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STABILITY AND BOUNDEDNESS RESULTS ON CERTAIN NONLINEAR VECTOR DIFFERENTIAL EQUATIONS OF FOURTH ORDER

РЕЗУЛЬТАТИ ПРО СТІЙКІСТЬ ТА ОБМЕЖЕНІСТЬ ДЕЯКИХ НЕЛІНІЙНИХ ВЕКТОРНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ЧЕТВЕРТОГО ПОРЯДКУ

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We consider the equation

$$X^{(4)} + \Phi(X'')X''' + F(X, X')X'' + G(X') + H(X) = P(t, X, X', X'', X''')$$

in two cases: $P \equiv 0$ and $P \neq 0$. In the case $P \equiv 0$, the asymptotic stability of the zero solution $X = 0$ of the equation is investigated; in the case $P \neq 0$ the boundedness of all solutions of the equation are proved.

Розглядається рівняння

$$X^{(4)} + \Phi(X'')X''' + F(X, X')X'' + G(X') + H(X) = P(t, X, X', X'', X''')$$

у двох випадках: $P \equiv 0$ та $P \neq 0$. У випадку $P \equiv 0$ вивчається асимптотична стійкість нульового розв'язку $X = 0$ рівняння; у випадку $P \neq 0$ доведено обмеженість усіх розв'язків рівняння.

1. Introduction. It is well known that a study of qualitative properties of solutions, in particular, an investigation of stability and boundedness of them is a very important problem in the theory and applications of differential equations. In the last three decades, a great effort has been made to study stability and boundedness of solutions of nonlinear ordinary differential equations of higher order, second-, third-, fourth-, fifth- and sixth-order, see e.g. [1 – 38] and the references cited therein for some related works existing on stability and boundedness of the solutions. In the above works, perhaps, due to the effectiveness of the method, the authors dealt with the problems by using the Lyapunov's second (or direct) method [39] and obtained criteria for stability and boundedness of solutions of the equations under consideration. It is worth mentioning the opinions of some authors about the method. In [40], Iggidr and Sallet expressed that "The most efficient tool for the study of the stability of a given nonlinear system is provided by Lyapunov theory". Next, in [16], Qian stated that "So far, the most effective method to study the stability of nonlinear differential equations is still the Lyapunov's direct method" . Of course, when one applies this method, finding a proper Lyapunov function in general is a big challenge. In spite of the existence of many works on the stability and boundedness of solutions for various second-, third-, fourth- and fifth-order nonlinear scalar differential equations, there are only a few results about certain fourth-order nonlinear vector differential equations on the topic. Namely, one may refer to Abou-El-Ela and Sadek [1], Sadek [19] and Tunç [24] for some

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recent publications on these topics. Recently, the case when $n = 1$ was considered in [20, 27] for the differential equation

$$x^{(4)} + \varphi(x'')x''' + f(x,x')x'' + g(x') + h(x) = p(t,x,x',x'',x''').$$

Here, we consider the fourth-order nonlinear vector differential of the form

$$X^{(4)} + \Phi(X'')X''' + F(X,X')X'' + G(X') + H(X) = P(t,X,X',X'',X'''), \qquad (1)$$

where $X \in \mathfrak{R}^n$, Φ and F are $(n\times)n$-symmetric matrix functions; $G\colon \mathfrak{R}^n \to \mathfrak{R}^n$, $H\colon \mathfrak{R}^n \to \mathfrak{R}^n$ and $P\colon \mathfrak{R}^+ \times \mathfrak{R}^n \times \mathfrak{R}^n \times \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n$. Let Φ, F, G, H and P be continuous and so constructed such that the uniqueness theorem is valid. Equation (1) is the vector version for systems of real fourth-order nonlinear differential equations of the form

$$x_i^{(4)} + \sum_{k=1}^{n} \phi_{ik}(x_1'', x_2'', \ldots, x_n'')x_k''' + \sum_{k=1}^{n} f_{ik}(x_1, x_2, \ldots, x_n; x_1', x_2', \ldots, x_n')x_k'' +$$

$$+ g_i(x_1', x_2', \ldots, x_n') + h_i(x_1', x_2', \ldots, x_n') =$$

$$= p_i(t; x_1, x_2, \ldots, x_n; x_1', x_2', \ldots, x_n'; x_1'', x_2'', \ldots, x_n''; x_1''', x_2''', \ldots, x_n'''), \quad i = 1, 2, \ldots, n.$$

We shall assume, as basic throughout in what follows, that the derivatives $\dfrac{\partial \phi_{ik}}{\partial x_j''}, \dfrac{\partial f_{ik}}{\partial x_j}, \dfrac{\partial f_{ik}}{\partial x_j'}$, $\dfrac{\partial g_{ik}}{\partial x_j'}$ and $\dfrac{\partial h_{ik}}{\partial x_j}$, $j, k = 1, 2, \ldots, n$, exist and are continuous. Now, we write equation (1) as the following equivalent system:

$$X' = Y, \quad Y' = Z, \quad Z' = W,$$
$$W' = \Phi(Z)W - F(X,Y)Z - G(Y) - H(X) + P(t,X,Y,Z,W), \qquad (2)$$

which was obtained as usual by setting $X' = Y$, $X'' = Y$, $X''' = W$ in (1). It will also be assumed throughout the paper that the Jacobian matrices $J_H(X), J_G(Y), J_\Phi(Z), J(F(X,Y)Y|X)$ and $J(F(X,Y)Y|Y)$ corresponding to $H(X), G(Y), \Phi(Z)$ and $F(X,Y)$, respectively, are symmetric and given by

$$J_H(X) = \left(\frac{\partial h_i}{\partial x_j}\right), \quad J_G(Y) = \left(\frac{\partial g_i}{\partial y_j}\right), \quad J_\Phi(Z) = \left(\frac{\partial \phi_i}{\partial z_j}\right),$$

$$J(F(X,Y)Y|X) = \left(\frac{\partial}{\partial x_j}\sum_{k=1}^{n} f_{ik}y_k\right) = \left(\sum_{k=1}^{n}\frac{\partial f_{ik}}{\partial x_j}y_k\right),$$

$$J(F(X,Y)Y|Y) = \left(\frac{\partial}{\partial y_j}\sum_{k=1}^{n} f_{ik}y_k\right) = F(X,Y) + \left(\sum_{k=1}^{n}\frac{\partial f_{ik}}{\partial y_j}y_k\right).$$

Moreover, the symbol $\langle X,Y\rangle$ will be used to denote the usual scalar product corresponding to any pair X,Y in $\Re^n$, that is, $\langle X,Y\rangle = \sum_{i=1}^{n} x_i y_i$; thus $\langle X,Y\rangle = \|X\|^2$, and $\lambda_i(A),\ i = 1,2,\ldots,n$, are eigenvalues of the $(n\times n)$-matrix A.

The motivation for the present work has come from the papers of Chin [5], Shi-zhong, Zheng-rong and Yuan-hong [20], Tunç [27], Wu and Xiong [38] and the papers mentioned above. The results obtained here are also an n-dimensional analogue of the results in [20, 27, 38]. Our aim is to obtain similar results and to generalize, revise, and improve some results established in the papers just stated above [5, 20, 27, 38] to the equation (1). It should also be noted that the domain of attraction of the zero solution $X = 0$ of system (2) (for $P \equiv 0$) in the following first result is not going to be determined here.

2. The stability and boundedness results of solutions of system (2). In this section, we study the stability and the boundedness of solutions of system (2) by using the Lyapunov's second (or direct) direct method. The following theorems make the main results.

In the case $P\equiv 0$, we have the following result.

Theorem 1. *In addition to the basic assumptions imposed on Φ, F, G and H, suppose that there are positive constants a, b, c, d, δ and ε such that the following conditions are satisfied:*

(i) $H(0) = G(0) = 0$;

(ii) $abc - c^2 - a^2 d \ge abc - c\,\|J_G(Y)\| - ad \left\| \int\limits_0^1 \Phi(\sigma Z)d\sigma \right\| \ge \delta > 0$ *for all Y and Z;*

(iii) eigenvalues of the matrices $J_H(X)$ and $(dI - J_H(X))$, that is, and $\lambda_i\,(J_H(X))$ and $\lambda_i[dI - J_H(X))$, respectively, satisfy $\lambda_i\,(J_H(X)) \ge d$ and

$$0 \le \lambda_i\,(dI - J_H(X)) \le \frac{\sqrt{\varepsilon\delta a}}{8} \quad \textit{for all} \quad X,\ \ i = 1,2,...,n,$$

in which $\varepsilon \le \dfrac{\delta}{2acD}$, $D = ab + \dfrac{bc}{d}$ and I is the $(n\times n)$-identity matrix;

(iv) eigenvalues of the matrices $J_G(Y)$ and $(J_G(Y) - cI)$, that is, $\lambda_i(J_G(Y)$ and $\lambda_i(J_G(Y) - cI)$, respectively, satisfy $\lambda_i(J_G(Y)) \ge c$ and

$$0 \le \lambda_i\,(J_G(Y) - cI) < \frac{\delta}{8c}\sqrt{\frac{d}{2ac}} \quad \textit{for all} \quad Y, \quad i = 1,2,\ldots,n;$$

(v) the matrix $F(X,Y)$ is symmetric, eigenvalues of the matrices $F(X,Y)$ and the matrix $(F(X,Y) - bI)$, that is, $\lambda_i\,(F(X,Y))$ and $\lambda_i(F(X,Y) - bI)$, respectively, satisfy $\lambda_i\,(F(X,Y)) \ge b$ and

$$0 \le \lambda_i\,(F(X,Y) - bI) \le \frac{a}{8}\sqrt{\frac{\varepsilon\delta}{c}} \quad \textit{for all} \quad X \ \textit{and} \ Y, \quad 1 = 1,2,\ldots,n;$$

(vi) the matrix $\Phi(Z)$ is symmetric, eigenvalues of the matrices $\Phi(Z)$ and $(\Phi(Z) - aI)$, that is, $\lambda_i(\Phi(Z))$ and $\lambda_i(\Phi(Z) - aI)$, respectively, satisfy $\lambda_i(\Phi(Z)) \ge a$ and

$$0 \le \lambda_i\,(\Phi(Z) - aI) \le \frac{\delta}{32a}\frac{1}{\sqrt{acd}} \quad \textit{for all} \quad Z,\ \ i = 1,2,\ldots,n;$$

(vii) $J\left(F(X,Y)Y\,|X\right)$ *is negative definite for all* X *and* Y.
Then the zero solution of system (2) is asymptotically stable.

Remark 1. From conditions (ii), (iv) and (vi) of Theorem 1 we can obtain

$$\left\| \int_0^1 \Phi(\sigma Z)d\sigma \right\| < \frac{bc}{d}, \qquad \|J_G(Y)\| < ab.$$

Remark 2. In the case $n = 1$, equation (1) reduces to a scalar ordinary nonlinear differential equation of the form

$$x^{(4)} + \varphi(\ddot{x})\dddot{x} + f(x,\dot{x})\ddot{x} + g(\dot{x}) + h(x) = p(t,x,\dot{x},\ddot{x},\dddot{x}).$$

When we take $\varphi(\ddot{x}) = a$, $f(x,\dot{x}) = b$, $g(\dot{x}) = c\dot{x}$, $h(x) = dx$ and $p(t,x,\dot{x},\ddot{x},\dddot{x}) = 0$ in the above equation, then the equation clearly reduces to a linear constant coefficient differential equation and conditions (i) – (vii) of Theorem 1 reduce to the corresponding Routh – Hurwitz criterion.

Remark 3. For the case $n = 1$, Theorem 1 includes the first results of Tunç (Theorem 1 [27]) except that some minor modifications arise in the conditions established here and in Theorem 1 [24], Ezeilo [9, 10], Harrow [12] and Wu and Xiong [38], and also improves the results in [9, 10, 12] and [38] except for the restriction on $F(X,Y)$, that is, $0 \le \lambda_i\left(F(X,Y) - bI\right) \le \frac{a}{8}\sqrt{\frac{\varepsilon\delta}{c}}$, $i = 1, 2, ..., n$. Namely, the results in [9, 10, 12] and [38] were obtained for certain scalar ordinary differential equations of fourth order, which are special cases of differential equation (1). When, we compare the assumptions established in Theorem 1 here with that constituted in [9, 10, 12] and [38], our assumptions are less restrictive than those established in [9, 10, 12] and [38] except for some minor modifications. For the sake of the brevity, we would not like to give details of the comparison. Finally, Theorem 1 also revises the first theorem in [20].

In the case $P \neq 0$ we have the following result.

Theorem 2. *Suppose the following are satisfied:*
(i) all the conditions of Theorem 1 hold,
(ii) $\|P(t,X,Y,Z,W)\| \le (\bar{a} + \|Y\| + \|Z\| + \|W\|)\,\theta(t)$, *where* $\theta(t)$ *is a nonnegative and continuous function of* t, *and satisfies* $\int_0^t \theta(s)ds \le \bar{b} < \infty$ *for all* $t \ge 0$, $\bar{a}$ *and* $\bar{b}$ *are positive constants. Then there exists a positive constant* k *such that any solution* $(X(t), Y(t), Z(t), W(t))$ *of system (2) determined by*

$$X(0) = X_0, \quad Y(0) = Y_0, \quad Z(0) = Z_0, \quad W(0) = W_0$$

satisfies, for all $t \ge 0$,

$$\|X(t)\| \le k, \quad \|Y(t)\| \le k, \quad \|Z(t)\| \le k, \quad \|W(t)\| \le k.$$

Remark 4. Theorem 2 revises the second result in [20] , and also gives an n-dimensional generalization for the results obtained in [9, 13, 20, 27].

Now, define the Lyapunov function $v_0 = v_0(X, Y, Z, W)$ as:

$$2v_0 = 2\beta \int_0^1 \langle H(\sigma X), X\rangle \, d\sigma + 2\beta \int_0^1 \langle \sigma F(X, \sigma Y)Y, Y\rangle \, d\sigma -$$

$$- \alpha \langle dY, Y\rangle + 2 \int_0^1 \langle G(\sigma Y), Y\rangle \, d\sigma + \alpha \langle bZ, Z\rangle +$$

$$+ 2 \int_0^1 \langle \sigma \Phi(\sigma Z)Z, Z\rangle \, d\sigma - \beta \langle Z, Z\rangle + \alpha \langle W, W\rangle + 2 \langle H(X), Y\rangle +$$

$$+ 2\alpha \langle H(X), Z\rangle + 2\alpha \langle G(Y), Z\rangle +$$

$$+ 2\beta \int_0^1 \langle \Phi(\sigma Z)Z, Y\rangle \, d\sigma + 2\beta \langle Y, W\rangle + 2 \langle Z, W\rangle , \tag{3}$$

where

$$\alpha = \varepsilon + \frac{1}{a}, \qquad \beta = \varepsilon + \frac{d}{c}. \tag{4}$$

The following lemmas will be needed in the proofs of our main results.

Lemma 1. *Let A be a real symmetric $(n \times n)$-matrix and*

$$a' \geq \lambda_i(A) \geq a > 0, \quad i = 1, 2, ..., n,$$

where a', a are constants.
Then

$$a' \langle X, X\rangle \geq \langle AX, X\rangle \geq a \langle X, X\rangle$$

and

$$a'^2 \langle X, X\rangle \geq \langle AX, AX\rangle \geq a^2 \langle X, X\rangle .$$

Proof. See [24].

Lemma 2.

$$\frac{d}{dt}\int_0^1 \langle H(\sigma X), X\rangle\, d\sigma = \langle H(X), Y\rangle ,$$

$$\frac{d}{dt}\int_0^1 \langle G(\sigma Y), Y\rangle\, d\sigma = \langle G(Y), Z\rangle ,$$

$$\frac{d}{dt}\int_0^1 \langle \sigma F(X,\sigma Y)Y, Y\rangle\, d\sigma = \langle F(X,Y)Z, Y\rangle + \int_0^1 \langle \sigma J\left(F(X,\sigma Y)Y\,|X\right)Y, Y\rangle d\sigma,$$

$$\frac{d}{dt}\int_0^1 \langle \sigma\Phi(\sigma Z)Z, Z\rangle\, d\sigma = \langle \Phi(Z)W, Z\rangle ,$$

$$\frac{d}{dt}\int_0^1 \langle \Phi(\sigma Z)Z, Y\rangle\, d\sigma = \langle \Phi(Z)W, Y\rangle + \int_0^1 \langle \Phi(\sigma Z)Z, Z\rangle\, d\sigma.$$

Proof. We have

$$\frac{d}{dt}\int_0^1 \langle H(\sigma X), X\rangle\, d\sigma = \int_0^1 \sigma\,\langle J_H(\sigma X)Y, X\rangle\, d\sigma + \int_0^1 \langle H(\sigma X), Y\rangle\, d\sigma =$$

$$= \int_0^1 \sigma\frac{\partial}{\partial\sigma}\,\langle H(\sigma X), Y\rangle\, d\sigma + \int_0^1 \langle H(\sigma X), Y\rangle\, d\sigma =$$

$$= \sigma\,\langle H(\sigma X), Y\rangle\Big|_0^1 = \langle H(X), Y\rangle ,$$

$$\frac{d}{dt}\int_0^1 \langle G(\sigma Y), Y\rangle\, d\sigma = \int_0^1 \sigma\,\langle J_G(\sigma Y)Z, Y\rangle\, d\sigma + \int_0^1 \langle G(\sigma Y), Z\rangle\, d\sigma =$$

$$= \int_0^1 \sigma\frac{\partial}{\partial\sigma}\,\langle G(\sigma Y), Z\rangle\, d\sigma + \int_0^1 \langle G(\sigma Y), Z\rangle\, d\sigma =$$

$$= \sigma\,\langle G(\sigma Y), Z\rangle\Big|_0^1 = \langle G(Y), Z\rangle ,$$

$$\frac{d}{dt}\int_0^1 \langle \sigma F(X,\sigma Y)Y,Y\rangle\, d\sigma = \int_0^1 \langle \sigma F(X,\sigma Y)Z,Y\rangle\, d\sigma + \int_0^1 \langle \sigma F(X,\sigma Y)Y,Z\rangle\, d\sigma +$$

$$+\int_0^1 \left\langle \sigma J(F(X,\sigma Y)Y\big|X)Y,Y\right\rangle d\sigma +$$

$$+\int_0^1 \left\langle \sigma^2 J(F(X,\sigma Y)Z\big|Y)Y,Y\right\rangle d\sigma =$$

$$= \int_0^1 \langle \sigma F(X,\sigma Y)Z,Y\rangle\, d\sigma + \int_0^1 \left\langle \sigma J(F(X,\sigma Y)Y\big|X)Y,Y\right\rangle d\sigma +$$

$$+\int_0^1 \sigma\frac{\partial}{\partial\sigma}\langle \sigma F(X,\sigma Y)Z,Y\rangle\, d\sigma =$$

$$= \sigma^2 \langle F(X,Y)Z,Y\rangle\Big|_0^1 + \int_0^1 \left\langle \sigma J(F(X,\sigma Y)Y\big|X)Y,Y\right\rangle d\sigma =$$

$$= \langle F(X,Y)Z,Y\rangle + \int_0^1 \left\langle \sigma J(F(X,\sigma Y)Y\big|X)Y,Y\right\rangle d\sigma,$$

$$\frac{d}{dt}\int_0^1 \langle \sigma\Phi(\sigma Z)Z,Z\rangle\, d\sigma = \int_0^1 \sigma\langle \Phi(\sigma Z)Z,W\rangle\, d\sigma + \int_0^1 \sigma^2\langle J_\Phi(\sigma Z)ZW,Z\rangle\, d\sigma +$$

$$+\int_0^1 \langle \sigma\Phi(\sigma Z)W,Z\rangle\, d\sigma =$$

$$= \int_0^1 \sigma\frac{\partial}{\partial\sigma}\langle \sigma\Phi(\sigma Z)W,Z\rangle\, d\sigma + \int_0^1 \langle \sigma\Phi(\sigma Z)W,Z\rangle\, d\sigma =$$

$$= \sigma^2\langle \Phi(\sigma Z)W,Z\rangle\Big|_0^1 = \langle \Phi(Z)W,Z\rangle,$$

$$\frac{d}{dt}\int_0^1 \langle \Phi(\sigma Z)Z, Y\rangle\, d\sigma = \int_0^1 \langle \Phi(\sigma Z)Z, Z\rangle\, d\sigma + \int_0^1 \sigma \langle J_\Phi(\sigma Z)WZ, Y\rangle\, d\sigma +$$

$$+ \int_0^1 \langle \Phi(\sigma Z)W, Y\rangle\, d\sigma =$$

$$= \int_0^1 \langle \Phi(\sigma Z)Z, Z\rangle\, d\sigma + \int_0^1 \sigma \frac{\partial}{\partial \sigma} \langle \Phi(\sigma Z)W, Y\rangle\, d\sigma +$$

$$+ \int_0^1 \langle \Phi(\sigma Z)W, Y\rangle\, d\sigma =$$

$$= \sigma \langle \Phi(\sigma Z)W, Y\rangle \Big|_0^1 + \int_0^1 \langle \Phi(\sigma Z)Z, Z\rangle\, d\sigma =$$

$$= \langle \Phi(Z)W, Y\rangle + \int_0^1 \langle \Phi(\sigma Z)Z, Z\rangle\, d\sigma.$$

Lemma 3. *If the conditions of Theorem 1 hold, then there exists a positive constant* d_1 *such that*

$$v_0(X, Y, Z, W) \geq d_1(\|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2)$$

is valid for every solution of system (2).

Proof. Rewrite the function $v_0 = v_0(X, Y, Z, W)$ as follows:

$$2v_0 = \left(\frac{1}{c}\right) \|H(X) + cY + \alpha c Z\|^2 +$$

$$+ \left(\frac{1}{a}\right) \|W + aZ + \beta a Y\|^2 +$$

$$+ 2\alpha \langle G(Y), Z\rangle - 2\alpha c \langle Y, Z\rangle +$$

$$+ 2\beta \int_0^1 \langle \Phi(\sigma Z)Z, Y\rangle d\sigma - 2a\beta \langle Z, Y\rangle + \sum_{i=1}^{4} w_i, \qquad (5)$$

where

$$w_1 \equiv 2\beta \int_0^1 \langle H(\sigma X), X\rangle \, d\sigma - \frac{1}{c}\langle H(X), H(X)\rangle ,$$

$$w_2 \equiv 2\beta \int_0^1 \langle \sigma F(X, \sigma Y)Y, Y\rangle \, d\sigma - \left[\alpha d + \beta^2 a\right] \|Y\|^2 + 2\int_0^1 \langle G(\sigma Y), Y\rangle \, d\sigma - c\langle Y, Y\rangle ,$$

$$w_3 \equiv \left[\alpha b - \beta - \alpha^2 c\right] \|Z\|^2 + 2\int_0^1 \langle \sigma \Phi(\sigma Z)Z, Z\rangle \, d\sigma - a\langle Z, Z\rangle ,$$

$$w_4 \equiv \left[\alpha - \frac{1}{a}\right] \|W\|^2 .$$

Since $\frac{\partial}{\partial\sigma} H(\sigma X) = J_H(\sigma X)X, H(0) = 0$, we have

$$2\beta \int_0^1 \langle H(\sigma X), X\rangle d\sigma = 2\varepsilon \int_0^1 \int_0^1 \langle \sigma_1 J_H(\sigma_1\sigma_2 X)X, X\rangle d\sigma_2 d\sigma_1 +$$

$$+ \frac{2d}{c} \int_0^1 \int_0^1 \langle \sigma_1 J_H(\sigma_1\sigma_2 X)X, X\rangle d\sigma_2 d\sigma_1 \geq$$

$$\geq \varepsilon d \|X\|^2 + \frac{2d}{c} \int_0^1 \int_0^1 \langle \sigma_1 J_H(\sigma_1\sigma_2 X)X, X\rangle d\sigma_2 d\sigma_1 .$$

Now, because of $\frac{\partial}{\partial\sigma_1}\langle H(\sigma_1 X), H(\sigma_1 X)\rangle = 2\langle J_H(\sigma_1 X)X, H(\sigma_1 X)\rangle$ and $H(0) = 0$, it follows, on integrating both sides from $\sigma_1 = 0$ to $\sigma_1 = 1$, that

$$\langle H(X), H(X)\rangle = 2\int_0^1 \langle J_H(\sigma_1 X)X, H(\sigma_1 X)\rangle d\sigma_1 .$$

It is also evident that

$$\frac{\partial}{\partial\sigma_2}\langle H(\sigma_1\sigma_2 X), J_H(\sigma_1 X)X\rangle = \langle \sigma_1 J_H(\sigma_1\sigma_2 X)X, J_H(\sigma_1 X)X\rangle .$$

By integrating both sides from $\sigma_2 = 0$ to $\sigma_2 = 1$, we get

$$\langle H(\sigma_1 X), J_H(\sigma_1 X)X\rangle = \int_0^1 \langle \sigma_1 J_H(\sigma_1\sigma_2 X)X, J_H(\sigma_1 X)X\rangle \, d\sigma_2.$$

Hence,

$$w_1 \geq \varepsilon d \|X\|^2 + \frac{2d}{c}\int_0^1\int_0^1 \langle \sigma_1 J_H(\sigma_1\sigma_2 X)X, X\rangle d\sigma_2 d\sigma_1 -$$

$$-\frac{1}{c}\int_0^1\int_0^1 \langle \sigma_1 J_H(\sigma_1\sigma_2 X)X, J_H(\sigma_1 X)X\rangle d\sigma_1 d\sigma_2 \geq$$

$$\geq \varepsilon d \|X\|^2 + \frac{1}{c}\int_0^1\int_0^1 \langle \sigma_1 J_H(\sigma_1\sigma_2 X)X, [dI - J_H(\sigma_1 X)]\, X\rangle d\sigma_1 d\sigma_2 \geq$$

$$\geq \varepsilon d \|X\|^2 \quad \text{by (iii)}.$$

It follows from conditions (ii), (iv), (v) of Theorem 1 and (4) that

$$w_2 \geq \left[\beta b - \alpha d - \beta^2 a\right] \|Y\|^2 + 2\int_0^1\int_0^1 \langle \sigma_1 J_G(\sigma_1\sigma_2 Y)Y, Y\rangle d\sigma_2 d\sigma_1 - c\,\langle Y, Y\rangle \geq$$

$$\geq \left[\beta b - \alpha d - \beta^2 a\right] \|Y\|^2 \geq \left(\frac{\delta d}{2ac^2}\right)\|Y\|^2.$$

By a similar estimation, conditions (ii), (vi) of Theorem 1 and (4) show that

$$w_3 \geq \left(\frac{\delta}{2a^2 c}\right)\|Z\|^2 \quad \text{and} \quad w_4 = \varepsilon \|W\|^2.$$

On using the estimates for w_1, w_2, w_3 and w_4 in (5) we have that

$$2v_0 \geq \varepsilon d \|X\|^2 + \left(\frac{\delta d}{2ac^2}\right)\|Y\|^2 + \left(\frac{\delta}{2a^2 c}\right)\|Z\|^2 + \varepsilon \|W\|^2 + 2\alpha\,\langle G(Y), Z\rangle -$$

$$- 2\alpha c\,\langle Y, Z\rangle + 2\beta\int_0^1 \langle \Phi(\sigma Z)Z, Y\rangle d\sigma - 2a\beta\,\langle Z, Y\rangle. \tag{6}$$

Now, consider the terms

$$w_5 \equiv \left(\frac{\delta d}{4ac^2}\right)\|Y\|^2 + 2\alpha\langle G(Y), Z\rangle - 2\alpha c\langle Y, Z\rangle + \left(\frac{\delta}{8a^2c}\right)\|Z\|^2$$

and

$$w_6 \equiv \left(\frac{\delta d}{16ac^2}\right)\|Y\|^2 + 2\beta\int_0^1 \langle\Phi(\sigma Z)Z, Y\rangle d\sigma - 2a\beta\langle Z, Y\rangle + \left(\frac{\delta}{16a^2c}\right)\|Z\|^2$$

which are contained in (6). Clearly, the conditions (iv) and (vi) of Theorem 1 imply that

$$w_5 = \left(\frac{\delta d}{4ac^2}\right)\|Y\|^2 + 2\alpha\int_0^1 \langle (J_G(\sigma Y) - cI)\,Y, Z\rangle d\sigma + \left(\frac{\delta}{8a^2c}\right)\|Z\|^2 \geq$$

$$\geq \left(\frac{\delta d}{4ac^2}\right)\|Y\|^2 - \left(\frac{\delta}{2ac}\sqrt{\frac{d}{2ac}}\right)\|Y\|\,\|Z\| + \left(\frac{\delta}{8a^2c}\right)\|Z\|^2 =$$

$$= \left[\frac{1}{2c}\sqrt{\frac{\delta d}{a}}\,\|Y\| - \frac{1}{2a}\sqrt{\frac{\delta}{2c}}\,\|Z\|\right]^2 \geq 0$$

and

$$w_6 = \left(\frac{\delta d}{16ac^2}\right)\|Y\|^2 + 2\beta\int_0^1 \langle (\Phi(\sigma Z) - aI)\,Z, Y\rangle d\sigma + \left(\frac{\delta}{16a^2c}\right)\|Z\|^2 \geq$$

$$\geq \left(\frac{\delta d}{16ac^2}\right)\|Y\|^2 - \left(\frac{\delta}{8ac}\sqrt{\frac{d}{ac}}\right)\|Y\|\,\|Z\| + \left(\frac{\delta}{16a^2c}\right)\|Z\|^2 =$$

$$= \left[\frac{1}{4c}\sqrt{\frac{\delta d}{a}}\,\|Y\| - \frac{1}{4a}\sqrt{\frac{\delta}{c}}\,\|Z\|\right]^2 \geq 0.$$

Hence

$$v_0 \geq \varepsilon d\,\|X\|^2 + \left(\frac{3\delta d}{16ac^2}\right)\|Y\|^2 + \left(\frac{5\delta}{16a^2c}\right)\|Z\|^2 + \left(\frac{\varepsilon}{2}\right)\|W\|^2 \geq$$

$$\geq d_1(\|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2),$$

where $d_1 = \min\left\{\varepsilon d, \frac{3\delta d}{16ac^2}, \frac{5\delta}{16a^2c}, \frac{\varepsilon}{2}\right\}$.

This completes the proof of Lemma 3.

Now, let $(X(t), Y(t), Z(t), W(t))$ be an arbitrary solution of system (2). Define $\bar{v}_0(t) = v_0(X(t), Y(t), Z(t), W(t))$. We can easily prove the following lemma.

Lemma 4. *Assume that all the conditions of Theorem 1 are satisfied. Then*

$$\dot{\bar{v}}_0(t) \leq 0 \quad \textit{for all} \quad t \geq 0$$

and especially

$$\dot{\bar{v}}_0(t) = \frac{d}{dt} v_0(X, Y, Z, W) < 0 \quad \textit{provided} \quad \|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2 > 0.$$

Proof. An easy calculation, from (3), (2), Lemma 1 and (vii), yields that

$$\frac{d}{dt} v_0(X, Y, Z, W) \leq -w_7 - w_8 - w_9 - \alpha \langle F(X, Y)Z, W\rangle + \alpha b \langle Z, W\rangle +$$

$$+ \alpha \langle J_H(X)Y, Z\rangle - \alpha d \langle Y, Z\rangle, \tag{7}$$

where

$$w_7 = \beta \langle Y, G(Y)\rangle - \langle J_H(X)Y, Y\rangle,$$

$$w_8 = \left[\langle Z, F(X, Y)Z\rangle - \alpha \langle J_G(Y)Z, Z\rangle - \beta \left\| \int_0^1 \Phi(\sigma Z) d\sigma \right\| \langle Z, Z\rangle \right],$$

$$w_9 = \alpha \langle \Phi(Z)W, W\rangle - \langle W, W\rangle.$$

It is clear that

$$W_7 \geq (\varepsilon c) \|Y\|^2, \quad \text{by (iii) and (iv).} \tag{8}$$

From conditions (ii), (iv) and (v) of Theorem 1, we obtain

$$w_8 \geq \left[b - \alpha \|J_G(Y)\| - \beta \left\| \int_0^1 \Phi(\sigma Z) d\sigma \right\| \right] \langle Z, Z\rangle =$$

$$= \left(\frac{1}{ac}\right) \left[abc - c \|J_G(Y)\| - ad \left\| \int_0^1 \Phi(\sigma Z) d\sigma \right\| \right] \|Z\|^2 -$$

$$- \varepsilon \left[\|J_G(Y)\| + \left\| \int_0^1 \Phi(\sigma Z) d\sigma \right\| \right] \|Z\|^2 \geq$$

$$\geq \left(\frac{\delta}{ac}\right) \|Z\|^2 - \varepsilon D \|Z\|^2 \geq \left(\frac{\delta}{2ac}\right) \|Z\|^2. \tag{9}$$

Similarly, condition (vi) of Theorem 1 shows that

$$w_9 \geq (\varepsilon a)\|W\|^2 . \tag{10}$$

On substituting the estimates (8) – (10) into (7) we get

$$\frac{d}{dt}v_0(X,Y,Z,W) \leq -(\varepsilon c)\|Y\|^2 - \left(\frac{\delta}{2ac}\right)\|Z\|^2 - (\varepsilon a)\|W\|^2 - \alpha\langle F(X,Y)Z,W\rangle +$$

$$+\alpha b\langle Z,W\rangle + \alpha\langle J_H(X)Y,Z\rangle - \alpha d\langle Y,Z\rangle . \tag{11}$$

Let

$$w_{10} = \left(\frac{\varepsilon c}{4}\right)\|Y\|^2 - \alpha\langle (J_H(X)-dI)\,Y,Z\rangle + \left(\frac{\delta}{16ac}\right)\|Z\|^2$$

and

$$w_{11} = \left(\frac{\varepsilon a}{4}\right)\|W\|^2 - \alpha\langle (F(X,Y)-bI)\,Z,W\rangle + \left(\frac{\delta}{16ac}\right)\|Z\|^2 .$$

By using conditions (iii), (v) of Theorem 1 and (4) we find

$$w_{10} \geq \left(\frac{\varepsilon c}{4}\right)\|Y\|^2 - \left(\frac{\sqrt{\varepsilon\delta a}}{4a}\right)\|Y\|\,\|Z\| + \left(\frac{\delta}{16ac}\right)\|Z\|^2 = \left[\frac{\sqrt{\varepsilon c}}{2}\|Y\| - \frac{1}{4}\sqrt{\frac{\delta}{ac}}\|Z\|\right]^2 \geq 0$$

and

$$w_{11} \geq \left(\frac{\varepsilon a}{4}\right)\|W\|^2 - \left(\frac{1}{4}\sqrt{\frac{\varepsilon\delta}{c}}\right)\|Z\|\,\|W\| + \left(\frac{\delta}{16ac}\right)\|Z\|^2 = \left[\frac{\sqrt{\varepsilon a}}{2}\|W\| - \frac{1}{4}\sqrt{\frac{\delta}{ac}}\|Z\|\right]^2 \geq 0.$$

Combining the inequalities for w_{11} and w_{10} in (11) we obtain

$$\frac{d}{dt}v_0(X,Y,Z,W) \leq -\left(\frac{3\varepsilon c}{4}\right)\|Y\|^2 - \left(\frac{3\delta}{8ac}\right)\|Z\|^2 - \left(\frac{3\varepsilon a}{4}\right)\|W\|^2 .$$

Thus, it is evident that

$$\dot{v}_0(t) \leq 0$$

and especially

$$\dot{v}_0(t) < 0 \quad \text{whenever} \quad \|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2 > 0.$$

Proof of Theorem 1. From Lemma 3 and Lemma 4, we see that the function $v_0(X,Y,Z,W)$ is Lyapunov for system (2). Hence, the zero solution of system (2) is asymptotically stable [38].

This completes the proof of Theorem 1.

Proof of Theorem 2. The proof here is based essentially on the method devised by Antosiewicz [41]. Consider the function $\bar{v}_0$ defined as above. Then under the conditions of Theorem 2, the conclusion of Lemma 3 can be obtained, that is

$$\bar{v}_0 \geq d_1(\|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2), \tag{12}$$

and since $P \neq 0$, the conclusion of Lemma 4 can be revised as follows:

$$\dot{\bar{v}}_0 \leq -\left(\frac{3\varepsilon c}{4}\right)\|Y\|^2 - \left(\frac{3\delta}{8ac}\right)\|Z\|^2 - \left(\frac{3\varepsilon a}{4}\right)\|W\|^2 + \langle \alpha W + Z + \beta Y, P\rangle \leq$$

$$\leq (\alpha\|W\| + \|Z\| + \beta\|Y\|)\,\|P(t, X, Y, Z, W)\| \leq$$

$$\leq (\alpha\|W\| + \|Z\| + \beta\|Y\|)\,(A + \|Y\| + \|Z\| + \|W\|)\,\theta(t).$$

Let $d_2 = \max\{\alpha, 1, \beta\}$. Using the inequalities

$$\|W\| \leq 1 + \|W\|^2 \quad \text{and} \quad 2\|Y\|\,\|W\| \leq \|Y\|^2 + \|W\|^2$$

we find

$$\dot{\bar{v}}_0 \leq d_2\left[3\bar{a} + (\bar{a} + 3)\left(\|Y\|^2 + \|Z\|^2 + \|W\|^2\right)\right]\theta(t).$$

Hence,

$$\dot{\bar{v}}_0 \leq d_3\left[1 + \left(\|Y\|^2 + \|Z\|^2 + \|W\|^2\right)\right]\theta(t), \tag{13}$$

where $d_3 = \max\{3d_2\bar{a}, d_2(\bar{a} + 3)\}$.

It follows from (12) and (13) that

$$\dot{\bar{v}}_0 \leq d_3\theta(t) + d_4\bar{v}_0\theta(t), \tag{14}$$

where $d_4 = d_3/d_1$. Integrating both sides of (14) from 0 to t, $t \geq 0$, leads to the inequality

$$\bar{v}_0(t) - \bar{v}_0(0) \leq d_3\int_0^t \theta(s)ds + d_4\int_0^t \bar{v}_0(s)\theta(s)ds.$$

On putting $d_5 = \bar{v}_0(0) + d_3\bar{b}$, we obtain

$$\bar{v}_0(t) \leq d_5 + d_4\int_0^t \bar{v}_0(s)\theta(s)ds.$$

Gronwall – Bellman inequality, see [42], yields

$$\bar{v}_0(t) \le d_5 \exp\left(d_4 \int\limits_0^t \theta(s) ds \right).$$

The proof of Theorem 2 is now complete.

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