BOOK REVIEW:
TOPOLOGICAL DEGREE APPROACH TO BIFURCATION PROBLEMSITALIC
BY MICHAL FEČKAN

КНИЖКОВИЙ ОГЛЯД:
МІХАЛ ФЕЧКАН.
МЕТОД ТОПОЛОГІЧНОГО СТУПЕНЯ ДЛЯ БІФУРКАЦІЙНИХ ЗАДАЧ

Many phenomena in physics, biology, chemistry, economics, and other branches of science can be modeled by either differential or difference equations depending on parameters. As parameters are changing, the qualitative properties of such equations are changing as well. Roughly speaking, such parameter depending problems gave rise to the bifurcation theory. Bifurcation methods involve topological, analytical and variational approaches, which are explained in several well-known books and papers on nonlinear analysis (cf. [1, 4, 5, 7, 8, 13, 18]). Topological methods are usually applied if nonlinearities of the studied equations are not smooth enough or even in the case where they are discontinuous. Nowadays there is a great interest in nonsmooth and discontinuous equations due to a broad scope of their applications (cf. [16, 17]). Such kinds of bifurcation problems are studied also in this book with the topics covering non-smooth mechanical systems (cf. [3, 10]), dynamical systems on infinite lattices (cf. [20, 21]), nonlinear beam and string partial differential equations (cf. [12, 13]), and discontinuous wave partial differential equations (cf. [2]). The main approach applied in the book is a combination of the perturbation method used in the theory of dynamical systems (cf. [9, 11, 12]), the decomposition method of Lyapunov – Schmidt known from the theory of nonlinear analysis (cf. [7, 8, 13]), and the theory of topological degrees of Brouwer and Leray – Schauder (cf. [5, 18] as well).

The book consists of the following chapters: 1. Introduction, 2. Theoretical Background, 3. Bifurcation of Periodic Solutions, 4. Bifurcation of Chaotic Solutions, 5. Topological Transverlity, 6. Traveling waves on lattices, 7. Periodic Oscillations of Wave Equations, 8. Topological Degree for Wave Equations.

Now we briefly summarize their contents: Chapter 1 presents an illustrative example on the bifurcation of chaotic solutions in an apparatus containing a periodically forced slender beam clamped to a rigid framework which supports two magnets. The whole apparatus is periodically forced by an electromagnetic vibration generator. Oscillations of the beam are described by the well-known periodically forced and damped Duffing equation. Then the above-mentioned mathematical methods are used to derive a Melnikov bifurcation function which determines chaotic wedge-shaped regions in the parametric space of the perturbed Duffing equation. This example is used to outline basic features applied in the book.

Chapter 2 presents known fundamental mathematical tools from linear and nonlinear functional analysis, differential topology, the theory of multivalued mappings, and dynamical systems, which are latter applied in proofs of results in the book.

In the first part of Chapter 3, the author proves existence of infinitely many subharmonic solutions for weakly discontinuous differential equations, which have periods tending to infinity and bifurcate from homoclinic trajectories. This is the first step to show existence of a chaos in discontinuous systems. Then singularly perturbed discontinuous systems are studied as well. Moreover, bifurcations of periodic solutions from periodic trajectories are also studied for weakly discontinuous differential equations. Several types of systems of differential equations are dealt with similarly to differential equations with dry frictions, weakly coupled nonlinear oscillators, and forced systems with relay hysteresis. In this way, some classical bifurcation results such as saddle-node and Poincaré – Andronov bifurcations of periodic solutions (cf. [11, 19]) are extended to nonsmooth differential equations. All theoretical achievements are illustrated with several concrete examples.

The method of Chapter 3 is extended further in Chapter 4 to almost-periodically and quasiperiodically forced differential inclusions. It is supposed that the unperturbed systems possess either a single solution to hyperbolic equilibria, or a manifold of solutious. Melnikov type conditions are derived to prove existence of a chaos for perturbed systems. Again, several illustrative examples are given. A brief review of recent results on homoclinic bifurcations for other types of discontinuous differential equations is presented at the end of the chapter. Both cases occur if either the discontinuity surface is transversally crossesed by a homoclinic solution or a part of homoclinic solution slides on that discontinuity surface.

In Chapter 5, the classical Smale – Birkhoff homoclinic theorem (cf. [11]) is extended to show existence of topologically transversal homoclinic and heteroclinic points of diffeomorphisms, i.e., to the case where intersections of stable and unstable manifolds of hyperbolic fixed points of the diffeomorphism are only topologically transversal. This still leads to chaos for diffeomorphisms. It is again necessary to use topological degree methods. Bifurcations of such points are studied as well. Then topological transversality is extended to show that periodic points of reversible diffeomorphisms accumulate on homoclinic points, while their periods tend to infinity. This blow-up phenomenon is known as the blue sky catastrophe (cf. [9]). The final part of the chapter discusses the blue sky catastrophe for infinite chains of reversible oscillators, showing an accumulation of either breather or traveling wave types solutions.

Chapter 6 continues with a study of ordinary differential equations on one-dimensional infinite lattices. But now these are spatially discretized partial differential equations (cf. [21]). Both the sine-Gordon and the Klein – Gordon discretized lattice equations are studied. There a persistence of kink traveling waves as solutions of partial differential equations under spatial discretization is studied. For this purpose, first a traveling wave equation on a lattice is considered as an evolution equation on some Banach space. Then this infinite dimensional evolution equation is reduced to a finite dimensional equation by using the well-known center manifold method (cf. [11, 12, 19]). To study the reduced traveling wave equation on the center manifold, a bifurcation result for periodic solutions of certain singularly perturbed ordinary differential equations are derived. The singular parameter is the sise of the discretization step. Both the homoclinic and the heteroclinic traveling wave solutions are considered. The final part of the chapter is devoted to a review of existence results on traveling waves for ordinary differential equations on two-dimensional lattices.

Undamped abstract wave differential equations on a Hilbert space are investigated in Chapter 7. Existence of periodic and subharmonic solutions are studied. It is well-known that in this case, in general, infinitely many resonant terms appear, which is known as the problem of small divisors (cf. [15]). Diophantine-type inequalities are introduced to avoid these resonances. Then using analytical and topological methods, bifurcations of subharmonic solutions from homoclinic ones, and bifurcations of periodic solutions from periodic ones are established for these undamped abstract wave differential equations. Applications are given to several types of periodically forced nonlinear undamped beam equations. In the next part of the chapter, a similar approach is used to study weakly nonlinear wave equations with a derivation of bifurcation function for periodic solutions. Concrete Diophantine-type inequalities arising in the above-mentioned applications are solved in the final part of this chapter by using some tools from the number theory (cf. [6, 14]).

The final Chapter 8 proceeds with a study of undamped wave partial differential equations that are, however, discontinuous. Moreover infinitely many resonances occur as well. In order to find periodic solutions for such discontinuous partial differential equation, in the first part of this chapter, a topological degree theory is extended to monotone multivalued mappings; it is a combination of the multivalued Browder – Skrypnik topological degree method [4, 22] and Mawhin's coincidence index theory [18]. In this way, two classical bifurcation results, the Krasnoselskii bifurcation theorem and bifurcations from infinity [1, 7, 18], are extended to multivalued operator equations on a Hilbert space. Applying these abstract bifurcation theorems, the existence of periodic solutions with large amplitudes is proved for periodically forced and discontinuous undamped semilinear wave equations. At the end of this chapter there is an outline of a possibility to extend the approach developed in this book for showing existence of a chaos in weakly discontinuous and periodically forced semilinear wave equations.

The book is mainly based on results contained in author's papers. These results have not been previously collected in a book, which makes the book original. It can be recommended not only to mathematicians but also to other scientists specializing in physics, biology, chemistry and economics, to those who are interesting in nonlinear oscillations and chaos theory and their applications. The book is readably written and well organized, contains complete proofs, which makes it suitable for reading by a PLD student.

1. *Berger M. S.* Nonlinearity and functional analysis. — New York: Acad. Press, 1977.
2. *Berkowits J., Tienari M.* Topological degree theory for some classes of multis with applications to hyperbolic and elliptic problems involving discontinuous nonlinearities // Dynam Syst. and Appl. — 1996. — **5**. — P. 1 – 18.
3. *Brogliato B.* Nonsmooth impact mechanics: models, dynamics, and control // Lect. Notes Contr. and Inform. Sci. — Berlin: Springer, 1996. — **220**.
4. *Browder F. E.* Degree theory for nonlinear mappings // Proc. Symp. Pure Math. — Providence, R.I.: Amer. Math. Soc., 1986. — **45**, Pt 1. — P. 203 – 226.
5. *Brown R. F.* A topological introduction to nonlinear analysis. — Boston: Birhkhäuser, 1993.
6. *Cassels J. W. S.* An introduction to Diophantine approximation. — Cambridge Univ. Press, 1957.
7. *Chow S. N., Hale J. K.* Methods of bifurcation theory. — New York: Springer, 1982.
8. *Deimling K.* Nonlinear Functional Analysis. — Berlin: Springer, 1985.
9. *Devaney R.* An introduction to chaotic dynamical systems. — Menlo Park: Benjamin/Cummings, CA, 1986.
10. *Fidlin A.* Nonlinear oscillations in mechanical engineering. — Berlin: Springer, 2006.
11. *Guckenheimer J., Holmes P.* Nonlinear oscillations, dynamical systems, and bifurcations of vector fields. — New York: Springer, 1983.
12. *Iooss G., Aelmeyer M.* Topics in bifurcation theory and applications. — Singapore: World Sci. Publ. Co., 1992.
13. *Kielhöfer H.* Bifurcation theory an introduction with applications to PDEs. — New-York: Springer, 2004.
14. *Khinchin A. Ya.* Continued fractions. — New York: Dover Publ., 1997.

15. *Kuksin S. B.* Nearly integrable infinite-dimensional hamiltonian systems // Lect. Notes Math. — Berlin: Springer, 1993. — **1556**.

16. *Kunze M., Küpper T.* Non-smooth dynamical systems: an overview // Ergodic Theory, Analysis and Efficient Simulation of Dynamical Systems / Ed. B. Fiedler. — Berlin: Springer, 2001. — P. 431 – 452.

17. *Kuznetsov Yu. A., Rinaldi S. A., Gragnani A.* One-parametric bifurcations in planar Filippov systems // Int. J. Bif. Chaos. — 2003. — **13**. — P. 2157 – 2188.

18. *Mawhin J.* Topological degree methods in nonlinear boundary value problems // CBMS Reg. Conf. Ser. Math. — Providence, RI: Amer. Math. Soc., 1979. — **40**.

19. *Medveď M.* Fundamentals of dynamical systems and bifurcation theory. — Bristol: Adam Hilger, 1992.

20. *Samoilenko A. M., Teplinskij Yu. V.* Countable systems of differential equations. — Brill Acad. Publ., 2003.

21. *Scott A.* Nonlinear sciences: emergence and dynamics of coherent structures. — 2nd ed. — Oxford Univ. Press, 2003.

22. *Skrypnik I. V.* Methods of investigation of nonlinear elliptic boundary value problems. — Moscow: Nauka, 1990 (in Russian).

O. A. Boichuk