) has only the trivial solution. Let u be a solution to problem (2.1). Using Remark 2.2, we have $0 \in \widetilde{V}_{ab}^-(h)$. Therefore, according to Remark 2.1, the problem

$$\alpha'(t) = \ell([u]_-)(t), \tag{3.24}$$

$$\alpha(a) = h(\alpha) \tag{3.25}$$

has a unique solution α and the relation

$$\alpha(t) \geq 0 \quad \text{for} \quad t \in [a,b] \tag{3.26}$$

holds. From (1.1), (1.2), (3.24), (3.25), and the assumption $-\ell \in P_{ab}$, we get the relations

$$v'(t) = \ell([u]_+)(t) \leq 0 \quad \text{for a.e.} \quad t \in [a,b], \quad v(a) = h(v),$$

where

$$v(t) = u(t) + \alpha(t) \quad \text{for} \quad t \in [a,b]. \tag{3.27}$$

Consequently, using the inclusion $0 \in \widetilde{V}_{ab}^-(h)$, we obtain $v(t) \geq 0$ for $t \in [a,b]$, and thus

$$-u(t) \leq \alpha(t) \quad \text{for} \quad t \in [a,b]. \tag{3.28}$$

Taking now relation (3.26) into account, inequality (3.28) implies

$$[u(t)]_- \leq \alpha(t) \quad \text{for} \quad t \in [a,b].$$

Therefore, in view of the assumption $-\ell \in P_{ab}$, equation (3.24) yields

$$\alpha'(t) \geq \ell(\alpha)(t) \quad \text{for a.e.} \quad t \in [a,b]. \tag{3.29}$$

Consequently, α is a nonnegative function satisfying the conditions (3.25) and (3.29). Hence, the assumption $\ell \in U_{ab}^-(h)$ implies $\alpha \equiv 0$, and thus relation (3.28) yields

$$u(t) \geq 0 \quad \text{for} \quad t \in [a,b]. \tag{3.30}$$

Since $-u$ is also solution to the homogeneous problem (2.1), according to the above-proved we have $-u(t) \geq 0$ for $t \in [a,b]$. Consequently, $u \equiv 0$, i.e., the homogeneous problem (2.1) has only the trivial solution. By virtue of the Fredholm property of the problem (2.2) (see, e.g., [2, 10]), the problem

$$\gamma'(t) = \ell(\gamma)(t), \quad \gamma(a) = h(\gamma) + 1 - h(1) \tag{3.31}$$

has a unique solution γ. Setting

$$\bar{\gamma}(t) = \gamma(t) - 1 \quad \text{for} \quad t \in [a,b],$$

we get from (3.31) the relations

$$\bar{\gamma}'(t) \leq \ell(\bar{\gamma})(t) \quad \text{for a.e.} \quad t \in [a,b], \quad \bar{\gamma}(a) = h(\bar{\gamma}).$$

Now, analogously as above one can show that $\bar{\gamma}(t) \geq 0$ for $t \in [a,b]$. Therefore, in view of the assumption $h(1) > 1$, it follows from (3.31) that γ is a positive function satisfying inequalities (3.2) and (3.3).

The theorem is proved.

Proof of Theorem 3.3. Let u be a nonnegative solution to the problem (1.1), (1.2). We shall show that $u \equiv 0$. Suppose that, on the contrary, $u \not\equiv 0$. Put

$$x_0 = \int_a^b \ell_0(1)(s)\,ds, \quad y_0 = \int_a^b \ell_1(1)(s)\,ds, \tag{3.32}$$

$$M = \max\{u(t) : \ t \in [a,b]\}, \quad m = \min\{u(t) : \ t \in [a,b]\}, \tag{3.33}$$

and choose $t_M, t_m \in [a,b]$ such that

$$u(t_M) = M, \quad u(t_m) = m. \tag{3.34}$$

Obviously,

$$M > 0, \quad m \geq 0, \tag{3.35}$$

and either

$$t_m < t_M \tag{3.36}$$

or

$$t_m \geq t_M. \tag{3.37}$$

First suppose that (3.36) holds. The integrations of (1.1) from a to t_m and from t_M to b, in view of (3.33) – (3.35) and the assumptions $\ell_0, \ell_1 \in P_{ab}$, yield

$$u(a) - m \leq \int_a^{t_m} \ell_1(u)(s)\,ds - \int_a^{t_m} \ell_0(u)(s)\,ds \leq M \int_a^{t_m} \ell_1(1)(s)\,ds, \tag{3.38}$$

$$M - u(b) \leq \int_{t_M}^b \ell_1(u)(s)\,ds - \int_{t_M}^b \ell_0(u)(s)\,ds \leq M \int_{t_M}^b \ell_1(1)(s)\,ds. \tag{3.39}$$

Moreover, on account of (3.33) and the assumptions $h_0, h_1 \in PF_{ab}$, condition (1.2) implies

$$u(a) - \lambda u(b) \geq h_0(u) - h_1(u) \geq m h_0(1) - M h_1(1). \tag{3.40}$$

We get from (3.38) – (3.40) the inequality

$$M\left(\lambda - h_1(1)\right) - m\left(1 - h_0(1)\right) \leq M \left(\int_a^{t_m} \ell_1(1)(s)\,ds + \lambda \int_{t_M}^b \ell_1(1)(s)\,ds \right),$$

i.e.,

$$M\left((\lambda - h_1(1))\min\left\{1, \frac{1}{\lambda}\right\} - y_0\right) \leq m\,(1 - h_0(1))\min\left\{1, \frac{1}{\lambda}\right\}. \tag{3.41}$$

Now suppose that (3.37) holds. The integration of (1.1) from t_M to t_m, in view of (3.33) – (3.35) and the assumptions $\ell_0, \ell_1 \in P_{ab}$, results in

$$M - m \leq \int_{t_M}^{t_m} \ell_1(u)(s)\,ds - \int_{t_M}^{t_m} \ell_0(u)(s)\,ds \leq M \int_{t_M}^{t_m} \ell_1(1)(s)\,ds. \tag{3.42}$$

It is not difficult to verify that, by virtue of (3.5) and (3.42), inequality (3.41) is true.

We have proved that, in both cases (3.36) and (3.37), inequality (3.41) is satisfied. On the other hand, integration of (1.1) from a to b, in view of (3.32), (3.33), and the assumptions ℓ_0, $\ell_1 \in P_{ab}$, yields

$$u(a) - u(b) \leq \int_a^b \ell_1(u)(s)\,ds - \int_a^b \ell_0(u)(s)\,ds \leq My_0 - mx_0,$$

i.e.,

$$mx_0 \leq My_0 + u(b) - u(a). \tag{3.43}$$

Moreover, condition (1.2) implies

$$u(b) - u(a) \leq u(b)\,(1 - \lambda) - h_0(u) + h_1(u), \tag{3.44}$$

$$u(b) - u(a) \leq u(a)\left(\frac{1}{\lambda} - 1\right) - \frac{1}{\lambda}\,h_0(u) + \frac{1}{\lambda}\,h_1(u). \tag{3.45}$$

First suppose that $\lambda \leq 1$. Inequalities (3.43) and (4.44), together with (3.33) and the assumptions $h_0, h_1 \in PF_{ab}$, result in

$$mx_0 \leq My_0 + M\,(1 - \lambda) - mh_0(1) + Mh_1(1). \tag{3.46}$$

Hence, by virtue of (3.6), (3.32), and (3.35), we get from (3.41) and (3.46) the relation $m > 0$ and the inequality

$$(\lambda - h_1(1) - y_0)\,(x_0 + h_0(1)) \leq (y_0 + 1 - \lambda + h_1(1))\,(1 - h_0(1)),$$

which, in view of (3.6) and (3.32), contradicts (3.7).

Now suppose that $\lambda > 1$. The inequalities (3.43) and (3.45), together with (3.33) and the assumptions $h_0, h_1 \in PF_{ab}$, imply

$$mx_0 \leq My_0 - m\,\frac{\lambda - 1}{\lambda} - \frac{1}{\lambda}\,mh_0(1) + \frac{1}{\lambda}\,Mh_1(1). \tag{3.47}$$

Hence, by virtue of (3.6), (3.32), and (3.35), we get from (3.41) and (3.47) the relation $m > 0$ and the inequality

$$\left(1 - \frac{1}{\lambda}\, h_1(1) - y_0\right) \left(x_0 + \frac{\lambda - 1}{\lambda} + \frac{1}{\lambda}\, h_0(1)\right) \le \left(y_0 + \frac{1}{\lambda}\, h_1(1)\right) \frac{1 - h_0(1)}{\lambda},$$

which, in view of (3.6) and (3.32), contradicts (3.7).

The contradictions obtained prove the relation $u \equiv 0$, and thus $\ell \in U_{ab}^{-}(h)$.

The theorem is proved.

Proof of Theorem 3.4. Let u be a nonnegative solution to the problem (1.1), (1.2). We shall show that $u \equiv 0$. Suppose that, on the contrary, $u \not\equiv 0$. Define the numbers x_0, y_0 and M, m by formulae (3.32) and (3.33), respectively, and choose $t_M, t_m \in [a,b]$ such that relations (3.34) hold. Obviously, condition (3.35) is true and either the relation (3.36) or (3.37) is satisfied.

First suppose that (3.36) holds. Analogously to the proof of Theorem 3.3, the validity of inequality (3.41) can be proved. Consequently, in view of (2.3) and (3.35), we get

$$M\left((\lambda - h_1(1)) \min\left\{1, \frac{1}{\lambda}\right\} - y_0\right) \le m\,(\lambda - h_1(1)) \min\left\{1, \frac{1}{\lambda}\right\}. \tag{3.48}$$

Now suppose that (3.37) holds. Analogously to the proof of Theorem 3.3, it can be shown that (3.42) is satisfied. Consequently, it is not difficult to verify that, by virtue of (3.1) and (3.42), inequality (3.48) is true.

We have proved that, in both cases (3.36) and (3.37), inequality (3.48) is satisfied. On the other hand, analogously to the proof of Theorem 3.3, inequalities (3.43) – (3.45) can be derived.

First suppose that

$$\lambda \ge 1, \quad y_0 < \frac{(h(1) - 1)\left(1 - \frac{1}{\lambda} h_1(1)\right)}{\lambda - 1 + h_0(1)}.$$

Relations (3.43) and (3.44), together with (3.33) and the assumptions $h_0, h_1 \in PF_{ab}$, result in

$$m x_0 \le M y_0 - m(\lambda - 1) - m h_0(1) + M h_1(1). \tag{3.49}$$

Hence, by virtue of (3.6), (3.32), and (3.35), we get, from (3.48) and (3.49), the relation $m > 0$ and the inequality

$$\left(1 - \frac{1}{\lambda} h_1(1) - y_0\right) (x_0 + \lambda - 1 + h_0(1)) \le (y_0 + h_1(1)) \left(1 - \frac{1}{\lambda} h_1(1)\right),$$

which, in view of (3.6) and (3.32), contradicts (3.9) with ω given by (3.10).

Now suppose that

$$\lambda \ge 1, \quad y_0 \ge \frac{(h(1) - 1)\left(1 - \frac{1}{\lambda} h_1(1)\right)}{\lambda - 1 + h_0(1)}.$$

The inequalities (3.43) and (3.45), together with (3.33) and the assumptions $h_0, h_1 \in PF_{ab}$, result in

$$m x_0 \le M y_0 - m\, \frac{\lambda - 1}{\lambda} - \frac{1}{\lambda}\, m h_0(1) + \frac{1}{\lambda}\, M h_1(1). \tag{3.50}$$

Hence, by virtue of (3.6), (3.32), and (3.35), we get, from (3.48) and (3.50), the relation $m > 0$ and the inequality

$$\left(1 - \frac{1}{\lambda}\,h_1(1) - y_0\right)\left(x_0 + \frac{\lambda - 1}{\lambda} + \frac{1}{\lambda}\,h_0(1)\right) \le \left(y_0 + \frac{1}{\lambda}\,h_1(1)\right)\frac{\lambda - h_1(1)}{\lambda},$$

which, in view of (3.6) and (3.32), contradicts (3.9) with ω given by (3.10).

Now suppose that

$$\lambda < 1, \quad y_0 < \frac{(h(1) - 1)\,(\lambda - h_1(1))}{h_0(1)}.$$

The inequalities (3.43) and (3.45), together with (3.33) and the assumptions $h_0,\, h_1 \in PF_{ab}$, result in

$$mx_0 \le My_0 + M\,\frac{1 - \lambda}{\lambda} - \frac{1}{\lambda}\,mh_0(1) + \frac{1}{\lambda}\,Mh_1(1). \tag{3.51}$$

Hence, by virtue of (3.6), (3.32), and (3.35), we get, from (3.48) and (3.51), the relation $m > 0$ and the inequality

$$(\lambda - h_1(1) - y_0)\left(x_0 + \frac{1}{\lambda}\,h_0(1)\right) \le \left(y_0 + \frac{1 - \lambda}{\lambda} + \frac{1}{\lambda}\,h_1(1)\right)(\lambda - h_1(1))\,,$$

which, in view of (3.6) and (3.32), contradicts (3.9) with ω given by (3.10).

Finally suppose that

$$\lambda < 1, \quad y_0 \ge \frac{(h(1) - 1)\,(\lambda - h_1(1))}{h_0(1)}.$$

The inequalities (3.43) and (3.44), together with (3.33) and the assumptions $h_0,\, h_1 \in PF_{ab}$, result in

$$mx_0 \le My_0 + M\,(1 - \lambda) - mh_0(1) + Mh_1(1). \tag{3.52}$$

Hence, by virtue of (3.6), (3.32), and (3.35), we get, from (3.48) and (3.52), the relation $m > 0$ and the inequality

$$(\lambda - h_1(1) - y_0)\,(x_0 + h_0(1)) \le (y_0 + 1 - \lambda + h_1(1))\,(\lambda - h_1(1))\,,$$

which, in view of (3.6) and (3.32), contradicts (3.9) with ω given by (3.10).

The contradictions obtained prove the relation $u \equiv 0$ and thus $\ell \in U^-_{ab}(h)$.

The theorem is proved.

Proof of Theorem 3.5. By virtue of the inequality (3.12) and the assumption $\ell \in P_{ab}$, Lemma 3.1 guarantees that $\ell \in \mathcal{S}_{ab}(a)$.

Let u be a nonpositive solution to the problem (1.1), (3.11). It is not difficult to verify that

$$u(a) < 0. \tag{3.53}$$

Indeed, if $u(a) = 0$ then inequality (1.1), in view of the inclusion $\ell \in \mathcal{S}_{ab}(a)$, yields $u(t) \ge 0$ for $t \in [a, b]$. Hence we get $u \equiv 0$, which contradicts relation (3.11).

Put

$$w(t) = \gamma(a)u(t) - u(a)\gamma(t) \quad \text{for} \quad t \in [a, b].$$

We immediately obtain, from (1.1), (3.12), (3.53), and the assumption $\gamma(a) > 0$, the relations

$$w'(t) \geq \ell(w)(t) \quad \text{for a.e.} \quad t \in [a, b], \tag{3.54}$$

$$w(a) = 0. \tag{3.55}$$

Therefore, the inclusion $\ell \in \mathcal{S}_{ab}(a)$ implies

$$w(t) \geq 0 \quad \text{for} \quad t \in [a, b]. \tag{3.56}$$

On the other hand, it follows from (3.11), (3.13), (3.53), (3.56), and the assumptions $\gamma(a) > 0$ and $h \in PF_{ab}$ that

$$w(a) > h(w) \geq 0,$$

which contradicts relation (3.55).

The contradiction obtained proves that there is no nonpositive solution to the problem (1.1), (3.11), and thus $\ell \in \widetilde{U}_{ab}^{+}(h)$.

The theorem is proved.

4. Main results. In this sections, we give main results of the paper, which are efficient conditions under which the operator ℓ belongs to the set $\widetilde{V}_{ab}^{-}(h)$. The results are formulated in Sections 4.1 – 4.3, their proofs are presented in Section 4.5.

We first give a rather theoretical statement.

Proposition 4.1. *Let $h \in F_{ab}$. Then $\ell \in \widetilde{V}_{ab}^{-}(h)$ if and only if $\ell \in U_{ab}^{-}(h)$ and there exists $\bar{\ell} \in P_{ab}$ such that $\ell + \bar{\ell} \in \widetilde{V}_{ab}^{-}(h)$.*

Now we present a general result.

Theorem 4.1. *Let $\ell \in S_{ab}(b) \cap \widetilde{U}_{ab}^{+}(h_0)$. Then $\ell \in \widetilde{V}_{ab}^{-}(h)$ if and only if there exists a function $\gamma \in \widetilde{C}([a, b]; \mathbb{R}_+)$ satisfying the conditions (3.2) and (3.3).*

4.1. The case $\ell \in P_{ab}$. The following statements can be proved in the case where $\ell \in P_{ab}$.

Theorem 4.2. *Let $\ell \in P_{ab} \cap \widetilde{U}_{ab}^{+}(h_0)$ be a b-Volterra operator and condition (3.4) hold. Then $\ell \in \widetilde{V}_{ab}^{-}(h)$ if and only if $\ell \in S_{ab}(b)$.*

Corollary 4.1. *Let $\ell \in P_{ab}$ be a b-Volterra operator and condition (3.4) be fulfilled. If, moreover, there exists a function $\gamma \in \widetilde{C}([a, b];]0, +\infty[)$ such that the conditions (3.12) and*

$$\gamma(a) \geq h_0(\gamma) \tag{4.1}$$

hold, then $\ell \in \widetilde{V}_{ab}^{-}(h)$.

Corollary 4.2. *Let $\ell \in P_{ab}$ be a b-Volterra operator and*

$$h(1) > 1, \quad h_0(1) < 1. \tag{4.2}$$

Assume that

$$h_0(\varphi_1) > 0 \tag{4.3}$$

and there exist $m, k \in \mathbb{N}$ *such that* $m > k$ *and*

$$\varrho_m(t) \le \varrho_k(t) \quad \textit{for} \quad t \in [a,b], \tag{4.4}$$

where $\varrho_1 \equiv 1$ *and*

$$\varrho_{i+1}(t) \stackrel{\mathrm{df}}{=} \frac{h_0(\varphi_i)}{1 - h_0(1)} + \varphi_i(t) \quad \textit{for} \quad t \in [a,b], \quad i \in \mathbb{N}, \tag{4.5}$$

$$\varphi_i(t) \stackrel{\mathrm{df}}{=} \int_a^t \ell(\varrho_i)(s)ds \quad \textit{for} \quad t \in [a,b], \quad i \in \mathbb{N}. \tag{4.6}$$

Then $\ell \in \widetilde{V}_{ab}^-(h)$.

Remark 4.1. It follows from Corollary 4.2 (for $k = 1$ and $m = 2$) that if $\ell \in P_{ab}$ is a b-Volterra operator, condition (4.2) is fulfilled, and relation (4.3) holds with φ_1 given by formula (4.6), then $\ell \in \widetilde{V}_{ab}^-(h)$ provided that

$$\int_a^b \ell(1)(s)ds \le 1 - h_0(1).$$

Corollary 4.3. *Let* $\ell \in P_{ab}$ *be a* b*-Volterra operator and condition (4.2) be fulfilled. Then the operator* ℓ *belongs to the set* $\widetilde{V}_{ab}^-(h)$ *provided that* $\ell \in \widetilde{V}_{ab}^+(h_0)$.

Remark 4.2. Recall that efficient conditions guaranteeing the validity of the inclusion $\ell \in$ $\in \widetilde{V}_{ab}^+(h_0)$ are stated in [14].

4.2. The case $-\ell \in P_{ab}$. The following statements can be proved in the case where $-\ell \in$ $\in P_{ab}$.

Theorem 4.3. *Let* $-\ell \in P_{ab}$ *and condition (3.4) be fulfilled. Then* $\ell \in \widetilde{V}_{ab}^-(h)$ *if and only if* $\ell \in U_{ab}^-(h)$.

Corollary 4.4. *Let* $-\ell \in P_{ab}$ *and condition (3.4) be fulfilled. Assume that at least one of the following conditions is satisfied:*

(a) there exist $m, k \in \mathbb{N}$ *and a constant* $\delta \in [0,1[$ *such that* $m > k$ *and*

$$\varrho_m(t) \le \delta \varrho_k(t) \quad \textit{for} \quad t \in [a,b], \tag{4.7}$$

where $\varrho_1 \equiv 1$, $\varrho_{i+1} \equiv \vartheta(\varrho_i)$ *for* $i \in \mathbb{N}$, *and*

$$\vartheta(v)(t) \stackrel{\mathrm{df}}{=} \frac{\widetilde{h}(v)}{h(1) - 1} - \frac{z(v)(a)}{h(1) - 1} - z(v)(t) \quad \textit{for} \quad t \in [a,b], \quad v \in C([a,b];\mathbb{R}), \tag{4.8}$$

$$\widetilde{h}(v) \stackrel{\mathrm{df}}{=} h(z(v)), \quad z(v)(t) \stackrel{\mathrm{df}}{=} \int_t^b \ell(v)(s)\, ds \quad \textit{for} \quad t \in [a,b], \quad v \in C([a,b];\mathbb{R}); \tag{4.9}$$

(b) there exists $\bar{\ell} \in P_{ab}$ such that

$$h(z_0) > z_0(a), \tag{4.10}$$

$$z_0(a)\,(1 - h(z_1)) + h(z_0)z_1(a) < h(z_0), \tag{4.11}$$

and the inequality

$$\ell(1)(t)\vartheta(v)(t) - \ell(\vartheta(v))(t) \le \bar{\ell}(v)(t) \quad \textit{for a.e.} \quad t \in [a,b] \tag{4.12}$$

holds on the set $\{v \in C([a,b];\mathbb{R}_+) : v(a) = h(v)\}$, *where the operator* ϑ *is defined by formulae (4.8) and (4.9),*

$$z_0(t) = \exp\left(\int_t^b |\ell(1)(s)|ds\right) \quad \textit{for} \quad t \in [a,b], \tag{4.13}$$

$$z_1(t) = \int_t^b \bar{\ell}(1)(s)\exp\left(\int_t^s |\ell(1)(\xi)|\,d\xi\right)ds \quad \textit{for} \quad t \in [a,b]. \tag{4.14}$$

Then $\ell \in \widetilde{V}_{ab}^-(h)$.

Remark 4.3. Let $-\ell \in P_{ab}$ and the condition (3.4) be fulfilled. Then it follows from Corollary 4.4(a) (for $k = 1$ and $m = 2$) that $\ell \in \widetilde{V}_{ab}^-(h)$ provided

$$\int_a^b |\ell(1)(s)|\,ds < 1 - \frac{1 + h_1(1)}{\lambda + h_0(1)}.$$

Moreover, it follows from Corollary 4.4(b) (with $\bar{\ell} \equiv 0$) that $\ell \in \widetilde{V}_{ab}^-(h)$ provided that ℓ is a b-Volterra operator and the condition (4.10)

$$z_0(a) < h(z_0)$$

holds, where the function z_0 is given by formula (4.13).

4.3. The case $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in P_{ab}$. The following statements can be proved in the case where the operator is regular, i.e., admits the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in P_{ab}$.

Theorem 4.4. *Let* $\ell = \ell_0 - \ell_1$, *where* $\ell_0, \ell_1 \in P_{ab}$,

$$h_1(1) < \lambda, \quad h_0(1) \le 1, \tag{4.15}$$

and

$$\int_a^b \ell_0(1)(s)ds \le (1 - h_0(1))\min\left\{1, \frac{1}{\lambda}\right\}. \tag{4.16}$$

Then $\ell \in \widetilde{V}^-_{ab}(h)$ *if and only if* $\ell \in U^-_{ab}(h)$.

Theorem 4.5. *Let* $\ell = \ell_0 - \ell_1$, *where* $\ell_0, \ell_1 \in P_{ab}$, $h \in F_{ab}$, *and condition (2.3) hold. If*

$$\ell_0 \in \widetilde{V}^-_{ab}(h), \quad -\ell_1 \in \widetilde{V}^-_{ab}(h), \tag{4.17}$$

then $\ell \in \widetilde{V}^-_{ab}(h)$.

4.4. Further remarks. Introduce the operator $\varphi : C([a,b];\mathbb{R}) \to C([a,b];\mathbb{R})$ by setting

$$\varphi(w)(t) \stackrel{\text{df}}{=} w(a+b-t) \quad \text{for} \quad t \in [a,b], \quad w \in C([a,b];\mathbb{R}).$$

Let

$$\widehat{\ell}(w)(t) \stackrel{\text{df}}{=} -\ell\left(\varphi(w)\right)(a+b-t) \quad \text{for a.e.} \quad t \in [a,b] \quad \text{and all} \quad w \in C([a,b];\mathbb{R}),$$

$$\widehat{h}(w) \stackrel{\text{df}}{=} \frac{1}{\lambda} v(b) - \frac{1}{\lambda} h_0\left(\varphi(w)\right) + \frac{1}{\lambda} h_1\left(\varphi(w)\right) \quad \text{for} \quad w \in C([a,b];\mathbb{R}).$$

It is clear that if u is a solution to the problem (1.1), (1.2) then the function $v \stackrel{\text{df}}{=} -\varphi(u)$ is a solution to the problem

$$v'(t) \geq \widehat{\ell}(v)(t), \quad v(a) \geq \widehat{h}(v), \tag{4.18}$$

and vice versa, if v is a solution to the problem (4.18) then the function $u \stackrel{\text{df}}{=} -\varphi(v)$ is a solution to the problem (1.1), (1.2).

Consequently, the relation

$$\ell \in \widetilde{V}^+_{ab}(h) \Leftrightarrow \widehat{\ell} \in \widetilde{V}^-_{ab}\left(\widehat{h}\right)$$

holds.

Therefore, efficient conditions guaranteeing the validity of the inclusion $\ell \in \widetilde{V}^+_{ab}(h)$ can be immediately derived from the results stated in Sections 4.1 – 4.3. For example, Corollary 4.1 of Section 4.1 immediately yields the following.

Corollary 4.5. *Let* $-\ell \in P_{ab}$ *be an* a*-Volterra operator and*

$$h(1) < 1, \quad h_1(1) \leq \lambda.$$

If, moreover, there exists a function $\gamma \in \widetilde{C}([a,b];\,]0,+\infty[)$ *such that the conditions (3.2) and*

$$\lambda\gamma(b) \geq h_1(\gamma)$$

hold then $\ell \in \widetilde{V}^+_{ab}(h)$.

4.5. Proofs. To prove statements formulated in Sections 4 and 4.5 we need the following lemmas.

Lemma 4.1. *Let* $h \in F_{ab}$ *and* $\ell \in U^-_{ab}(h)$. *Then* $\ell + \bar{\ell} \in U^-_{ab}(h)$ *for every* $\bar{\ell} \in P_{ab}$.

Proof. It follows immediately from Definition 3.1.

Lemma 1. ([6], Theorem 1.6). *Let $\ell \in P_{ab}$ be a b-Volterra operator and there exist a function $\gamma \in \widetilde{C}([a,b];\mathbb{R}_+)$ satisfying the conditions (3.12) and*

$$\gamma(t) > 0 \quad \textit{for} \quad t \in]a,b].$$

Then $\ell \in \mathcal{S}_{ab}(b)$.

Proof of Proposition 4.1. First suppose that $\ell \in \widetilde{V}_{ab}^-(h)$. Then, according to Remark 3.1, we have $\ell \in U_{ab}^-(h)$. Moreover, it is clear that the inclusion $\ell + \bar{\ell} \in \widetilde{V}_{ab}^-(h)$ is true with $\bar{\ell} \equiv 0$.

Now suppose that $\ell \in U_{ab}^-(h)$ and there exists an operator $\bar{\ell} \in P_{ab}$ such that $\ell + \bar{\ell} \in \widetilde{V}_{ab}^-(h)$. Let u be a solution to the problem (1.1), (1.2). We shall show that the function u is nonpositive.

According to the assumption $\ell + \bar{\ell} \in \widetilde{V}_{ab}^-(h)$ and Remark 2.1, the problem

$$\alpha'(t) = \left(\ell + \bar{\ell}\right)(\alpha)(t) - \bar{\ell}([u]_+)(t), \tag{4.19}$$

$$\alpha(a) = h(\alpha) \tag{4.20}$$

has a unique solution α and the relation

$$\alpha(t) \geq 0 \quad \text{for} \quad t \in [a,b] \tag{4.21}$$

holds. From (1.1), (1.2), (4.19), (4.20), and the assumption $\bar{\ell} \in P_{ab}$, we get the relations

$$v'(t) \geq (\ell + \bar{\ell})(v)(t) \quad \text{for a.e.} \quad t \in [a,b], \quad v(a) \geq h(v),$$

where

$$v(t) = u(t) - \alpha(t) \quad \text{for} \quad t \in [a,b]. \tag{4.22}$$

Consequently, using the inclusion $\ell + \bar{\ell} \in \widetilde{V}_{ab}^-(h)$, we obtain $v(t) \leq 0$ for $t \in [a,b]$, and thus

$$u(t) \leq \alpha(t) \quad \text{for} \quad t \in [a,b]. \tag{4.23}$$

Taking now relation (4.21) into account, inequality (4.23) implies

$$[u(t)]_+ \leq \alpha(t) \quad \text{for} \quad t \in [a,b].$$

Therefore, in view of the assumption $\bar{\ell} \in P_{ab}$, equation (4.19) yields

$$\alpha'(t) \geq (\ell + \bar{\ell})(\alpha)(t) - \bar{\ell}(\alpha)(t) = \ell(\alpha)(t) \quad \text{for a.e.} \quad t \in [a,b]. \tag{4.24}$$

Consequently, α is a nonnegative function satisfying the conditions (4.20) and (4.24). Hence, the assumption $\ell \in U_{ab}^-(h)$ implies $\alpha \equiv 0$, and thus relation (4.23) yields

$$u(t) \leq 0 \quad \text{for} \quad t \in [a,b]. \tag{4.25}$$

Therefore, the inclusion $\ell \in \widetilde{V}_{ab}^-(h)$ is true.

The proposition is proved.

Proof of Theorem 4.1. First suppose that $\ell \in \widetilde{V}_{ab}^{-}(h)$. According to Remark 2.1, the problem

$$\gamma'(t) = \ell(\gamma)(t), \quad \gamma(a) = h(\gamma) - 1 \tag{4.26}$$

has a unique solution γ and, moreover, the relation

$$\gamma(t) \geq 0 \quad \text{for} \quad t \in [a, b] \tag{4.27}$$

holds. Obviously, the function γ satisfies the conditions (3.2) and (3.3).

Now suppose that there exists a function $\gamma \in \widetilde{C}([a, b]; \mathbb{R}_+)$ satisfying the conditions (3.2) and (3.3). We shall show that $\ell \in \widetilde{V}_{ab}^{-}(h)$. Let u be a solution to the problem (1.1), (1.2). It is clear that either

$$u(b) > 0 \tag{4.28}$$

or

$$u(b) \leq 0. \tag{4.29}$$

Assume that condition (4.28) holds. Put

$$w(t) = \gamma(b)u(t) - u(b)\gamma(t) \quad \text{for} \quad t \in [a, b].$$

We get, from (1.1), (3.2), and (4.28), the relations

$$w'(t) \geq \ell(w)(t) \quad \text{for a.e.} \quad t \in [a, b], \tag{4.30}$$

$$w(b) = 0. \tag{4.31}$$

Therefore, the assumption $\ell \in \mathcal{S}_{ab}(b)$ yields

$$w(t) \leq 0 \quad \text{for} \quad t \in [a, b]. \tag{4.32}$$

On the other hand, it follows from (1.2), (3.3), (4.28), (4.31), (4.32), and the assumption $h_1 \in \in PF_{ab}$ that

$$w(a) > \lambda w(b) + h_0(w) - h_1(w) \geq h_0(w).$$

Consequently, the function w is a nonpositive solution to the problem

$$w'(t) \geq \ell(w)(t), \quad w(a) > h_0(w),$$

which contradicts the assumption $\ell \in \widetilde{U}_{ab}^{+}(h_0)$.

The contradiction obtained proves that u satisfies condition (4.29). Taking now (1.1) and (4.29) into account, the assumption $\ell \in \mathcal{S}_{ab}(b)$ implies relation (4.25), and thus $\ell \in \widetilde{V}_{ab}^{-}(h)$.

The theorem is proved.

Proof of Theorem 4.2. First suppose that $\ell \in \mathcal{S}_{ab}(b)$. It is clear that, in view of (2.3) and the assumption $\ell \in P_{ab}$, the function $\gamma \equiv 1$ satisfies the conditions (3.2) and (3.3). Hence, by virtue of Theorem 4.1, we get $\ell \in \widetilde{V}_{ab}^{-}(h)$.

Now assume that $\ell \in \widetilde{V}_{ab}^-(h)$. Suppose that, on the contrary, $\ell \notin \mathcal{S}_{ab}(b)$. Then there exists a solution u to the inequality (1.1) satisfying the relations $u(b) = c$ and

$$u(t_0) > 0, \tag{4.33}$$

where $c \leq 0$ and $t_0 \in]a, b[$. According to the assumption $\ell \in \widetilde{V}_{ab}^-(h)$ and Remark 2.1, the problem

$$u_0'(t) = \ell(u_0)(t), \tag{4.34}$$

$$u_0(a) = h(u_0) - 1 \tag{4.35}$$

has a unique solution u_0 and, moreover, the relation

$$u_0(t) \geq 0 \quad \text{for} \quad t \in [a, b] \tag{4.36}$$

holds. It is not difficult to verify that

$$u_0(b) > 0. \tag{4.37}$$

Indeed, suppose that (4.37) does not hold. Then, in view of (4.36), we find $u_0(b) = 0$. Hence, by virtue of (4.36) and the assumption $h_1 \in PF_{ab}$, the condition (4.35) implies

$$u_0(a) = \lambda u_0(b) + h_0(u_0) - h_1(u_0) - 1 < h_0(u_0),$$

which, together with (4.34) and (4.36), contradicts the assumption $\ell \in \widetilde{U}_{ab}^+(h_0)$. The contradiction obtained proves the validity of relation (4.37).

Since $\ell \notin \mathcal{S}_{ab}(b)$, if follows from Lemma 4.2, on account of (4.34), (4.36), (4.37), and the assumption $\ell \in P_{ab}$, that there exists $a_0 \in]a, b[$ such that

$$u_0(t) = 0 \quad \text{for} \quad t \in [a, a_0], \tag{4.38}$$

$$u_0(t) > 0 \quad \text{for} \quad t \in]a_0, b]. \tag{4.39}$$

Denote by $\widetilde{\ell}$ the restriction of the operator ℓ to the space $C([a_0, b]; \mathbb{R})$. By virtue of the conditions (4.34) and (4.39), we get

$$u_0'(t) = \widetilde{\ell}(u_0)(t) \quad \text{for a.e.} \quad t \in [a_0, b], \quad u_0(t) > 0 \quad \text{for} \quad t \in]a_0, b],$$

and thus Lemma 4.2 guarantees validity of the inclusion $\widetilde{\ell} \in \mathcal{S}_{a_0 b}(b)$. It follows from inequality (1.1) and condition (4.34) that

$$w'(t) \geq \widetilde{\ell}(w)(t) \quad \text{for a.e.} \quad t \in [a_0, b], \quad w(b) = 0, \tag{4.40}$$

where

$$w(t) = u(t) - \frac{c}{u_0(b)} u_0(t) \quad \text{for} \quad t \in [a_0, b].$$

Since $\widetilde{\ell} \in \mathcal{S}_{a_0 b}(b)$, relations (4.40) result in $w(t) \leq 0$ for $t \in [a_0, b]$, i.e.,

$$u(t) \leq \frac{c}{u_0(b)}\, u_0(t) \quad \text{for} \quad t \in [a_0, b].$$

From the latter inequality, (4.33), and (4.39) we get

$$a < t_0 < a_0. \tag{4.41}$$

Now we put

$$v(t) = u(t) + (u(a) - h(u))\, u_0(t) \quad \text{for} \quad t \in [a, b]. \tag{4.42}$$

It is clear that

$$v'(t) \geq \ell(v)(t) \quad \text{for a.e.} \quad t \in [a, b], \quad v(a) = h(v).$$

Consequently, by virtue of the assumption $\ell \in \widetilde{V}_{ab}^{-}(h)$, the inequality $v(t) \leq 0$ holds for $t \in$ $\in [a, b]$. Finally, in view of (4.38) and (4.41), relation (4.42) yields

$$0 \geq v(t_0) = u(t_0) + (u(a) - h(u))\, u_0(t_0) = u(t_0),$$

which contradicts the inequality (4.33).

The contradiction obtained proves the validity of the inclusion $\ell \in \mathcal{S}_{ab}(b)$.

The theorem is proved.

Proof of Corollary 4.1. According to Lemma 4.2, inequality (3.12) yields $\ell \in \mathcal{S}_{ab}(b)$. On the other hand, by virtue of the conditions (3.12), (3.4), and (4.1), using Theorem 3.5 we get $\ell \in \widetilde{U}_{ab}^{+}(h_0)$. Consequently, the assertion of the corollary follows from Theorem 4.2.

Proof of Corollary 4.2. Put

$$\gamma(t) = \sum_{j=k+1}^{m} \varrho_j(t) \quad \text{for} \quad t \in [a, b].$$

In view of condition (4.3), where the function φ_1 is given by the formula (4.6), we get $\gamma \in$ $\in \widetilde{C}([a, b];]0, +\infty[\,)$. On the other hand, by virtue of the relations (4.4) – (4.6) and the assumption $\ell \in P_{ab}$, it is clear that the function γ satisfies the conditions (3.12) and (4.1). Consequently, the assumptions of Corollary 4.1 are satisfied.

Proof of Corollary 4.3. According to the assumption $\ell \in P_{ab} \cap \widetilde{V}_{ab}^{+}(h_0)$, Theorem 2.1 in [14] guarantees that there exists a function $\gamma \in \widetilde{C}([a, b];]0, +\infty[)$ satisfying the conditions (3.12) and

$$\gamma(a) > h_0(\gamma).$$

Consequently, the assumptions of Corollary 4.1 are satisfied.

Proof of Theorem 4.3. The validity of the theorem follows immediately from Proposition 4.1 (with $\bar{\ell} \equiv -\ell$) and Remark 2.2.

Proof of Corollary 4.4. (a) It is not difficult to verify that the function γ, defined by the formula

$$\gamma(t) = \sum_{j=1}^{m} \varrho_j(t) - \delta \sum_{j=1}^{k} \varrho_j(t) \quad \text{for} \quad t \in [a,b],$$

is positive and satisfies the conditions (3.2) and (3.3). Consequently, the assertion of the corollary follows from Theorems 3.2 and 4.3.

(b) According to relations (4.10) and (4.11), there exists $\varepsilon > 0$ such that

$$\gamma_0 \left(\varepsilon - h(z_1)\right) z_0(a) + \gamma_0 h(z_0) z_1(a) \leq 1, \tag{4.43}$$

where $\gamma_0 = \left(h(z_0) - z_0(a)\right)^{-1}$. Put

$$\gamma(t) = \gamma_0 \Bigg[(\varepsilon - h(z_1)) \exp\left(\int_t^b |\ell(1)(s)|\, ds\right) + \\ + \exp\left(\int_a^b |\ell(1)(s)|\, ds\right) \int_a^t \bar{\ell}(1)(s) \exp\left(\int_t^s |\ell(1)(\xi)|\, d\xi\right) ds + \\ + h(z_0) \int_t^b \bar{\ell}(1)(s) \exp\left(\int_t^s |\ell(1)(\xi)|\, d\xi\right) ds \Bigg] \quad \text{for} \quad t \in [a,b],$$

where the functions z_0 and z_1 are defined by the formulae (4.13) and (4.14), respectively. It is not difficult to verify that γ is a solution to the problem

$$\gamma'(t) = \ell(1)(t)\gamma(t) - \bar{\ell}(1)(t), \tag{4.44}$$

$$\gamma(a) = h(\gamma) - \varepsilon. \tag{4.45}$$

Using the inequalities (3.4) and (4.10), and the assumptions $h_0, h_1 \in PF_{ab}$, it follows from the definition of the function γ that $\gamma(b) > 0$, and thus the relation $\gamma(t) > 0$ holds for $t \in [a,b]$. Since $-\ell, \bar{\ell} \in P_{ab}$, equality (4.44) implies $\gamma(t) \leq \gamma(a)$ for $t \in [a,b]$. Taking now inequality (4.43) into account, the conditions (4.44) and (4.45) result in

$$\gamma'(t) \leq \ell(1)(t)\gamma(t) - \bar{\ell}(\gamma)(t) \quad \text{for a.e.} \quad t \in [a,b], \quad \gamma(a) < h(\gamma).$$

Consequently, Theorem 4.3 guarantees validity of inclusion

$$\widetilde{\ell} \in \widetilde{V}_{ab}^{-}(h), \tag{4.46}$$

where

$$\widetilde{\ell}(v)(t) \stackrel{\text{df}}{=} \ell(1)(t)v(t) - \bar{\ell}(v)(t) \quad \text{for a.e.} \quad t \in [a,b] \quad \text{and all} \quad v \in C([a,b];\mathbb{R}).$$

Since $-\ell \in P_{ab}$, in order to prove the inclusion $\ell \in \widetilde{V}_{ab}^{-}(h)$ it is sufficient to show that $\ell \in U_{ab}^{-}(h)$ (see Theorem 4.3). Hence, let u be a nonnegative solution to the problem (1.1), (1.2). We shall show that $u \equiv 0$. Put

$$w(t) = \vartheta(v)(t) \quad \text{for} \quad t \in [a,b], \tag{4.47}$$

where the operator ϑ is defined by the formulae (4.8) and (4.6), and

$$v(t) = u(t) + \frac{u(a) - h(u)}{h(1) - 1} \quad \text{for} \quad t \in [a,b].$$

Obviously,

$$v(t) \geq u(t) \quad \text{for} \quad t \in [a,b]$$

and

$$v'(t) \geq \ell(v)(t) \quad \text{for a.e.} \quad t \in [a,b], \quad v(a) = h(v), \tag{4.48}$$

$$w'(t) = \ell(v)(t) \quad \text{for a.e.} \quad t \in [a,b], \quad w(a) = h(w). \tag{4.49}$$

It follows from (4.48) and (4.49) that

$$y'(t) \geq 0 \quad \text{for a.e.} \quad t \in [a,b], \quad y(a) = h(y),$$

where $y(t) = v(t) - w(t)$ for $t \in [a,b]$. By virtue of Remark 2.2, we have $0 \in \widetilde{V}_{ab}^{-}(h)$. Consequently, $y(t) \leq 0$ for $t \in [a,b]$, i.e.,

$$0 \leq u(t) \leq v(t) \leq w(t) \quad \text{for} \quad t \in [a,b]. \tag{4.50}$$

On the other hand, using (4.12), (4.47) – (4.50), and the assumptions $-\ell, \bar{\ell} \in P_{ab}$, we get

$$\begin{aligned} w'(t) &= \ell(v)(t) \geq \ell(1)(t)w(t) + \ell(w)(t) - \ell(1)(t)w(t) = \\ &= \ell(1)(t)w(t) + \ell(\vartheta(v))(t) - \ell(1)(t)\vartheta(v)(t) \geq \ell(1)(t)w(t) - \bar{\ell}(v)(t) \geq \\ &\geq \ell(1)(t)w(t) - \bar{\ell}(w)(t) = \widetilde{\ell}(w)(t) \quad \text{for a.e.} \quad t \in [a,b]. \end{aligned}$$

Taking now (4.46) and (4.49) into account, we find $w(t) \leq 0$ for $t \in [a,b]$. Hence, the relation (4.50) implies $u \equiv 0$, and thus $\ell \in \widetilde{V}_{ab}^{-}(h)$.

The corollary is proved.

Proof of Theorem 4.4. Assume that $\ell \in U_{ab}^{-}(h)$. Since $\ell_1 \in P_{ab}$, Lemma 4.1 guarantees that $\ell_0 = \ell + \ell_1 \in U_{ab}^{-}(h)$.

We shall show that $\ell_0 \in \widetilde{V}_{ab}^{-}(h)$. Assume that, on the contrary, there exists a solution u to the inequality

$$u'(t) \geq \ell_0(u)(t) \tag{4.51}$$

satisfying condition (1.2), which is not nonpositive on the interval $[a,b]$. Then, in view of the above-proved inclusion $\ell_0 \in U_{ab}^{-}(h)$, it is clear that u assumes both positive and negative values, i.e.,

$$M > 0, \quad m > 0, \tag{4.52}$$

where
$$M = \max\{u(t) : t \in [a,b]\}, \quad m = -\min\{u(t) : t \in [a,b]\}. \tag{4.53}$$
Now we choose $t_M, t_m \in [a,b]$ such that
$$u(t_M) = M, \quad u(t_m) = -m. \tag{4.54}$$
Obviously, either
$$t_M < t_m \tag{4.55}$$
or
$$t_M > t_m. \tag{4.56}$$

If (4.55) holds then the integration of (4.51) from t_M to t_m, in view of (4.52), (4.53) and the assumption $\ell_0 \in P_{ab}$, results in
$$M + m \le -\int_{t_M}^{t_m} \ell_0(u)(s)\,ds \le m\int_a^b \ell_0(1)(s)\,ds.$$
Hence, by virtue of (4.16) and the second inequality in (4.52), we get $M \le 0$, which contradicts the first inequality in (4.52).

If (4.56) holds then the integrations of (4.51) from a to t_m and from t_M to b, in view of (4.52), (4.53) and the assumption $\ell_0 \in P_{ab}$, yield
$$u(a) + m \le -\int_a^{t_m} \ell_0(u)(s)\,ds \le m\int_a^{t_m} \ell_0(1)(s)\,ds, \tag{4.57}$$
$$M - u(b) \le -\int_{t_M}^b \ell_0(u)(s)\,ds \le m\int_{t_M}^b \ell_0(1)(s)\,ds. \tag{4.58}$$
On the other hand, the condition (1.2), on account of (4.53) and the assumptions $h_0, h_1 \in PF_{ab}$, implies
$$u(a) - \lambda u(b) \ge h_0(u) - h_1(u) \ge -mh_0(1) - Mh_1(1). \tag{4.59}$$
Now we get, from (4.57) – (4.59), the inequality
$$M(\lambda - h_1(1)) + m(1 - h_0(1)) \le m\left(\int_a^{t_m} \ell_0(1)(s)\,ds + \lambda\int_{t_M}^b \ell_0(1)(s)\,ds\right),$$
i.e.,
$$M(\lambda - h_1(1))\min\left\{1, \frac{1}{\lambda}\right\} + m(1 - h_0(1))\min\left\{1, \frac{1}{\lambda}\right\} \le m\int_a^b \ell_0(1)(s)\,ds.$$

Hence, by virtue of (3.1), (4.16), and the second inequality in (4.52), we find $M \le 0$, which contradicts the first inequality in (4.52).

The contradictions obtained prove the validity of the inclusion $\ell_0 \in \widetilde{V}_{ab}^-(h)$.

Now we put $\bar{\ell} \equiv \ell_1$. Since $\ell \in U_{ab}^-(h)$ and $\ell + \bar{\ell} = \ell_0 \in \widetilde{V}_{ab}^-(h)$, Proposition 4.1 yields $\ell \in \widetilde{V}_{ab}^-(h)$.

The validity of the converse implication follows immediately from Remark 3.1.

The theorem is proved.

Proof of Theorem 4.5. It is easy to verify that $\ell \in U_{ab}^-(h)$. Indeed, the assumption $-\ell_1 \in \widetilde{V}_{ab}^-(h)$ yields $-\ell_1 \in U_{ab}^-(h)$ (see Remark 3.1), and thus, in view of Lemma 4.1, we get $\ell = -\ell_1 + \ell_0 \in U_{ab}^-(h)$.

Now we put $\bar{\ell} \equiv \ell_1$. Then it is clear that $\bar{\ell} \in P_{ab}$ and $\ell + \bar{\ell} = \ell_0 \in \widetilde{V}_{ab}^-(h)$. Consequently, Proposition 4.1 yields $\ell \in \widetilde{V}_{ab}^-(h)$.

The theorem is proved.

5. Differential inequalities with argument deviations. In this section, we give some corollaries of the main results for operators with argument deviations. More precisely, efficient criteria are proved below for validity of the inclusion $\ell \in \widetilde{V}_{ab}^-(h)$ in the case where the operator ℓ is given by one of the following formulae:

$$\ell(v)(t) \stackrel{\mathrm{df}}{=} p(t)v(\tau(t)) \quad \text{for a.e.} \quad t \in [a,b] \quad \text{and all} \quad v \in C([a,b];\mathbb{R}), \tag{5.1}$$

$$\ell(v)(t) \stackrel{\mathrm{df}}{=} -g(t)v(\mu(t)) \quad \text{for a.e.} \quad t \in [a,b] \quad \text{and all} \quad v \in C([a,b];\mathbb{R}), \tag{5.2}$$

and

$$\ell(v)(t) \stackrel{\mathrm{df}}{=} p(t)v(\tau(t)) - g(t)v(\mu(t)) \quad \text{for a.e.} \quad t \in [a,b] \quad \text{and all} \quad v \in C([a,b];\mathbb{R}). \tag{5.3}$$

Here we suppose that $p, g \in L([a,b];\mathbb{R}_+)$ and $\tau, \mu : [a,b] \to [a,b]$ are measurable functions.

Throughout this section, the following notation is used:

$$\mu_* = \operatorname{ess\,inf}\{\mu(t) : t \in [a,b]\}, \quad \tau^* = \operatorname{ess\,sup}\{\tau(t) : t \in [a,b]\}, \tag{5.4}$$

and

$$\alpha(t) = \exp\left(\int_t^b g(s)\,ds\right), \quad \beta(t) = \exp\left(\int_a^t p(s)\,ds\right) \quad \text{for} \quad t \in [a,b]. \tag{5.5}$$

We first formulate all the results, their proofs are given later, in Section 5.1 below.

Theorem 5.1. *Let condition (3.1) be fulfilled and*

$$h(1) < 1. \tag{5.6}$$

Assume that

$$0 < \int_a^b p(s)\,ds \le (1 - h_0(1)) \min\left\{1, \frac{1}{\lambda}\right\} \tag{5.7}$$

and

$$\operatorname{ess\,inf}\left\{\int\limits_t^{\tau(t)} p(s)\,ds \,:\, t\in[a,b]\right\} > \eta_*, \tag{5.8}$$

where

$$\eta_* = \inf\left\{\frac{1}{x}\ln\frac{x\beta^x(\tau^*)}{\beta^x(\tau^*)+(h(\beta^x)-1)\,(1-h(1))^{-1}} \,:\, x>0,\ h(\beta^x)>1\right\}. \tag{5.9}$$

Then the operator ℓ given by formula (5.1) belongs to the set $\widetilde{V}_{ab}^{-}(h)$.

Corollary 5.1. *Let the inequalities (3.1) and (5.6) be fulfilled. Assume that condition (5.7) is satisfied and*

$$\operatorname{ess\,inf}\left\{\int\limits_t^{\tau(t)} p(s)\,ds \,:\, t\in[a,b]\right\} > \xi_*, \tag{5.10}$$

where

$$\xi_* = \inf\left\{\frac{||p||_L}{y}\ln\frac{ye^y\,(1-h(1))}{||p||_L\,(e^y-1)\,(1-h_0(1))} \,:\, y>\ln\frac{1-h_0(1)}{\lambda-h_1(1)}\right\}. \tag{5.11}$$

Then the operator ℓ defined by formula (5.1) belongs to the set $\widetilde{V}_{ab}^{-}(h)$.

Theorem 5.2. *Let the conditions (3.1) and (5.6) be fulfilled. Assume that $\tau(t)\ge t$ for a.e. $t\in[a,b]$,*

$$\int\limits_a^b p(s)ds > \ln\frac{1-h_0(1)}{\lambda-h_1(1)}, \tag{5.12}$$

and at least one of the following conditions is satisfied:

(a) $h_0(z_0)>0$ and

$$\max\left\{\frac{h_0(z_1)+(1-h_0(1))\,z_1(t)}{h_0(z_0)+(1-h_0(1))\,z_0(t)} \,:\, t\in[a,b]\right\} < 1-\frac{h_0(z_0)}{1-h_0(1)}, \tag{5.13}$$

where

$$z_0(t) = \int\limits_a^t p(s)\,ds \quad \textit{for} \quad t\in[a,b], \tag{5.14}$$

$$z_1(t) = \int\limits_a^t p(s)\left(\int\limits_a^{\tau(s)} p(\xi)d\xi\right)ds \quad \textit{for} \quad t\in[a,b]; \tag{5.15}$$

(b)

$$h_0(\beta) < 1, \tag{5.16}$$

$$\frac{h_0(\gamma_0)}{1-h_0(\beta)}\,\beta(b)+\gamma_0(b) < 1, \tag{5.17}$$

where

$$\gamma_0(t) = \int\limits_a^t p(s) \left(\int\limits_s^{\tau(s)} p(\xi) d\xi \right) \exp \left(\int\limits_s^t p(\eta) d\eta \right) ds \quad \textit{for} \quad t \in [a, b]; \tag{5.18}$$

(c) $h_0(1) \neq 0$

$$\operatorname{ess\,sup} \left\{ \int\limits_t^{\tau(t)} p(s) ds : t \in [a, b] \right\} < \kappa^*, \tag{5.19}$$

where

$$\kappa^* = \sup \left\{ \frac{\|p\|_L}{x} \ln \frac{x e^x (1 - h_0(1))}{\|p\|_L (e^x - 1)} : 0 < x < \ln \frac{1}{h_0(1)} \right\}. \tag{5.20}$$

Then the operator ℓ *given by the formula (5.1) belongs to the set* $\widetilde{V}_{ab}^-(h)$.

Theorem 5.3. *Let the conditions (2.3) and (4.2) be fulfilled. Assume that* $\tau(t) \geq t$ *for a.e.* $t \in [a, b]$ *and at least one of the following conditions is satisfied:*

(a) the inequality (5.13) holds, where the functions z_0 *and* z_1 *are defined by the formulae (5.14) and (5.15), respectively;*

(b) the inequalities (5.16) and (5.17) hold, where the function γ_0 *is given by the formula (5.18);*

(c) $h_0(1) \neq 0$ *and the condition (5.19) holds, where the number* κ^* *is defined by the formula (5.20).*

Then the operator ℓ *given by the formula (5.1) belongs to the set* $\widetilde{V}_{ab}^-(h)$.

Remark 5.1. If $h_0(z_0) > 0$, where z_0 is defined by formula (5.14), then the strict inequality (5.13) in Theorem 5.3(a) can be weakened. More precisely, the following assertion is true.

Theorem 5.4. *Let the conditions (2.3) and (4.2) be fulfilled. Assume that* $\tau(t) \geq t$ *for a.e.* $t \in [a, b]$,

$$h_0(z_0) > 0,$$

and

$$\max \left\{ \frac{h_0(z_1) + (1 - h_0(1))\, z_1(t)}{h_0(z_0) + (1 - h_0(1))\, z_0(t)} : t \in [a, b] \right\} \leq 1 - \frac{h_0(z_0)}{1 - h_0(1)}, \tag{5.21}$$

where the functions z_0 *and* z_1 *are defined by the formulae (5.14) and (5.15), respectively. Then the operator* ℓ *given by the formula (5.1) belongs to the set* $\widetilde{V}_{ab}^-(h)$.

Theorem 5.5. *Let the conditions (2.3) and (3.4) be fulfilled. Assume that*

$$\operatorname{ess\,sup} \left\{ \int\limits_{\mu(t)}^t g(s)\, ds : t \in [a, b] \right\} < \omega^*, \tag{5.22}$$

where

$$\omega^* = \sup \left\{ \frac{1}{x} \ln \frac{x \alpha^x(\mu_*)}{\alpha^x(\mu_*) - f(x)} : x > 0, \widehat{h}(\alpha^x) > \alpha^x(a) \right\},$$

$$f(x) \stackrel{\text{df}}{=} \frac{\widehat{h}(\alpha^x) - \alpha^x(a)}{h(1) - 1} \quad \textit{for} \quad x > 0, \tag{5.23}$$

$$\widehat{h}(v) \stackrel{\text{df}}{=} \min\{h(1), h(v)\} \quad \textit{for} \quad v \in C([a,b];\mathbb{R}).$$

Then the operator ℓ given by formula (5.2) belongs to the set $\widetilde{V}_{ab}^{-}(h)$.

Corollary 5.2. *Let the conditions (2.3) and (3.4) be fulfilled. Assume that $g \not\equiv 0$ and*

$$\operatorname{ess\,sup}\left\{ \int_{\mu(t)}^{t} g(s)ds : t \in [a,b] \right\} < \xi^*,$$

where

$$\xi^* = \sup\left\{ \frac{||g||_L}{y} \ln \frac{ye^y\,(h(1)-1)}{||g||_L(e^y-1)\,(\lambda + h_0(1))} : 0 < y < \ln \frac{\lambda + h_0(1)}{1 + h_1(1)} \right\}. \tag{5.24}$$

Then the operator ℓ defined by the formula (5.2) belongs to the set $\widetilde{V}_{ab}^{-}(h)$.

Theorem 5.6. *Let the conditions (2.3) and (4.2) be fulfilled. Assume that $g \not\equiv 0$ and*

$$\max\left\{ \frac{z_1(a) - h(z_1) + (h(1)-1)\,z_1(t)}{z_0(a) - h(z_0) + (h(1)-1)\,z_0(t)} : t \in [a,b] \right\} < 1 - \frac{z_0(a) - h(z_0)}{h(1)-1}, \tag{5.25}$$

where

$$z_0(t) = \int_t^b g(s)\,ds \quad \textit{for} \quad t \in [a,b],$$

$$z_1(t) = \int_t^b g(s) \left(\int_{\mu(s)}^b g(\xi)\,d\xi \right) ds \quad \textit{for} \quad t \in [a,b].$$

Then the operator ℓ given by formula (5.2) belongs to the set $\widetilde{V}_{ab}^{-}(h)$.

Theorem 5.7. *Let the conditions (2.3) and (3.4) be fulfilled. Assume that the inequalities (4.10) and (4.11) are satisfied, where*

$$\begin{gathered} z_0(t) = \exp\left(\int_t^b g(s)\,ds \right) \quad \textit{for} \quad t \in [a,b], \\ z_1(t) = \int_t^b g(s)\sigma(s) \left(\int_{\mu(s)}^s g(\xi)\,d\xi \right) \exp\left(\int_t^s g(\eta)\,d\eta \right) ds \quad \textit{for} \quad t \in [a,b], \end{gathered} \tag{5.26}$$

and

$$\sigma(t) = \frac{1}{2}\,(1 + \operatorname{sgn}\,(t - \mu(t))) \quad \textit{for a.e.} \quad t \in [a, b]. \tag{5.27}$$

Then the operator ℓ *given by formula (5.2) belongs to the set* $\widetilde{V}_{ab}^{-}(h)$.

Theorem 5.8. *Let the conditions (3.1) and (3.5) be fulfilled. If*

$$\int\limits_a^b g(s)\,ds < (\lambda - h_1(1)) \min\left\{1, \frac{1}{\lambda}\right\} \tag{5.28}$$

and

$$\frac{(1 - h_0(1)) \min\left\{1, \frac{1}{\lambda}\right\}}{(\lambda - h_1(1)) \min\left\{1, \frac{1}{\lambda}\right\} - \int_a^b g(s)\,ds} - 1 < \int_a^b p(s)\,ds \leq (1 - h_0(1)) \min\left\{1, \frac{1}{\lambda}\right\},$$

then the operator ℓ *given by formula (5.3) belongs to the set* $\widetilde{V}_{ab}^{-}(h)$.

Theorem 5.9. *Let the conditions (2.3) and (3.4) be fulfilled. Assume that the inequality (5.28) is satisfied and*

$$\omega\left(\int\limits_a^b g(s)\,ds\right) < \int\limits_a^b p(s)\,ds \leq (1 - h_0(1)) \min\left\{1, \frac{1}{\lambda}\right\},$$

where the function ω *is defined by formula (3.10). Then the operator* ℓ *given by formula (5.3) belongs to the set* $\widetilde{V}_{ab}^{-}(h)$.

Corollary 5.3. *Let the conditions (3.1) and (3.4) be fulfilled. Assume that either*

$$h(1) \leq 1, \quad \frac{1 - h_0(1)}{\lambda - h_1(1)} - 1 < \int\limits_a^b p(s)\,ds \leq (1 - h_0(1)) \min\left\{1, \frac{1}{\lambda}\right\}$$

or

$$h(1) > 1, \quad \int\limits_a^b p(s)\,ds \leq (1 - h_0(1)) \min\left\{1, \frac{1}{\lambda}\right\}.$$

Then the operator ℓ *given by formula (5.1) belongs to the set* $\widetilde{V}_{ab}^{-}(h)$.

Theorem 5.10. *Let the conditions (2.3) and (3.4) be fulfilled. Assume that the functions* p, τ *satisfy condition (5.7) or the assumptions of Theorems 5.3 or 5.4, whereas the functions* g, μ *fulfil the assumptions of Theorems 5.5, 5.6 or 5.7. Then the operator* ℓ *given by formula (5.3) belongs to the set* $\widetilde{V}_{ab}^{-}(h)$.

5.1. Proofs. We give the following lemmas before we prove statements formulated above.

Lemma 5.1. *Let the functional* h *be defined by formula (1.3), where* $\lambda > 0$ *and* $h_0, h_1 \in PF_{ab}$ *are such that the conditions (3.1) and (5.6) are fulfilled. Let, moreover, the operator* ℓ *be defined by the formula (5.1),* $p \not\equiv 0$, *and condition (5.8) be satisfied, where the number* η_* *is defined by* *formula (5.9). Then there exists a function* $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ *satisfying the inequalities (3.2) and (3.3).*

Proof. According to (5.8) with η^* given by (5.9), there exist $x_0 > 0$ and $\varepsilon > 0$ such that

$$h(\beta^{x_0}) \geq 1 + \varepsilon \tag{5.29}$$

and the relation

$$\int\limits_t^{\tau(t)} p(s)\,ds \geq \frac{1}{x_0} \ln \frac{x_0 \beta^{x_0}(\tau^*)}{\beta^{x_0}(\tau^*) + \left(h(\beta^{x_0}) - 1 - \varepsilon\right)\left(1 - h(1)\right)^{-1}} \quad \text{for a.e.} \quad t \in [a,b] \tag{5.30}$$

holds. Put

$$\delta = \frac{h(\beta^{x_0}) - 1 - \varepsilon}{1 - h(1)}. \tag{5.31}$$

By virtue of the conditions (5.6) and (5.29), we get $\delta \geq 0$. Hence, relation (5.30) yields

$$e^{x_0 \int\limits_t^{\tau(t)} p(s)\,ds} \geq \frac{x_0 \beta^{x_0}(\tau^*)}{\beta^{x_0}(\tau^*) + \delta} \geq \frac{x_0 \beta^{x_0}(\tau(t))}{\beta^{x_0}(\tau(t)) + \delta} \quad \text{for a.e.} \quad t \in [a,b].$$

Consequently, we have

$$x_0 e^{x_0 \int\limits_a^t p(s)\,ds} \leq e^{x_0 \int\limits_a^{\tau(t)} p(s)\,ds} + \delta \quad \text{for a.e.} \quad t \in [a,b]. \tag{5.32}$$

Now we put

$$\gamma(t) = e^{x_0 \int\limits_a^t p(s)\,ds} + \delta \quad \text{for} \quad t \in [a,b].$$

It is clear that $\gamma(t) > 0$ for $t \in [a,b]$ and, using condition (5.32), we get

$$\ell(\gamma)(t) = p(t)\left(e^{x_0 \int\limits_a^{\tau(t)} p(s)\,ds} + \delta\right) \geq x_0 p(t) e^{x_0 \int\limits_a^t p(s)\,ds} = \gamma'(t) \quad \text{for a.e.} \quad t \in [a,b],$$

i.e., inequality (3.2) holds. On the other hand, in view of equality (5.31) and the assumption $h(1) < 1$, inequality (3.3) is satisfied.

The lemma is proved.

Lemma 5.2. *Let the operator* ℓ *be defined by formula (5.2),* $h \in F_{ab}$ *satisfies condition (2.3), and let inequality (5.22) be fulfilled, where the number* ω^* *is defined by formulae (5.23). Then there exists a function* $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ *satisfying the inequalities (3.2) and (3.3).*

Proof. According to (5.22) with ω^* given by (5.23), there exist $x_0 > 0$ and $\varepsilon > 0$ such that

$$\widehat{h}(\alpha^{x_0}) \geq \alpha^{x_0}(a) + \varepsilon \tag{5.33}$$

and the inequality

$$\int\limits_{\mu(t)}^{t} g(s)ds \leq \frac{1}{x_0} \ln \frac{x_0 \alpha^{x_0}(\mu_*)}{\alpha^{x_0}(\mu_*) - \left(\widehat{h}(\alpha^{x_0}) - \alpha^{x_0}(a) - \varepsilon\right)(h(1)-1)^{-1}} \tag{5.34}$$

holds for a.e. $t \in [a,b]$. Put

$$\delta = \frac{\widehat{h}(\alpha^{x_0}) - \alpha^{x_0}(a) - \varepsilon}{h(1)-1}. \tag{5.35}$$

By virtue of the conditions (2.3), (5.23), and (5.33), we get $\delta \in [0,1[$. Hence, relation (5.34) yields

$$e^{x_0 \int\limits_{\mu(t)}^{t} g(s)\,ds} \leq \frac{x_0 \alpha^{x_0}(\mu_*)}{\alpha^{x_0}(\mu_*) - \delta} \leq \frac{x_0 \alpha^{x_0}(\mu(t))}{\alpha^{x_0}(\mu(t)) - \delta} \quad \text{for a.e.} \quad t \in [a,b].$$

Consequently, we have

$$e^{x_0 \int\limits_{\mu(t)}^{b} g(s)\,ds} - \delta \leq x_0 e^{x_0 \int\limits_{t}^{b} g(s)\,ds} \quad \text{for a.e.} \quad t \in [a,b]. \tag{5.36}$$

Now we put

$$\gamma(t) = e^{x_0 \int\limits_{t}^{b} g(s)\,ds} - \delta \quad \text{for} \quad t \in [a,b].$$

It is clear that $\gamma(t) > 0$ for $t \in [a,b]$ and, using condition (5.36), we get

$$\ell(\gamma)(t) = -g(t)\left(e^{x_0 \int\limits_{\mu(t)}^{b} g(s)\,ds} - \delta\right) \geq -x_0 g(t) e^{x_0 \int\limits_{t}^{b} g(s)\,ds} = \gamma'(t) \quad \text{for a.e.} \quad t \in [a,b],$$

i.e., inequality (3.2) holds. On the other hand, in view of (5.23), (5.35), and the assumption $h(1) > 1$, the inequality (3.3) is satisfied.

The lemma is proved.

Now we are in position to prove Theorems 5.1 – 5.10.

Proof of Theorem 5.1. Let the operator ℓ be defined by formula (5.1). It is clear that $\ell \in P_{ab}$ and condition (5.7) implies validity of relation (4.16) with $\ell_0 \equiv \ell$. According to Lemma 5.1, there exists a function $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ satisfying the conditions (3.2) and (3.3), which guarantees validity of the inclusion $\ell \in U_{ab}^{-}(h)$ (see Theorem 3.1). Consequently, in view of Theorem 4.4 (with $\ell_0 \equiv \ell$ and $\ell_1 \equiv 0$), we get $\ell \in \widetilde{V}_{ab}^{-}(h)$.

The theorem is proved.

Proof of Corollary 5.1. It is not difficult to verify that

$$\frac{x\beta^x(\tau^*)}{\beta^x(\tau^*) + (h(\beta^x) - 1)\,(1 - h(1))^{-1}} \leq$$

$$\leq \frac{x\beta^x(b)}{\beta^x(b) + \big((\lambda - h_1(1))\,e^{x\|p\|_L} + h_0(1) - 1\big)\,(1 - h(1))^{-1}} =$$

$$= \frac{xe^{x\|p\|_L}\,(1 - h(1))}{\big(e^{x\|p\|_L} - 1\big)\,(1 - h_0(1))}$$

for every $x > 0$ such that $(\lambda - h_1(1))\,e^{x\|p\|_L} > 1 - h_0(1)$. Therefore, the relation $\eta_* \leq \xi_*$ holds, where η_* and ξ_* are defined by the formulae (5.9) and (5.11), respectively. Consequently, the assertion of the corollary follows immediately from Theorem 5.1.

Proof of Theorem 5.2. Let the operator ℓ be defined by formula (5.1). It is clear that $\ell \in P_{ab}$ and ℓ is a b-Volterra operator. According to Theorems 4.1 and 4.2, and Corollary 4.2 in [14], we conclude that each of the conditions (a) – (c) guarantees validity of the inclusion $\ell \in \widetilde{V}^+_{ab}(h_0)$. Moreover, by virtue of Theorem 2.1 in [14], there exists a function $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ satisfying inequality (3.12). Therefore, Lemma 4.2 guarantees that $\ell \in S_{ab}(b)$. Furthermore, the above-proved inclusion $\ell \in \widetilde{V}^+_{ab}(h_0)$ yields $\ell \in \widetilde{U}^+_{ab}(h_0)$ (see Remark 3.3).

On the other hand, since we suppose that $\tau(t) \geq t$ for a. e. $t \in [a,b]$, condition (5.12) implies validity of condition (5.10), where ξ_* is defined by formula (5.11). Therefore, analogously to the proof of Corollary 5.1 it can be shown that relation (5.8) is satisfied with η_* given by formula (5.9), and thus, according to Lemma 5.1, there exists a function $\gamma \in \widetilde{C}([a,b];]0,+\infty[\,)$ satisfying the conditions (3.2) and (3.3).

Consequently, by virtue of Theorem 4.1, we get $\ell \in \widetilde{V}^-_{ab}(h)$.

The theorem is proved.

Proof of Theorem 5.3. Let the operator ℓ be defined by formula (5.1). It is clear that $\ell \in P_{ab}$ and ℓ is a b-Volterra operator. According to Theorems 4.1 and 4.2, and Corollary 4.2 in [14], we conclude that each of the conditions (a) – (c) guarantees validity of the inclusion $\ell \in \widetilde{V}^+_{ab}(h_0)$. Therefore, the assumptions of Corollary 4.3 are satisfied.

The theorem is proved.

Proof of Theorem 5.4. Let the operator ℓ be defined by formula (5.1). It is clear that $\ell \in P_{ab}$ and ℓ is a b-Volterra operator. Using condition (5.21), it is not difficult to verify that

$$\varrho_3(t) \leq \varrho_2(t) \quad \text{for} \quad t \in [a,b],$$

where the functions ϱ_2 and ϱ_3 are defined by the formulae (4.5) and (4.6). Consequently, the assumptions of Corollary 4.2 are satisfied with $k = 2$ and $m = 3$.

The theorem is proved.

Proof of Theorem 5.5. The assertion of the theorem follows immediately from Lemma 5.2 and Theorem 4.3.

Proof of Corollary 5.2. It is not difficult to verify that

$$\frac{x\alpha^x(\mu_*)}{\alpha^x(\mu_*) - \left(\widehat{h}(\alpha^x) - \alpha^x(a)\right)(h(1)-1)^{-1}} \geq$$

$$\geq \frac{x\alpha^x(a)}{\alpha^x(a) - (\lambda + h_0(1) - (1+h_1(1))\,\alpha^x(a))\,(h(1)-1)^{-1}} =$$

$$= \frac{xe^{x\|g\|_L}\,(h(1)-1)}{\left(e^{x\|g\|_L}-1\right)(\lambda + h_0(1))}$$

for every $x > 0$ such that $\lambda + h_0(1) > (1+h_1(1))\,e^{x\|g\|_L}$. Therefore, the relation $\xi^* \leq \omega^*$ holds, where ω^* and ξ^* are defined by the formulae (5.23) and (5.24), respectively. Consequently, validity of the corollary follows immediately from Theorem 5.5.

Proof of Theorem 5.6. Let the operator ℓ be defined by formula (5.2). It is clear that $-\ell \in$ $\in P_{ab}$. According to condition (5.25), there exists $\delta \in [0,1[$ such that the inequality

$$\frac{z_1(a) - h(z_1)}{h(1)-1} + z_1(t) \leq \left(\delta - \frac{z_0(a)-h(z_0)}{h(1)-1}\right)\left(\frac{z_0(a)-h(z_0)}{h(1)-1} + z_0(t)\right)$$

holds for $t \in [a,b]$. However, it means that

$$\varrho_3(t) \leq \delta\varrho_2(t) \quad \text{for} \quad t \in [a,b],$$

where the functions ϱ_2 and ϱ_3 are defined in Corollary 4.4 (a). Consequently, the assumptions of Corollary 4.4(a) are satisfied with $k = 2$ and $m = 3$.

The theorem is proved.

Proof of Theorem 5.7. Let the operators ℓ and $\bar{\ell}$ be defined by formulae (5.2) and

$$\bar{\ell}(v)(t) \overset{\text{df}}{=} g(t)\sigma(t)\left(\int\limits_{\mu(t)}^{t} g(s)v(\mu(s))ds\right) \quad \text{for a.e.} \quad t \in [a,b], \quad \text{all} \quad v \in C([a,b];\mathbb{R}),$$

respectively, where the function σ is given by formula (5.27). It is clear that $-\ell \in P_{ab}$, $\bar{\ell} \in P_{ab}$, and

$$\ell(1)(t)\vartheta(v)(t) - \ell(\vartheta(v))(t) = g(t)\int\limits_{\mu(t)}^{t} g(s)v(\mu(s))ds \leq$$

$$\leq \bar{\ell}(v)(t) \quad \text{for a.e.} \quad t \in [a,b] \quad \text{and all} \quad v \in C([a,b];\mathbb{R}_+),$$

where the operator ϑ is defined by formulae (4.8) and (4.9), and thus condition (4.12) holds on the set $C([a,b];\mathbb{R}_+)$. Therefore, the assumptions of Corollary 4.4(b) are satisfied.

The theorem is proved.

Proof of Theorem 5.8. Let the operator ℓ be defined by formula (5.3),

$$\ell_0(v)(t) \stackrel{\text{df}}{=} p(t)v(\tau(t)) \quad \text{for a.e.} \quad t \in [a,b] \quad \text{and all} \quad v \in C([a,b];\mathbb{R}), \tag{5.37}$$

and

$$\ell_1(v)(t) \stackrel{\text{df}}{=} g(t)v(\mu(t)) \quad \text{for a.e.} \quad t \in [a,b] \quad \text{and all} \quad v \in C([a,b];\mathbb{R}). \tag{5.38}$$

It is clear that $\ell_0, \ell_1 \in P_{ab}$ and $\ell = \ell_0 - \ell_1$. Therefore, validity of the theorem follows from Theorems 3.3 and 4.4.

Proof of Theorem 5.9. Let the operators ℓ, ℓ_0, and ℓ_1 be defined by formulae (5.3), (5.37), and (5.38), respectively. It is clear that $\ell_0, \ell_1 \in P_{ab}$ and $\ell = \ell_0 - \ell_1$. Therefore, the assertion of the theorem follows from Theorems 3.4 and 4.4.

Proof of Corollary 5.3. Validity of the corollary follows immediately from Theorems 5.8 and 5.9 with $g \equiv 0$.

Proof of Theorem 5.10. Let the operators ℓ, ℓ_0, and ℓ_1 be defined by the formulae (5.3), (5.37), and (5.38), respectively. It is clear that $\ell_0, \ell_1 \in P_{ab}$ and $\ell = \ell_0 - \ell_1$. Therefore, the assertion of the theorem follows immediately from Theorem 4.5, Theorems 5.3 – 5.7, and Corollary 5.3.

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