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INSTABILITY OF SOLUTIONS FOR CERTAIN NONLINEAR VECTOR DIFFERENTIAL EQUATIONS OF FOURTH ORDER

НЕСТІЙКІСТЬ РОЗВ'ЯЗКІВ ДЕЯКИХ НЕЛІНІЙНИХ ВЕКТОРНИХ РІВНЯНЬ ЧЕТВЕРТОГО ПОРЯДКУ

C. Tunç

Yüzüncü Yil Univ.
65080, Van – Turkey
e-mail: cemtunc@yahoo.com

The main purpose of this paper is to give a result with an explanatory example which deals directly with the instability of the trivial solution of a certain nonlinear vector differential equation of fourth order. The result which will be established here improves and includes a well-known instability result in the literature established for a scalar nonlinear differential equation of fourth order.

Отримано результат із пояснювальним прикладом про нестійкість тривіального розв'язку деякого нелінійного векторного диференціального рівняння четвертого порядку, що покращує відомий результат про нестійкість для скалярного диференціального рівняння четвертого порядку.

1. Introduction. As is well-known, the area of differential equations is an old but durable subject that remains alive and useful to a wide variety of engineers, scientists, and mathematicians. The principles of differential equations are largely about the qualitative theory of ordinary differential equations. Qualitative theory refers to the study of the behavior of solutions, for example the investigation of stability, instability, boundedness of solutions and etc., without determining explicit formulas for the solutions. It originated with Poincare at the beginning of the twentieth century and has been the most important theme of ordinary differential equations in that century. During this period, very little attention is paid to techniques for finding analytic formulas for solutions. That is to say, there are actually very few equations (beyond linear equations with constant-coefficient and even there are difficulties if the order of the equation or system is high) for which we can do this. Therefore, it is very important to interpret qualitative behavior of solutions when there is no analytical expression for the solutions of any differential equation under investigation. So far, in the relevant literature, some methods have been improved to obtain information about qualitative behaviors of solutions of differential equations without solving them. Among them the Lyapunov's [1] second (or direct) method is most well-known. Namely, Lyapunov's second (or direct) method is one of the basic tools for investigating the qualitative behavior of solutions; stability, instability, boundedness, existence of periodic solutions and etc. of nonlinear differential equations of higher order. During the past fifty year, by using Lyapunov's [1] second (or direct) method, throughout hundreds of the papers, qualitative behaviors of solutions, stability, instability, boundedness, existence of periodic solutions and etc. for various ordinary scalar or vector nonlinear differential equations of second-, third-, fourth-, fifth-, sixth-, seventh- and eighth-order have been investigated extensively in the relevant literature. However, we will not discuss details of these works here. Now, we consider the constant-coefficient fourth order scalar differential equation

$$x^{(4)} + a_1\dddot{x} + a_2\ddot{x} + a_3\dot{x} + a_4x = 0. \tag{1}$$

As we know from the qualitative theory of ordinary differential equations that the trivial solution of the equation (1) is unstable if the associated (auxiliary) equation

$$\psi(r) = r^4 + a_1r^3 + a_2r^2 + a_3r + a_4 = 0 \tag{2}$$

has at the least one root with a positive real part. Naturally, the existence of such a root depends on (though not always on all of) the coefficients a_1, a_2, a_3 and a_4 of equation (2). The corresponding auxiliary equation (2) of equation (1) has no purely imaginary roots $r = i\lambda$ $(\lambda \neq 0)$ if

$$\mathcal{X}_1 \equiv \lambda^4 - a_2\lambda^2 + a_4 \neq 0$$

or if

$$\mathcal{X}_2 \equiv a_1\lambda^2 - a_3 \neq 0.$$

Note that $\mathcal{X}_1$ may be rearranged in the form

$$\mathcal{X}_1 = \left(\lambda^2 - \frac{1}{2}a_2\right)^2 + a_4 - \frac{1}{2}a_2^2$$

so that $\mathcal{X}_1 \neq 0$ holds if, for example, $a_4 > \frac{1}{4}a_2^2$. It is also easy to see that $\mathcal{X}_2 \neq 0$ will hold if

$$a_1a_3 < 0, \quad a_4 \neq 0,$$

where the condition on a_4 here being necessary in order to ensure that $\lambda \neq 0$. It should also be noted that it was shown in Ezeilo [2] that the existence of the relation $a_4 > \frac{1}{4}a_2^2$ between the coefficients a_2 and a_4 of the equation (1) is a sufficient condition that guarantees that the trivial solution $x = 0$ of the equation (1) is unstable. The above preamble is merely intended to give a background to the assumptions which play a dominant role in our treatment here and to give some indication of how these assumptions stand with respect to the Routh – Hurwitz criteria.

Now, the objective of this paper is to study the instability of the trivial solution $X = 0$ of nonlinear vector differential equations of the form

$$X^{(4)} + \Psi(\ddot{X})\dddot{X} + G\left(X, \dot{X}, \ddot{X}, \dddot{X}\right)\ddot{X} + \Theta(\dot{X}) + F(X) = 0, \tag{3}$$

in which $X \in \mathbb{R}^n$; Ψ and G are continuous $(n \times n)$-symmetric matrix-valued functions for the variables indicated explicitly; $\Theta : \mathbb{R}^n \to \mathbb{R}^n$, $F : \mathbb{R}^n \to \mathbb{R}^n$ and $\Theta(0) = F(0) = 0$, and it is also supposed that the functions Θ and F are continuous for all $X, \dot{X} \in \mathbb{R}^n$, respectively. In this case, the equation (3) can be rewritten as

$$\begin{gathered} \dot{X} = Y, \quad \dot{Y} = Z, \quad \dot{Z} = W, \\ \dot{W} = -\Psi(Z)W - G(X, Y, Z, W)Z - \Theta(Y) - F(X), \end{gathered} \tag{4}$$

which was obtained by taking $\dot{X} = Y, \ddot{X} = Z, \dddot{X} = W$ in the equation (3). Now, let $J(\Psi(Z)\,Z|Z)$, $J_\Theta(Y)$ and $J_F(X)$, respectively, denote the linear operators from the matrix $\Psi(Z)$ and vectors $\Theta(Y)$ and $F(X)$ to the matrices

$$J(\Psi(Z)Z|Z) = \left(\frac{\partial}{\partial z_j}\sum_{k=1}^{n}\psi_{ik}z_k\right) = \Psi(Z) + \left(\sum_{k=1}^{n}\frac{\partial\psi_{ik}}{\partial z_j}\,z_k\right),$$

$$J_\Theta(Y) = \left(\frac{\partial\theta_i}{\partial y_j}\right), \quad J_F(X) = \left(\frac{\partial f_i}{\partial x_j}\right), \quad i,j = 1,2,\ldots,n,$$

where $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$, $(z_1, z_2, \ldots, z_n)$, (ψ_{ik}), $(\theta_1, \theta_2, \ldots, \theta_n)$ and $(f_1, f_2, \ldots, f_n)$ are components of X, Y, Z, Ψ, Θ and F, respectively. Furthermore, it is also assumed throughout this paper that $F(X)$ and $\Theta(\dot{X})$ are gradient vector fields and the matrices $J_F(X)$, $J_\Theta(Y)$ and $J(\Psi(Z)\,Z|Z)$ exist and are symmetric and continuous.

It is also worth mentioning that although a substantial amount of results have been published with regard to stability of certain nonlinear differential equations of fourth order, comparatively only a few results have been published which deal directly with the instability of solutions of certain differential equations of fourth order. In particular, one can refer to the book of Reissig et al. [3] as a survey for some papers published on the stability of solutions of certain differential equations of fourth order and the papers of Ezeilo [2, 4, 5], Lu and Liao [6], Sadek [7], Skrapek [8], Tiryaki [9], Tunç [10–12], C. Tunç and E. Tunç [13] and references cited there for some investigations performed on the instability of solutions of certain fourth order linear and nonlinear differential equations. Now, to the best of our knowledge, so far in the relevant literature, the papers performed on the instability of solutions scalar differential equations of fourth order can be summarized as follows. First, in 1978, Ezeilo [2] has carried out a work on the instability of the trivial solution of the fourth-order scalar differential equation

$$x^{(4)} + a_1\dddot{x} + a_2\ddot{x} + a_3\dot{x} + f(x) = 0.$$

In 1979, Ezeilo [4] also extended his aforementioned instability result in [2] to a much more general scalar differential equation given by

$$x^{(4)} + a_1\dddot{x} + h(x,\dot{x},\ddot{x},\dddot{x})\,\ddot{x} + g(x)\,\dot{x} + f(x) = 0.$$

Later, in 1988, Tiryaki [9] interested in a fourth-order nonlinear scalar differential equation of the form

$$x^{(4)} + \psi(\ddot{x})\dddot{x} + \phi(\dot{x})\ddot{x} + \theta(\dot{x}) + f(x) = 0. \tag{5}$$

Under specified conditions imposed on the functions ψ, ϕ, θ and f, he established sufficient conditions that guarantee the instability of the trivial solution of the equation (5). After that, in 1980, Skrapek [8] considered fourth-order scalar differential equations of the form

$$x^{(4)} + f(x,\dot{x},\ddot{x},\dddot{x}) = 0.$$

He studied hypotheses on the function f for which the zero solution of the above fourth-order differential equation is unstable. Similarly, in 1993 by help of Lyapunov's second method, Lu and Liao [6] gave some sufficient conditions for the instability of solutions of a general fourth-order linear scalar differential equation with variable coefficient, at least one of the characteristic roots of which has positive real part. Finally, in 2000, Ezeilo [5] discussed a similar problem for the scalar differential equations

$$x^{(4)} + \psi(\ddot{x})\dddot{x} + g(x, \dot{x}, \ddot{x}, \dddot{x})\,\ddot{x} + \theta(\dot{x}) + f(x) = 0, \tag{6}$$

$$x^{(4)} + a_1\dddot{x} + g(x, \dot{x}, \ddot{x}, \dddot{x})\,\ddot{x} + h(x)\dot{x} + f(x, \dot{x}, \ddot{x}, \dddot{x}) = 0, \tag{7}$$

and

$$x^{(4)} + p(\dddot{x}, \ddot{x}) + q(x, \dot{x}, \ddot{x}, \dddot{x})\,\ddot{x} + a_3\dot{x} + a_4 x = 0, \tag{8}$$

respectively, where a_1, a_3 and a_4 are constants.

Now, with respect to our observations in the literature, so far instability of solutions of nonlinear vector differential equations of the form (3) was not investigated in the relevant literature, except for its scalar version, differential equation (6). It should also be clarified that nearly throughout all of the papers just stated above it was taken into consideration Krasovskii's criterion (see Krasovskii [14]) as a fundamental criterion through the proofs, and the Lyapunov's [1] second (or direct) method has been used as a basic tool to prove the results established there. In this paper, in view of Krasovskii's criterion [14], we use the same method to verify our main result. The motivation for the present paper has been inspired basically by the paper of Ezeilo [5] and that just mentioned above. It is worth mentioning that all the papers mentioned above do not contain an example on the topic. However, our main result includes an explanatory example.

Throughout this paper, the symbol $\langle X, Y\rangle$ is used to denote the usual scalar product in $\mathbb{R}^n$ for given any X, Y in $\mathbb{R}^n$, that is, $\langle X, Y\rangle = \sum_{i=1}^{n} x_i y_i$, thus $\|X\|^2 = \langle X, X\rangle$. It is also well-known that a real symmetric matrix $A = (a_{ij})$, $i, j = 1, 2, \ldots, n$, is said to be positive definite if and only if the quadratic form $X^T\,AX$ is positive definite, where $X \in \mathbb{R}^n$ and X^T denotes the transpose of X (see Bellman [15]).

2. Main result. Actually, we prove the following theorem.

Theorem. *Assume that the function G and the functions Ψ, Θ and F in equation (3), respectively, are continuous and continuously differentiable, and there exist constants a_1 and a_2 such that the following conditions are satisfied:*

$$\Psi,\ G,\ J_\Theta \quad \text{and} \quad J_F \quad \text{are symmetric such that} \quad \frac{\partial \psi_{ij}}{\partial z_k} = \frac{\partial \psi_{ik}}{\partial z_j}, \quad i, j, k = 1, 2, \ldots, n;$$

$$F(X) \neq 0 \quad \text{if} \quad X \neq 0, \quad \lambda_i(\Psi(Z)) \geq a_1 > 0, \quad \lambda_i(G(X, Y, Z, W)) \leq a_2$$

and

$$\lambda_i(J_F(X)) > \frac{1}{4}\,a_2^2, \quad i = 1, 2, \ldots, n,$$

for all $X, Y, Z, W \in \mathbb{R}^n$, *where* $\lambda_i(J_F(X))$ *and* $\lambda_i(G(X,Y,Z,W))$, $i = 1, 2 \ldots, n$, *are eigenvalues of* $G(X,Y,Z,W)$ *and* $J_F(X)$.

Then the trivial solution $X = 0$ *of equation (3) is unstable.*

Remark 1. In order to prove the above theorem, it will suffice, see Krasovskii [14], to show that there exists a continuous Lyapunov function $V = V(X,Y,Z,W)$ which has the following Krasovskii properties:

(K_1) In every neighborhood of $(0,0,0,0)$ there exists a point (ξ,η,ζ,μ) such that $V(\xi,\eta,\zeta,\mu) > 0$;

(K_2) the time derivative $\dot{V} = \dfrac{d}{dt}V(X,Y,Z,W)$ along solution paths of the system (4) is positive semi-definite;

(K_3) the only solution $(X,Y,Z,W) = (X(t),Y(t),Z(t),W(t))$ of the system (4) which satisfies $\dot{V} = 0$ $(t \geq 0)$ is the trivial solution $(0,0,0,0)$.

Remark 2. It should be clarified that there is no restriction on the eigenvalues of the matrix $J_\Theta(Y)$ obtained from $\Theta(Y)$.

Remark 3. Our result includes and improves the result of Ezeilo [5], which was established on the instability of zero solution of equation (6).

Proof. For the proof of the theorem, we define a Lyapunov function $V = V(X,Y,Z,W)$ given by

$$V = a_2\langle Y,Z\rangle + \langle W,Z\rangle + \langle F(X),Y\rangle + \int_0^1 \langle \Theta(\sigma Y),Y\rangle\,d\sigma + \int_0^1 \langle \Psi(\sigma Z)Z,Z\rangle\,d\sigma. \tag{9}$$

Observe that $V(0,0,0,0) = 0$. Indeed, in view of the assumptions of the theorem, we also have that

$$V(0,0,\varepsilon,\varepsilon) = \langle\varepsilon,\varepsilon\rangle + \int_0^1 \langle \Psi(\sigma\varepsilon)\varepsilon,\varepsilon\rangle\,d\sigma \geq \langle\varepsilon,\varepsilon\rangle + \int_0^1 \langle a_1\varepsilon,\varepsilon\rangle\,d\sigma =$$

$$= \langle\varepsilon,\varepsilon\rangle \frac{1}{2}\,a_1\langle\varepsilon,\varepsilon\rangle = \|\varepsilon\|^2 + \frac{1}{2}\,a_1\,\|\varepsilon\|^2 > 0$$

for all arbitrary $\varepsilon \neq 0$, $\varepsilon \in \mathbb{R}^n$, which verifies the property (K_1). Next let $(X,Y,Z,W) = (X(t),Y(t),Z(t),W(t))$ be an arbitrary solution of the system (4). In view of the function (9) and the system (4), a straightforward calculation gives that

$$\dot{V} = \frac{d}{dt}V(X,Y,Z,W) = a_2\langle Z,Z\rangle + \langle W,W\rangle + a_2\langle Y,W\rangle - \langle G(X,Y,Z,W)Z,Z\rangle +$$

$$+ \langle J_F(X)Y,Y\rangle - \langle \Psi(Z)W,Z\rangle - \langle \Theta(Y),Z\rangle +$$

$$+ \frac{d}{dt}\int_0^1 \langle \Theta(\sigma Y),Y\rangle\,d\sigma + \frac{d}{dt}\int_0^1 \langle \Psi(\sigma Z)Z,Z\rangle\,d\sigma. \tag{10}$$

Now, recall that

$$\frac{d}{dt}\int_0^1 \langle \Theta(\sigma Y), Y\rangle\, d\sigma = \int_0^1 \sigma\langle J_\Theta(\sigma Y)Z, Y\rangle\, d\sigma + \int_0^1 \langle \Theta(\sigma Y), Z\rangle\, d\sigma =$$

$$= \int_0^1 \sigma \frac{\partial}{\partial\sigma}\langle \Theta(\sigma Y), Z\rangle\, d\sigma + \int_0^1 \langle \Theta(\sigma Y), Z\rangle\, d\sigma =$$

$$= \sigma\, \langle \Theta(\sigma Y), Z\rangle\big|_0^1 = \langle \Theta(Y), Z\rangle \tag{11}$$

and

$$\frac{d}{dt}\int_0^1 \langle \Psi(\sigma Z)Z, Z\rangle\, d\sigma = \int_0^1 \langle \Psi(\sigma Z)Z, W\rangle\, d\sigma + \int_0^1 \langle \sigma\, J(\Psi(\sigma Z)Z|\sigma Z)\, W, Z\rangle\, d\sigma =$$

$$= \int_0^1 \langle \Psi(\sigma Z)Z, W\rangle\, d\sigma + \int_0^1 \sigma\langle J(\Psi(\sigma Z)Z|\sigma Z)Z, W\rangle\, d\sigma =$$

$$= \int_0^1 \langle \Psi(\sigma Z)Z, W\rangle\, d\sigma + \int_0^1 \sigma\frac{\partial}{\partial\sigma}\langle \Psi(\sigma Z)Z, W\rangle\, d\sigma =$$

$$= \sigma\langle \Psi(\sigma Z)Z, W\rangle\big|_0^1 = \langle \Psi(Z)Z, W\rangle. \tag{12}$$

Collecting the estimates (11) and (12) into (10), we get

$$\dot V = a_2\langle Z, Z\rangle + \langle W, W\rangle + a_2\langle Y, W\rangle - \langle G(X, Y, Z, W)Z, Z\rangle + \langle J_F(X)Y, Y\rangle. \tag{13}$$

In view of the assumption

$$\lambda_i(G(X, Y, Z, W)) \le a_2$$

of the theorem, it follows for the terms

$$a_2\langle Z, Z\rangle - \langle G(X, Y, Z, W)Z, Z\rangle,$$

which are contained in (13), that

$$a_2\langle Z, Z\rangle - \langle G(X, Y, Z, W)Z, Z\rangle \ge a_2\langle Z, Z\rangle - a_2\langle Z, Z\rangle = 0. \tag{14}$$

Hence, by using (14), we have

$$\dot V \ge \langle W, W\rangle + \langle Y, a_2 W\rangle + \langle J_F(X)Y, Y\rangle \ge$$

$$\ge \langle W, W\rangle - |a_2|\, \|Y\|\, \|W\| + \langle J_F(X)Y, Y\rangle. \tag{15}$$

In view of (15), the assumption $\lambda_i(J_F(X)) > \frac{1}{4}\,a_2^2$ of the theorem, we conclude that

$$\dot{V} \geq \left\| W + \frac{1}{2}\,a_2 Y \right\|^2 + \langle J_F(X)Y, Y\rangle - \frac{1}{4}\,a_2^2\langle Y, Y\rangle \geq$$

$$\geq \langle J_F(X)Y, Y\rangle - \frac{1}{4}\,a_2^2\langle Y, Y\rangle \geq 0.$$

Now, it is clear that

$$\dot{V}(0,0,0,W) = \langle W, W\rangle = \|W\|^2 \geq 0.$$

Hence, the above discussion shows that the function $\dot{V}$ is positive semi-definite. This case verifies the property (K_2). Next, let $(X(t), Y(t), Z(t), W(t))$ be an arbitrary solution of the system (3). We observe from the previous estimate of $\dot{V}$ in (13) that $\dot{V} = 0, t \geq 0$, necessarily implies that $Y(t) = 0$ for all $t \geq 0$ and, therefore, also that $X = \xi$ (a constant vector). In its turn this implies that

$$X = \xi, \quad Y = \dot{X}, \quad Z = \dot{Y} = 0, \quad W = \dot{Z} = 0 \quad \text{for all} \quad t \geq 0.$$

The substitution of the estimates $X = \xi$, $Y = \dot{X}$, $Z = \dot{Y} = 0$, $W = \dot{Z} = 0$ for all $t \geq 0$ into the system (4) leads to the result $F(\xi) = 0$ which, by the assumption of the theorem, implies (only) that $\xi = 0$. Thus $\dot{V} = 0, t \geq 0$, implies that

$$X = Y = Z = W = 0 \quad \text{for all} \quad t \geq 0.$$

Hence the property of (K_3) holds for the function V.

The theorem is thereby established.

Example. As a special case of the system (4), let us choose, for the case $n = 4$, Ψ, G, Θ and F that appeared in the system (4) to be the following:

$$\Psi(Z) = \begin{bmatrix} 1+z_1^2 & 0 & 0 & 0 \\ 0 & 1+z_2^2 & 0 & 0 \\ 0 & 0 & 1+z_3^2 & 0 \\ 0 & 0 & 0 & 1+z_4^2 \end{bmatrix},$$

$$G(X,Y,Z,W) = \begin{bmatrix} 1-x_1^2-y_1^2-z_1^2-w_1^2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2-\dfrac{2}{1+x_3^2+z_3^4} & 0 \\ 0 & 0 & 0 & \dfrac{1}{2}-\dfrac{3}{2+y_4^2+z_4^2+w_4^4} \end{bmatrix},$$

$$\Theta(Y) = \begin{bmatrix} 2y_1 + y_1^3 + y_1^5 \\ \\ y_2 + \dfrac{1}{3}\, y_2^3 \\ \\ 3y_3 + \dfrac{1}{5}\, y_3^5 \\ \\ 2y_4 + \dfrac{1}{3}\, y_4^3 \end{bmatrix},$$

and

$$F(X) = \begin{bmatrix} 3x_1 + \dfrac{1}{3}\, x_1^3 \\ \\ 2x_2 + \dfrac{1}{6}\, x_2^3 \\ \\ 4x_3 + x_3^3 \\ \\ 5x_4 + \dfrac{1}{3}\, x_4^3 \end{bmatrix}.$$

Then, clearly, eigenvalues of the matrix $\Psi(Z)$ are

$$\lambda_1(\Psi(Z)) = 1 + z_1^2, \quad \lambda_2(\Psi(Z)) = 1 + z_2^2, \quad \lambda_3(\Psi(Z)) = 1 + z_3^2, \quad \lambda_4(\Psi(Z)) = 1 + z_4^2.$$

Trivially,

$$\lambda_i(\Psi(Z)) \geq 1 = a_1 > 0, \quad i = 1, 2, \ldots, 4.$$

Next, observe that

$$\lambda_1(G(X,Y,Z,W)) = 1 - x^2 - y^2 - z^2 - w^2 \leq 1, \quad \lambda_2(G(X,Y,Z,W)) = 2,$$

$$\lambda_3(G(X,Y,Z,W)) = 2 - \frac{2}{1 + x^2 + z^4} \leq 2,$$

$$\lambda_4(G(X,Y,Z,W)) = \frac{1}{2} - \frac{3}{2 + y^2 + z^2 + w^2} \leq \frac{1}{2}.$$

Thus, one can choose $\lambda_i(G(X,Y,Z,W)) \leq a_2 = 2$, $i = 1, 2, \ldots, 4$. Similarly, it follows that $J_\Theta(Y)$ is a symmetric matrix, namely,

$$J_\Theta(Y) = \begin{bmatrix} 2 + 3y_1^2 + 5y_1^4 & 0 & 0 & 0 \\ 0 & 1 + y_2^2 & 0 & 0 \\ 0 & 0 & 3 + y_3^4 & 0 \\ 0 & 0 & 0 & 2 + y_4^2 \end{bmatrix},$$

hence it is readily verified that $J_\Theta(Y)$ is a symmetric matrix. Now, the derivative of the function $F(X)$ gives that

$$J_F(X) = \begin{bmatrix} 3+x_1^2 & 0 & 0 & 0 \\ 0 & 2+\dfrac{1}{2}x_2^2 & 0 & 0 \\ 0 & 0 & 4+3x_3^2 & 0 \\ 0 & 0 & 0 & 5+x_4^2 \end{bmatrix}.$$

Hence,

$$\lambda_1(J_F(X)) = 3 + x_1^2 \geq 3 > 0, \quad \lambda_2(J_F(X)) = 2 + \frac{1}{2}x_2^2 \geq 2 > 0,$$

$$\lambda_3(J_F(X)) = 4 + 3x_3^2 \geq 4 > 0, \quad \lambda_3(J_F(X)) = 5 + 3x_4^2 \geq 5 > 0.$$

Thus

$$\lambda_i(J_F(X)) \geq 2, \quad i = 1, 2, \ldots, 4.$$

Therefore, it can be concluded from the above discussion that

$$\lambda_i(J_F(X)) - \frac{1}{4}a_2^2 = \lambda_i(J_F(X)) - 1 > 0, \quad i = 1, 2, \ldots, 4.$$

Thus all the conditions of the theorem are satisfied.

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