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MAXIMUM RECOVERABLE WORK IN LINEAR THERMOELECTROMAGNETISM*

МАКСИМАЛЬНА ЕНЕРГІЯ , ЯКУ МОЖНА ПОВЕРНУТИ В ЛІНІЙНОМУ ТЕРМОЕЛЕКТРОМАГНЕТИЗМІ

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We give a general closed expression for the minimum free energy in terms of Fourier-transformed quantities for a thermoelectromagnetic conductor with memory effects for the electric current density and the heat flux, when the integrated histories of the electric field and of the temperature gradient are chosen to characterize the states of the material. An equivalent formulation is derived and applied to the discrete spectrum model material response.

Наведено загальний замкнений вираз для мінімальної вільної енергії в термінах перетворення Фур'є для провідника з ефектом запам'ятовування густини електричного струму та теплового потоку у випадку, коли для характеризації стану матеріалу розглядається інтегральна історія електричного струму та градієнта температури. Отримано і застосовано еквівалентне формулювання до моделі реакції матеріалу, що має дискретний спектр.

1. Introduction. In a recent work [1] we have studied the problem of finding an explicit form for the minimum free energy of a linear thermoelectromagnetic conductor characterized, in particular, by two constitutive equations for the electric current density and for the heat flux expressed by means of two local functionals of the histories of the electric field and the temperature gradient, respectively [2 – 4]. The investigation of such a problem is very important since the minimum free energy is related to the maximum recoverable work, that is, the maximum quantity of work we can obtain from the material at a given state.

The presence of thermal effects in electromagnetism was considered, in particular, in [5, 6] by Coleman & Dill, who also derived the restrictions imposed on the constitutive equations by the laws of thermodynamics. Also in [7] these problems were studied for the materials considered later on in [1], giving a theorem of uniqueness, existence, and asymptotic stability too.

In this work we follow Golden's lines of [8], where the minimum free energy is determined for a linear viscoelastic material in a scalar case, together with the procedure used in [9] for an analogous problem always in viscoelasticity, see also [10 – 12]. While in [1] the states of the thermoelectromagnetic material are characterized by the actual values of the electric and the magnetic fields, of the temperature and the histories of the electric field and of the temperature

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gradient, in the present work we consider the integrated histories of the electric field and of the temperature gradient instead of their histories. Thus, new appropriate definitions of the continuation of histories and processes are required.

The layout of this article is the following. After introducing the constitutive equations we recall the thermodynamic restrictions on them in Section 2, writing some fundamental relationships. In Section 3, we consider the definitions of states and processes and in the following Section 4 we give a first definition of equivalence between integrated histories. After considering the thermoelectromagnetic work in Section 5, we give an equivalent definition of equivalence by means of the boundedness of the work in Section 6. Then, in Section 7 we derive the expression of the maximum recoverable work, of which another equivalent form, derived in Section 8, is used to study the particular case of discrete spectrum materials in the last Section 9.

2. Preliminaries and basic equations. A rigid electromagnetic $\mathcal{B}$ occupies a region Ω, which is a bounded and simply-connected domain of the three-dimensional Euclidean space $\mathbf{R}^3$, with a smooth boundary $\partial\Omega$, whose outward normal unit is denoted by $\mathbf{n}$.

We are concerned with the linear theory of thermoelectromagnetism, which is characterized by the following constitutive equations:

$$\mathbf{D}(\mathbf{x},t) = \varepsilon\mathbf{E}(\mathbf{x},t) + \vartheta(\mathbf{x},t)\mathbf{a}, \qquad \mathbf{B}(\mathbf{x},t) = \mu\mathbf{H}(\mathbf{x},t), \tag{2.1}$$

$$\mathbf{J}(\mathbf{x},t) = \int\limits_0^{+\infty} \alpha(s)\mathbf{E}^t(\mathbf{x},s)ds, \quad \mathbf{q}(\mathbf{x},t) = -\int\limits_0^{+\infty} k(s)\mathbf{g}^t(\mathbf{x},s)ds, \tag{2.2}$$

$$h(\mathbf{x},t) = c\dot{\vartheta}(\mathbf{x},t) + \Theta_0\left[\mathbf{A}_1\cdot\dot{\mathbf{D}}(\mathbf{x},t)/\varepsilon + \mathbf{A}_2\cdot\dot{\mathbf{B}}(\mathbf{x},t)/\mu\right], \tag{2.3}$$

where $\mathbf{D}$ and $\mathbf{B}$ are the electric displacement and the magnetic induction, $\mathbf{E}$ and $\mathbf{H}$ denote the electric and magnetic fields, ϑ is the temperature relative to the absolute reference temperature Θ_0, uniform in Ω; moreover, the current density $\mathbf{J}$ and the heat flux $\mathbf{q}$ are expressed by two functionals of the histories, up to time t, of the electric field, $\mathbf{E}^t(\mathbf{x},s) = \mathbf{E}(\mathbf{x},t-s)\ \forall s\in\mathbf{R}^+ = [0,+\infty)$, and of the temperature gradient $\mathbf{g} = \nabla\vartheta$, $\mathbf{g}^t(\mathbf{x},s) = \mathbf{g}(\mathbf{x},t-s)\ \forall s\in\mathbf{R}^+$, respectively; finally, h denotes the rate at which heat is absorbed per unit volume.

On supposing that the body is homogeneous and isotropic, all the parameters, that is, the dielectric constant $\varepsilon > 0$, the magnetic permeability $\mu > 0$ and the specific heat $c > 0$, the vectors $\mathbf{a}$, $\mathbf{A}_1$ and $\mathbf{A}_2$ as well as the memory kernels α and k are constant for any point $\mathbf{x}\in\bar{\Omega}$.

In particular, the kernels α and k, called the electric and thermal conductivities, are two relaxation functions $\alpha:\mathbf{R}^+\to\mathbf{R}$ and $k:\mathbf{R}^+\to\mathbf{R}$ given by [2, 3]

$$\alpha(t) = \alpha_0 + \int\limits_0^t \alpha'(\tau)d\tau, \quad k(t) = k_0 + \int\limits_0^t k'(\tau)d\tau \quad \forall t\in\mathbf{R}^+, \tag{2.4}$$

where α_0 and k_0 denote the initial values, at time $t = 0$, of the two functions α and k, with $\lim\limits_{t\to+\infty} \mathrm{lim}\alpha(t) = 0$ and $\lim\limits_{t\to+\infty} \mathrm{lim}k(t) = 0$ under the hypothesis that they belong to $H^1(\mathbf{R}^+)$.

It is useful to introduce the integrated histories of $\mathbf{E}$ and $\mathbf{g}$, which are two functions $\bar{\mathbf{E}}^t(\mathbf{x},\cdot) : \mathbf{R}^+ \to \mathbf{R}^3$ and $\mathbf{g}^t(\mathbf{x},\cdot) : \mathbf{R}^+ \to \mathbf{R}^3$ defined as follows:

$$\bar{\mathbf{E}}^t(\mathbf{x},s) = \int\limits_0^s \mathbf{E}^t(\mathbf{x},\lambda)d\lambda, \quad \bar{\mathbf{g}}^t(\mathbf{x},s) = \int\limits_0^s \mathbf{g}^t(\mathbf{x},\lambda)d\lambda; \tag{2.5}$$

these quantities, taking account of the hypotheses assumed for α and k, allow us to rewrite the constitutive equations (2.2) in an equivalent form,

$$\mathbf{J}(\mathbf{x},t) = -\int\limits_0^{+\infty} \alpha'(s)\bar{\mathbf{E}}^t(\mathbf{x},s)ds, \quad \mathbf{q}(\mathbf{x},t) = \int\limits_0^{+\infty} k'(s)\bar{\mathbf{g}}^t(\mathbf{x},s)ds. \tag{2.6}$$

The constitutive equations (2.1)–(2.3) characterize the behaviour of a simple material [6, 13], for which we have deduced the restrictions that the thermodynamic principles place on the assumed constitutive equations in [7]. These principles state [14, 15] that the following two relations:

$$\oint [h(t) + \dot{\mathbf{D}}(t)\cdot\mathbf{E}(t) + \dot{\mathbf{B}}(t)\cdot\mathbf{H}(t) + \mathbf{J}(t)\cdot\mathbf{E}(t)]dt = 0, \tag{2.7}$$

$$\oint \left\{h(t)/[\Theta_0 + \vartheta(t)] + \mathbf{q}(t)\cdot\mathbf{g}(t)/[\Theta_0 + \vartheta(t)]^2\right\} dt \leq 0 \tag{2.8}$$

must hold for any cyclic process. In the last of these, the equality sign is referred only to reversible processes.

Here we have understood the dependence on the position $\mathbf{x} \in \Omega$ of all the functions; this will be done later on because the statements will be relative to any fixed $\mathbf{x} \in \Omega$.

Since we are concerned with a linear theory, only the linearized expression of (2.8) must be considered; thus, (2.8) becomes [16]

$$\frac{1}{\Theta_0^2}\oint \{h(t)[\Theta_0 - \vartheta(t)] + \mathbf{q}(t)\cdot\mathbf{g}(t)\}\, dt \leq 0 \tag{2.9}$$

and, hence, eliminating $\Theta_0 h(t)$ by means of (2.7), it follows that

$$\frac{1}{\Theta_0^2}\oint \left\{h(t)\vartheta(t) + \Theta_0[\dot{\mathbf{D}}(t)\cdot\mathbf{E}(t) + \dot{\mathbf{B}}(t)\cdot\mathbf{H}(t) + \mathbf{E}(t)\cdot\mathbf{J}(t)] - \mathbf{q}(t)\cdot\mathbf{g}(t)\right\} dt \geq 0. \tag{2.10}$$

This inequality must be satisfied by the constitutive equations; therefore, substituting (2.1) and (2.3) into it, we obtain an inequality, which, because of the arbitrarinesses of ϑ, $\mathbf{E}$, $\mathbf{g}$ and of the same ϑ with respect to $\dot{\mathbf{E}}$ and $\dot{\mathbf{H}}$, gives [7]

$$\mathbf{A}_1 = \mathbf{a}, \qquad \mathbf{A}_2 = \mathbf{0} \tag{2.11}$$

and other two inequalities, concerning with $\mathbf{J}$ and $\mathbf{q}$, which yield

$$\int_0^{+\infty} \alpha(s)\cos(\omega s)ds > 0, \qquad \int_0^{+\infty} k(s)\cos(\omega s)ds > 0 \quad \forall\omega \neq 0. \tag{2.12}$$

These results change the given form to the constitutive equation (2.3), which, taking account of $(2.1)_1$, reduces to

$$h(\mathbf{x},t) = \left(c + \Theta_0 \mathbf{a}^2/\varepsilon\right)\dot{\vartheta}(\mathbf{x},t) + \Theta_0 \mathbf{a}\cdot\dot{\mathbf{E}}(\mathbf{x},t) \tag{2.13}$$

so that (2.10) assumes the following form:

$$\oint \left[\frac{c}{\Theta_0}\dot{\vartheta}(t)\vartheta(t) + \frac{1}{\varepsilon}\dot{\mathbf{D}}(t)\cdot\mathbf{D}(t) + \frac{1}{\mu}\dot{\mathbf{B}}(t)\cdot\mathbf{B}(t) + \mathbf{J}(t)\cdot\mathbf{E}(t) - \frac{1}{\Theta_0}\mathbf{q}(t)\cdot\mathbf{g}(t)\right]dt \geq 0. \tag{2.14}$$

We recall that in thermoelectromagnetism the equations which are to be considered are Maxwell's ones besides the energy equation expressed by $h(\mathbf{x},t) = -\nabla\cdot\mathbf{q}(\mathbf{x},t) + r(\mathbf{x},t)$, where h is given by (2.13) and thus can be eliminated while r denotes the heat sources.

We now introduce the Fourier transform of any function $f : \mathbf{R} \to \mathbf{R}^n$, denoted by f_F and given by

$$f_F(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega\xi} f(\xi)d\xi = f_+(\omega) + f_-(\omega), \tag{2.15}$$

where

$$f_+(\omega) = \int_0^{+\infty} e^{-i\omega\xi} f(\xi)d\xi, \quad f_-(\omega) = \int_{-\infty}^{0} e^{-i\omega\xi} f(\xi)d\xi; \tag{2.16}$$

moreover, we consider the half-range Fourier cosine and sine transforms

$$f_c(\omega) = \int_0^{+\infty} \cos(\omega\xi)f(\xi)d\xi, \quad f_s(\omega) = \int_0^{+\infty} \sin(\omega\xi)f(\xi)d\xi, \tag{2.17}$$

which hold if the function f is defined only on $\mathbf{R}^+$ as well as f_+. In particular, we remember that if f is any function defined on $\mathbf{R}^+$, it can be extended on $\mathbf{R}$ in several ways. For this purpose we can consider its usual extension to $(-\infty, 0)$, where it vanishes identically, or its extensions made with an even ($f(\xi) = f(-\xi)\ \forall\xi < 0$) or an odd ($f(\xi) = -f(-\xi)\ \forall\xi < 0$) function thus obtaining the following relations:

$$f_F(\omega) = f_+(\omega) = f_c(\omega) - if_s(\omega), \quad f_F(\omega) = 2f_c(\omega), \quad f_F(\omega) = -2if_s(\omega), \tag{2.18}$$

which hold in the three cases, respectively.

We observe that the functions $f_{\pm}$, given by (2.16), can be considered as functions of $z \in \mathbf{C}$, that is defined in the complex plane $\mathbf{C}$; these new functions $f_{\pm}(z)$ are analytic for $z \in \mathbf{C}^{(\mp)}$, where we

$$\mathbf{C}^{(-)} = \left\{ z \in \mathbf{C} : \ \operatorname{Im} z \in \mathbf{R}^{--} \right\}, \quad \mathbf{C}^{(+)} = \left\{ z \in \mathbf{C} : \ \operatorname{Im} z \in \mathbf{R}^{++} \right\}, \tag{2.19}$$

$\mathbf{R}^{--}(\mathbf{R}^{++})$ denoting the strictly negative (positive) reals. We can extend, by hypothesis [8], the analyticity on $\mathbf{C}^{\mp}$, where

$$\mathbf{C}^{-} = \left\{ z \in \mathbf{C} : \ \operatorname{Im} z \in \mathbf{R}^{-} \right\}, \quad \mathbf{C}^{+} = \left\{ z \in \mathbf{C} : \ \operatorname{Im} z \in \mathbf{R}^{+} \right\}. \tag{2.20}$$

We shall use the notation $f_{(\pm)}(z)$ to denote that the function has zeros and singularities in $\mathbf{C}^{\pm}$.

Therefore we can rewrite (2.12) in terms of the definition $(2.17)_1$, that is, the thermodynamic restrictions are expressed by

$$\alpha_c(\omega) > 0, \quad k_c(\omega) > 0 \quad \forall \omega \in \mathbf{R}, \tag{2.21}$$

with the new hypotheses that $\alpha_c(0) > 0$ and $k_c(0) > 0$. Moreover, we recall the following results [1, 17]:

$$\alpha'_s(\omega) = -\omega \alpha_c(\omega), \quad k'_s(\omega) = -\omega k_c(\omega), \tag{2.22}$$

whence, if $\alpha'', k'' \in L^2(R^+)$ and $| \ \alpha'(0) \ | < +\infty$, $| \ k'(0) \ | < +\infty$, we have

$$\lim_{\omega \to \infty} \lim \omega \alpha'_s(\omega) = - \lim_{\omega \to \infty} \lim \omega^2 \alpha_c(\omega) = \alpha'(0) \leq 0, \tag{2.23}$$

$$\lim_{\omega \to \infty} \lim \omega k'_s(\omega) = - \lim_{\omega \to \infty} \lim \omega^2 k_c(\omega) = k'(0) \leq 0. \tag{2.24}$$

Still now we assume

$$\alpha'(0) < 0, \quad k'(0) < 0. \tag{2.25}$$

Let us introduce the electric and thermal conductivities

$$\nu^{(\alpha)}(t) = \int_0^t \alpha(\xi) d\xi, \quad \nu^{(k)}(t) = \int_0^t k(\xi) d\xi; \tag{2.26}$$

in particular, we have

$$\nu^{(\alpha)}_{\infty} = \int_0^{+\infty} \alpha(\xi) d\xi = \alpha_c(0) > 0, \quad \nu^{(k)}_{\infty} = \int_0^{+\infty} k(\xi) d\xi = k_c(0) > 0. \tag{2.27}$$

The static continuation, with duration $\tau \in \mathbf{R}^{++}$, of two given histories $\mathbf{E}^t(s)$ and $\mathbf{g}^t(s)$ $\forall s \in \mathbf{R}^+$ is defined by

$$\mathbf{E}^{t(\tau)} = \begin{cases} \mathbf{E}(t), & s \in [0,\tau], \\ \mathbf{E}^t(s-\tau), & s > \tau, \end{cases} \qquad \mathbf{g}^{t(\tau)} = \begin{cases} \mathbf{g}(t), & s \in [0,\tau], \\ \mathbf{g}^t(s-\tau), & s > \tau. \end{cases} \tag{2.28}$$

The integrated histories, which correspond to these continuations, are

$$\bar{\mathbf{E}}^{t+\tau}(s) = \begin{cases} \int\limits_0^s \mathbf{E}(t)d\eta = s\mathbf{E}(t),\ s \in [0,\tau], \\ \tau\mathbf{E}(t) + \int\limits_0^{s-\tau} \mathbf{E}^t(\rho)d\rho,\ s > \tau, \end{cases} \qquad \bar{\mathbf{g}}^{t+\tau}(s) = \begin{cases} \int\limits_0^s \mathbf{g}(t)d\eta = s\mathbf{g}(t),\ s \in [0,\tau], \\ \tau\mathbf{g}(t) + \int\limits_0^{s-\tau} \mathbf{g}^t(\rho)d\rho,\ s > \tau, \end{cases} \tag{2.29}$$

and must be considered in the expressions (2.6) to evaluate the current density and the heat flux yielded after the static continuations (2.28); thus, we obtain

$$\mathbf{J}(t+\tau) = \nu^{(\alpha)}(\tau)\mathbf{E}(t) - \int\limits_0^{+\infty} \alpha'(\tau+\rho)\bar{\mathbf{E}}^t(\rho)d\rho, \tag{2.30}$$

$$\mathbf{q}(t+\tau) = -\nu^{(k)}(\tau)\mathbf{g}(t) + \int\limits_0^{+\infty} k'(\tau+\rho)\bar{\mathbf{g}}^t(\rho)d\rho. \tag{2.31}$$

We observe that in the constitutive equations the present value of the temperature gradient has not the same role, which, on the contrary, that one of the electric field has, since this appears explicitly in the constitutive equation $(2.1)_1$ for the electric displacement; the presence of $\mathbf{g}(t)$ in (2.31) is analogous to the one of $\mathbf{E}(t)$ in (2.30), but this is due only to the static continuations of both the values $\mathbf{E}(t)$ and $\mathbf{g}(t)$.

Finally, we note that the asymptotic values (2.27) allow us to obtain the current density and the heat flux at time t when the constant histories $\mathbf{E}^\dagger = \mathbf{E}^t(s) = \mathbf{E}$ and $\mathbf{g}^\dagger = \mathbf{g}^t(s) = \mathbf{g}$ $\forall s \in \mathbf{R}^+$ are considered. In fact, from (2.6) or directly from (2.2) it follows that

$$\mathbf{J}(t) = \nu_\infty^{(\alpha)}\mathbf{E}, \qquad \mathbf{q}(t) = -\nu_\infty^{(k)}\mathbf{g},$$

which, on account of (2.27), express the physical results of a constant current density with the same versus of the electric field $\mathbf{E}$ and of a constant heat flux whose versus is opposite to the one of the temperature gradient $\mathbf{g}$.

3. States and processes for the thermoelectromagnetic body. The behaviour of our thermoelectromagnetic solid is characterized by the assumed constitutive equations (2.1), (2.3) and (2.6), which, as we have already noted, allows us to consider $\mathcal{B}$ as a simple material. Thus, $\mathcal{B}$ can be described in terms of states and processes.

Taking into account the relations (2.1) and (2.3), the function

$$\sigma(t) = (\mathbf{E}(t), \mathbf{H}(t), \vartheta(t), \bar{\mathbf{E}}^t, \bar{\mathbf{g}}^t) \tag{3.1}$$

can be assumed to express the thermodynamic state at time t at any fixed $\mathbf{x} \in \Omega$. We note that, contrary to the choice made in [1], the integrated histories of the electric field and of the temperature gradient are now chosen to express the memory effects on the instantaneous values of the current density and of the heat flux.

A map $P : [0, d) \to \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R} \times \mathbf{R}^3$, piecewise continuous on the time interval $[0, d) \subset \mathbf{R}$, defined as

$$P(\tau) = (\dot{\mathbf{E}}_P(\tau), \dot{\mathbf{H}}_P(\tau), \dot{\vartheta}_P(\tau), \mathbf{g}_P(\tau)) \quad \forall \tau \in [0, d) \tag{3.2}$$

is said to be a kinetic process of duration $d \in \mathbf{R}^+$. Here we have the time derivatives of the electric and magnetic fields $\mathbf{E}_P$ and $\mathbf{H}_P$ and of the temperature ϑ_P together with the temperature gradient at any instant τ of the time interval $[0, d)$. We shall denote by $P_{[t_1,t_2)}$ the restriction of the process P to the time interval $[t_1, t_2) \subset [0, d)$; moreover, the set of the admissible states will be denoted by Σ, while Π will denote the set of all admissible processes. Thus, we can introduce the function $\rho : \Sigma \times \Pi \to \Sigma$, defined by $\sigma^f = \rho(\sigma^i, P) \in \Sigma$, which maps an initial state $\sigma^i \in \Sigma$ and a process $P \in \Pi$ into the final state σ^f and it is said to be the evolution function; in particular, we call cycle the pair (σ, P) such that $\sigma(d) = \rho(\sigma(0), P) = \sigma(0)$.

The response of the material is given by the function

$$U(t) = (\mathbf{D}(t), \mathbf{B}(t), \mathbf{J}(t), \mathbf{q}(t)), \tag{3.3}$$

where the instantaneous values of $\mathbf{D}$, $\mathbf{B}$, $\mathbf{J}$ and $\mathbf{q}$ are given by (2.1) and (2.6), and therefore it depends on the pair (σ, P), that is, the output function

$$U = \tilde{U}(\sigma, P) \tag{3.4}$$

where $\tilde{U} : \Sigma \times \Pi \to \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3$.

We can introduce the linear functionals $\tilde{\mathbf{J}} : \Gamma_\alpha \to \mathbf{R}^3$ and $\tilde{\mathbf{q}} : \Gamma_k \to \mathbf{R}^3$, on account of (2.6), such that

$$\tilde{\mathbf{J}}(\bar{\mathbf{E}}^t) = -\int_0^{+\infty} \alpha'(s)\bar{\mathbf{E}}^t(s)ds, \qquad \tilde{\mathbf{q}}(\bar{\mathbf{g}}^t) = \int_0^{+\infty} k'(s)\bar{\mathbf{g}}^t(s)ds, \tag{3.5}$$

where, taking in mind (2.30), (2.31), the function spaces Γ_α and Γ_k are defined by

$$\Gamma_\alpha = \left\{ \bar{\mathbf{E}}^t : (0, +\infty) \to \mathbf{R}^3; \ \left| \int_0^{+\infty} \alpha'(s+\tau)\bar{\mathbf{E}}^t(s)\,ds \right| < +\infty \quad \forall \tau \geq 0 \right\}, \tag{3.6}$$

$$\Gamma_k = \left\{ \bar{\mathbf{g}}^t : (0, +\infty) \to \mathbf{R}^3; \ \left| \int_0^{+\infty} k'(s+\tau)\bar{\mathbf{g}}^t(s)\,ds \right| < +\infty \quad \forall \tau \geq 0 \right\}. \tag{3.7}$$

Any process $P = P(\tau)$, of duration d, is defined in the time interval $[0, d) \subset \mathbf{R}^+$ by means of (3.2); it may be applied at any instant $t \in \mathbf{R}^+$.

First, we suppose that P is applied at time $t = 0$, when the state is $\sigma(0) = (\mathbf{E}_*(0), \mathbf{H}_*(0), \vartheta_*(0), \bar{\mathbf{E}}_*^0, \bar{\mathbf{g}}_*^0) \in \Sigma$. In this case $\tau \equiv t$ and therefore (3.2) becomes $P(t) = (\dot{\mathbf{E}}_P(t), \dot{\mathbf{H}}_P(t), \dot{\vartheta}_P(t), \mathbf{g}_P(t)) \in \Pi$; the evolution function yields a set of states $\sigma(t) = (\mathbf{E}(t), \mathbf{H}(t), \vartheta(t), \bar{\mathbf{E}}^t, \bar{\mathbf{g}}^t)$ for any $t \in (0, d]$, characterized by

$$\mathbf{E}(t) = \mathbf{E}_*(0) + \int_0^t \dot{\mathbf{E}}_P(\xi) d\xi, \quad \mathbf{H}(t) = \mathbf{H}_*(0) + \int_0^t \dot{\mathbf{H}}_P(\xi) d\xi, \tag{3.8}$$

$$\vartheta(t) = \vartheta_*(0) + \int_0^t \dot{\vartheta}_P(\xi) d\xi, \tag{3.9}$$

where $\mathbf{E}(t) \equiv \mathbf{E}_P(\tau)$ and similarly for $\mathbf{H}$ and ϑ, and

$$\bar{\mathbf{E}}^t(s) = \begin{cases} \displaystyle\int_{t-s}^t \mathbf{E}(\eta) d\eta, & 0 \le s < t, \\ \displaystyle\bar{\mathbf{E}}_*^0(s-t) + \int_0^t \mathbf{E}(\eta) d\eta, & s \ge t, \end{cases} \tag{3.10}$$

$$\bar{\mathbf{g}}^t(s) = \begin{cases} \displaystyle\int_{t-s}^t \mathbf{g}_P(\eta) d\eta, & 0 \le s < t, \\ \displaystyle\bar{\mathbf{g}}_*^0(s-t) + \int_0^t \mathbf{g}_P(\eta) d\eta, & s \ge t, \end{cases} \tag{3.11}$$

where $\mathbf{E}(\eta)$ is given by $(3.8)_1$ and $\mathbf{g}_P(\eta)$ is assigned in the process.

Now, we apply the process P at time $t > 0$, when the initial state is $\sigma(t) = (\mathbf{E}(t), \mathbf{H}(t), \vartheta(t), \bar{\mathbf{E}}^t, \bar{\mathbf{g}}^t)$. The process $P(\tau) = (\dot{\mathbf{E}}_P(\tau), \dot{\mathbf{H}}_P(\tau), \dot{\vartheta}_P(\tau), \mathbf{g}_P(\tau))$, defined on $[0, d)$, is related to

$$\mathbf{E}_P : (0, d] \to \mathbf{R}^3, \qquad \mathbf{E}_P(\tau) = \mathbf{E}(t) + \int_0^\tau \dot{\mathbf{E}}_P(\eta) d\eta, \tag{3.12}$$

$$\mathbf{H}_P : (0, d] \to \mathbf{R}^3, \qquad \mathbf{H}_P(\tau) = \mathbf{H}(t) + \int_0^\tau \dot{\mathbf{H}}_P(\eta) d\eta, \tag{3.13}$$

$$\vartheta_P : (0, d] \to \mathbf{R}, \qquad \vartheta_P(\tau) = \vartheta(t) + \int_0^\tau \dot{\vartheta}_P(\eta) d\eta \tag{3.14}$$

for any $\tau \in (0, d]$. In particular, the process P assigns the temperature gradient

$$\mathbf{g}_P : [0, d) \to \mathbf{R}^3, \qquad \mathbf{g}_P(\tau) = \mathbf{g}(t+\tau). \tag{3.15}$$

Moreover such a process, starting at time t, to which corresponds the initial integrated histories $\bar{\mathbf{E}}^t$ and $\bar{\mathbf{g}}^t$ that appear in $\sigma(t)$, induces the continuation of the initial $\bar{\mathbf{E}}^t$ and $\bar{\mathbf{g}}^t$ by means of $\mathbf{E}_P(\tau) \equiv \mathbf{E}(t+\tau)$ and $(3.15)_2$, with $t + \tau \le t + d$, and expressed by

$$\bar{\mathbf{E}}^{t+d}(s) = (\mathbf{E}_P * \bar{\mathbf{E}})^{t+d}(s) = \begin{cases} \bar{\mathbf{E}}_P^d(s) = \displaystyle\int_0^s \mathbf{E}_P^d(\eta)d\eta, & 0 \le s < d, \\ \bar{\mathbf{E}}_P^d(d) + \bar{\mathbf{E}}^t(s-d), & s \ge d, \end{cases} \tag{3.16}$$

$$\bar{\mathbf{g}}^{t+d}(s) = (\mathbf{g}_P * \bar{\mathbf{g}})^{t+d}(s) = \begin{cases} \bar{\mathbf{g}}_P^d(s) = \displaystyle\int_0^s \mathbf{g}_P^d(\eta)d\eta, & 0 \le s < d, \\ \bar{\mathbf{g}}_P^d(d) + \bar{\mathbf{g}}^t(s-d), & s \ge d, \end{cases} \tag{3.17}$$

where

$$\bar{\mathbf{E}}_P^d(d) = \int_0^d \mathbf{E}_P(\eta)d\eta, \quad \bar{\mathbf{E}}^t(s-d) = \int_{t-(s-d)}^t \mathbf{E}(\xi)d\xi \quad \forall s \ge d, \tag{3.18}$$

$$\bar{\mathbf{g}}_P^d(d) = \int_0^d \mathbf{g}_P(\eta)d\eta, \quad \bar{\mathbf{g}}^t(s-d) = \int_{t-(s-d)}^t \mathbf{g}(\xi)d\xi \quad \forall s \ge d, \tag{3.19}$$

are the integrated histories corresponding to P in $[0, d)$ and the initial integrated histories, respectively for $\mathbf{E}$ and $\mathbf{g}$.

The current density and the heat flux, yielded by the application of a process P, of duration d, to a given initial state $\sigma(t)$, are $\tilde{\mathbf{J}}((\mathbf{E}_P * \bar{\mathbf{E}})^{t+d})$ and $\tilde{\mathbf{q}}((\mathbf{g}_P * \bar{\mathbf{g}})^{t+d})$ and can be evaluated by using (3.5) with (3.16) and (3.17). We derive these results in the case where a restriction of the process $P_{[0,\tau)}$, applied at time t when the state is $\sigma(t) = (\mathbf{E}(t), \mathbf{H}(t), \vartheta(t), \bar{\mathbf{E}}^t, \bar{\mathbf{g}}^t)$, is considered; in this case d must be substituted by τ in (3.16), (3.17) to evaluate (3.5), which thus give

$$\tilde{\mathbf{J}}(\bar{\mathbf{E}}^{t+\tau}) = -\int_0^{+\infty} \alpha'(s)(\mathbf{E}_P * \bar{\mathbf{E}})(t+\tau-s)ds =$$

$$= \alpha(\tau)\bar{\mathbf{E}}_P^\tau(\tau) - \int_0^\tau \alpha'(\eta)\bar{\mathbf{E}}_P^\tau(\eta)d\eta - \int_\tau^{+\infty} \alpha'(\rho)\bar{\mathbf{E}}^t(\rho-\tau)d\rho, \tag{3.20}$$

$$\tilde{\mathbf{q}}(\bar{\mathbf{g}}^{t+\tau}) = \int_0^{+\infty} k'(s)(\mathbf{g}_P * \bar{\mathbf{g}})(t+\tau-s)ds =$$

$$= -k(\tau)\bar{\mathbf{g}}_P^{\tau}(\tau) + \int_0^{\tau} k'(\eta)\bar{\mathbf{g}}_P^{\tau}(\eta)d\eta + \int_{\tau}^{+\infty} k'(\rho)\bar{\mathbf{g}}^{t}(\rho-\tau)d\rho. \tag{3.21}$$

4. Equivalence of states and of integrated histories. We can now introduce an equivalence relation in the state space Σ by means of the following definition [11].

Definition 4.1. *Two states $\sigma_j \in \Sigma$, $j = 1, 2$, are equivalent if*

$$\tilde{U}(\sigma_1, P) = \tilde{U}(\sigma_2, P) \qquad \forall P \in \Pi. \tag{4.1}$$

Thus, whatever may be the admissible process, two equivalent states yield the same response of the material and therefore they are indistinguishable. This definition satisfies the requirements of an equivalence relation, denoted by $\mathcal{R}$. Therefore, if Σ contains equivalent states, then we can introduce the corresponding quotient space $\Sigma_{\mathcal{R}}$, whose elements $\sigma_{\mathcal{R}}$ are the classes of equivalent states.

Definition 4.2. *We say that a state of the material is minimal if it is characterized by a minimum set of data.*

The introduction of the functionals in (3.5) allows us to give a new definition of equivalence relative to the integrated histories of the electric field and of the heat flux.

Definition 4.3. *Given two states $\sigma_j(t) = (\mathbf{E}_j(t), \mathbf{H}_j(t), \vartheta_j(t), \bar{\mathbf{E}}_j^t, \bar{\mathbf{g}}_j^t)$, $j = 1, 2$, corresponding to the same values of the magnetic field, $\mathbf{H}_j(t) = \mathbf{H}(t)$, $j = 1, 2$, and of the temperature, $\vartheta_j(t) = \vartheta(t)$, $j = 1, 2$, the integrated histories of the electric field, $\bar{\mathbf{E}}_j^t$, $j = 1, 2$, and of the temperature gradient, $\bar{\mathbf{g}}_j^t$, $j = 1, 2$, are said equivalent if for every $\mathbf{E}_P : (0, \tau] \to \mathbf{R}^3$, $\mathbf{g}_P : [0, \tau) \to \mathbf{R}^3$ and for every $\tau > 0$ the relations*

$$\mathbf{E}_1(t) = \mathbf{E}_2(t), \quad \tilde{\mathbf{J}}((\mathbf{E}_P * \bar{\mathbf{E}}_1)^{t+\tau}) = \tilde{\mathbf{J}}((\mathbf{E}_P * \bar{\mathbf{E}}_2)^{t+\tau}), \tag{4.2}$$

$$\tilde{\mathbf{q}}((\mathbf{g}_P * \bar{\mathbf{g}}_1)^{t+\tau}) = \tilde{\mathbf{q}}((\mathbf{g}_P * \bar{\mathbf{g}}_2)^{t+\tau}) \tag{4.3}$$

hold, whatever $\mathbf{H}_P : (0, \tau] \to \mathbf{R}^3$ and $\vartheta_P : (0, \tau] \to \mathbf{R}$ may be.

With this definition the integrated histories of $\mathbf{E}$ and $\mathbf{g}$, which yield the same current density and the same heat flux, respectively, are identified. Moreover, the conditions required in Definition 4.1 are now satisfied. In particular, we observe that for any process the values of $\mathbf{H}_P(\tau)$ and $\vartheta_P(\tau)$ do not depend on the two couples of the integrated histories $\bar{\mathbf{E}}_j^t$ and $\bar{\mathbf{g}}_j^t$, $j = 1, 2$; furthermore, we note that for both the integrated histories we must consider the same process P and hence the same value of the temperature gradient, $\mathbf{g}_P(\rho)$ $\forall \rho \in [0, \tau)$, which yields, in particular, $\mathbf{g}_P(0) = \mathbf{g}(t)$ by virtue of (3.15). Therefore, it follows that the equivalence of two integrated histories of $\mathbf{E}$ and $\mathbf{g}$ yield the same response of the material.

If we consider the continuation of the zero integrated histories of $\mathbf{E}$ and $\mathbf{g}$, $\bar{\mathbf{E}}^t(s) = \bar{\mathbf{0}}^\dagger(s) = \mathbf{0}$ and $\bar{\mathbf{g}}^t(s) = \bar{\mathbf{0}}^\dagger(s) = \mathbf{0}\ \forall s \in \mathbf{R}^+$, by means of a process $P_{[0,\tau)}$ applied at time t when the initial state is $\sigma(t) = (\mathbf{E}(t), \mathbf{H}(t), \vartheta(t), \bar{\mathbf{0}}^\dagger, \bar{\mathbf{0}}^\dagger)$, (3.16) and (3.17) yield

$$(\mathbf{E}_P * \bar{\mathbf{0}}^\dagger)^{t+\tau}(s) = \begin{cases} \bar{\mathbf{E}}_P^\tau(s), 0 \le s < \tau, \\ \bar{\mathbf{E}}_P^\tau(\tau), s \ge \tau, \end{cases} \quad (\mathbf{g}_P * \bar{\mathbf{0}}^\dagger)^{t+d}(s) = \begin{cases} \bar{\mathbf{g}}_P^\tau(s), 0 \le s < \tau, \\ \bar{\mathbf{g}}_P^\tau(\tau), s \ge \tau. \end{cases} \tag{4.4}$$

As a consequence of Definition 4.3, from (3.20), (3.21) it is easy to see that two integrated histories $\bar{\mathbf{E}}^t$ and $\bar{\mathbf{g}}^t$ are equivalent to their relative zero integrated histories if for every $\tau > 0$

$$\int_\tau^{+\infty} \alpha'(s)\bar{\mathbf{E}}^t(s-\tau)ds = \int_0^{+\infty} \alpha'(\tau+\xi)\bar{\mathbf{E}}^t(\xi)d\xi = \mathbf{0}, \tag{4.5}$$

$$\int_\tau^{+\infty} k'(s)\bar{\mathbf{g}}^t(s-\tau)ds = \int_0^{+\infty} k'(\tau+\xi)\bar{\mathbf{g}}^t(\xi)d\xi = \mathbf{0}. \tag{4.6}$$

Thus we see that both the relations (4.2), (4.3) and these last relations (4.5), (4.6) express the same equivalence between two couples of integrated histories. In fact, given the integrated histories $\bar{\mathbf{E}}_j^t$ and $\bar{\mathbf{g}}_j^t$, $j = 1, 2$, equivalent in the sense of Definition 4.3, obviously they must satisfy (4.2), (4.3), from which we deduce that

$$\int_\tau^{+\infty} \alpha'(s)\bar{\mathbf{E}}_1^t(s-\tau)ds = \int_\tau^{+\infty} \alpha'(s)\bar{\mathbf{E}}_2^t(s-\tau)ds, \tag{4.7}$$

$$\int_\tau^{+\infty} k'(s)\bar{\mathbf{g}}_1^t(s-\tau)ds = \int_\tau^{+\infty} k'(s)\bar{\mathbf{g}}_2^t(s-\tau)ds \tag{4.8}$$

must hold for all $\tau > 0$ too. Finally, it is enough to define $\bar{\mathbf{E}}^t(s-\tau) = \bar{\mathbf{E}}_1^t(s-\tau) - \bar{\mathbf{E}}_2^t(s-\tau)$ and $\bar{\mathbf{g}}^t(s-\tau) = \bar{\mathbf{g}}_1^t(s-\tau) - \bar{\mathbf{g}}_2^t(s-\tau)$ to verify that (4.7), (4.8) coincide with (4.5), (4.6) and conclude that the integrated histories obtained by the two differences are equivalent to their zero integrated histories.

5. Thermoelectromagnetic work. The local form of the second law of thermodynamics is expressed by (2.10), to which we have given the form (2.14) taking account of the constitutive equations of $\mathcal{B}$. These expressions allow us to deduce the work done on any process P [6, 13–16]. With reference to the second form (2.14), the work done on a process $P(\tau) = (\dot{\mathbf{E}}_P(\tau), \dot{\mathbf{H}}_P(\tau), \dot{\vartheta}_P(\tau), \mathbf{g}_P(\tau))$, defined for any $\tau \in [0, d)$ and applied at time t when the state is $\sigma^i(t) = (\mathbf{E}_i(t), \mathbf{H}_i(t), \vartheta_i(t), \bar{\mathbf{E}}_i^t, \bar{\mathbf{g}}_i^t)$, can be written as follows:

$$W(\sigma^i(t), P) = \tilde{W}(\mathbf{E}_i(t), \mathbf{H}_i(t), \vartheta_i(t), \bar{\mathbf{E}}_i^t, \bar{\mathbf{g}}_i^t; \dot{\mathbf{E}}_P, \dot{\mathbf{H}}_P, \dot{\vartheta}_P, \mathbf{g}_P) =$$

$$= \int_0^d \left[\frac{c}{\Theta_0} \dot{\vartheta}_P(\tau)\vartheta_P(\tau) + \frac{1}{\varepsilon}\dot{\mathbf{D}}(\mathbf{E}_P(\tau), \vartheta_P(\tau)) \cdot \mathbf{D}(\mathbf{E}_P(\tau), \vartheta_P(\tau)) + \frac{1}{\mu}\dot{\mathbf{B}}(\mathbf{H}_P(\tau)) \cdot \right.$$

$$\left. \cdot \mathbf{B}(\mathbf{H}_P(\tau)) + \tilde{\mathbf{J}}((\mathbf{E}_P * \bar{\mathbf{E}}_i)^{t+\tau}) \cdot \mathbf{E}_P(\tau) - \frac{1}{\Theta_0}\tilde{\mathbf{q}}((\mathbf{g}_P * \bar{\mathbf{g}}_i)^{t+\tau}) \cdot \mathbf{g}_P(\tau) \right] d\tau. \tag{5.1}$$

Here we have considered the variable $\tau \in [0, d)$ and therefore $\mathbf{E}_P(\tau), \mathbf{H}_P(\tau), \vartheta_P(\tau)$ are expressed by (3.12) – (3.14), $\mathbf{g}_P(\tau)$ is given by P and has the form (3.15) and finally $\mathbf{E}_P * \bar{\mathbf{E}}_i$, $\mathbf{g}_P * \bar{\mathbf{g}}_i$ are the continuations (3.16), (3.17). If we consider the variable $\xi \in [t, t+d)$ we can transform (5.1) as follows:

$$W(\sigma^i(t), P) = \int_t^{t+d} \left[\frac{c}{\Theta_0}\dot{\vartheta}(\xi)\vartheta(\xi) + \frac{1}{\varepsilon}\dot{\mathbf{D}}(\mathbf{E}(\xi), \vartheta(\xi)) \cdot \mathbf{D}(\mathbf{E}(\xi), \vartheta(\xi)) + \right.$$

$$\left. + \frac{1}{\mu}\dot{\mathbf{B}}(\mathbf{H}(\xi)) \cdot \mathbf{B}(\mathbf{H}(\xi)) + \tilde{\mathbf{J}}((\mathbf{E}_P * \bar{\mathbf{E}}_i)^{\xi}) \cdot \mathbf{E}(\xi) - \frac{1}{\Theta_0}\tilde{\mathbf{q}}((\mathbf{g}_P * \bar{\mathbf{g}}_i)^{\xi}) \cdot \mathbf{g}(\xi) \right] d\xi. \tag{5.2}$$

For the sake of simplicity we eliminate i in $\sigma^i(t)$ and from (5.2) we derive

$$W(\sigma(t), P) = \frac{1}{2}\left[\frac{c}{\Theta_0}\vartheta^2(t+d) + \frac{1}{\varepsilon}\mathbf{D}^2(t+d) + \frac{1}{\mu}\mathbf{B}^2(t+d) \right] - \frac{1}{2}\left[\frac{c}{\Theta_0}\vartheta^2(t) + \right.$$

$$\left. + \frac{1}{\varepsilon}\mathbf{D}^2(t) + \frac{1}{\mu}\mathbf{B}^2(t) \right] - \int_t^{t+d}\int_0^{+\infty} \alpha'(s)(\mathbf{E}_P * \bar{\mathbf{E}})^{\xi}(s)ds \cdot \mathbf{E}(\xi)d\xi -$$

$$- \frac{1}{\Theta_0}\int_t^{t+d}\int_0^{+\infty} k'(s)(\mathbf{g}_P * \bar{\mathbf{g}})^{\xi}(s)ds \cdot \mathbf{g}(\xi)d\xi, \tag{5.3}$$

where we have to use (3.16), (3.17) for the continuations of the integrated histories.

In Section 3 we have considered the particular case where the process is applied at time $t = 0$. Now we suppose at this initial instant the state to be

$$\sigma_0(0) = \sigma(0) = (\mathbf{0}, \mathbf{0}, 0, \bar{\mathbf{0}}^{\dagger}, \bar{\mathbf{0}}^{\dagger}), \tag{5.4}$$

that is, $\mathbf{E}(0) = \mathbf{0}$, $\mathbf{H}(0) = \mathbf{0}$, $\vartheta(0) = 0$, $\bar{\mathbf{E}}^0(s) = \bar{\mathbf{0}}^{\dagger}(s) = \mathbf{0}$ and $\bar{\mathbf{g}}^0(s) = \bar{\mathbf{0}}^{\dagger}(s) = \mathbf{0}$ $\forall s \in \mathbf{R}^+$; therefore, the ensuing fields, now denoted by $\mathbf{E}_0$, $\mathbf{H}_0$, ϑ_0 to distinguish this particular case, taking into account (3.8) – (3.11), assume the form

$$\mathbf{E}_0(t) = \int_0^t \dot{\mathbf{E}}_P(s)ds, \ \mathbf{H}_0(t) = \int_0^t \dot{\mathbf{H}}_P(s)ds, \ \vartheta_0(t) = \int_0^t \dot{\vartheta}_P(s)ds \tag{5.5}$$

and

$$(\mathbf{E}_P * \bar{\mathbf{0}}^\dagger)^t(s) = \begin{cases} \bar{\mathbf{E}}_0^t(s), 0 \le s < t, \\ \bar{\mathbf{E}}_0^t(t), s \ge t, \end{cases} \quad (\mathbf{g}_P * \bar{\mathbf{0}}^\dagger)^t(s) = \begin{cases} \bar{\mathbf{g}}_P^t(s), 0 \le s < t, \\ \bar{\mathbf{g}}_P^t(t), s \ge t. \end{cases} \tag{5.6}$$

With these hypotheses we are able to distinguish the work due only to the process and give the following definition.

Definition 5.1. *Let* $P(t) = (\dot{\mathbf{E}}_P(t), \dot{\mathbf{H}}_P(t), \dot{\vartheta}_P(t), \mathbf{g}_P(t))$ *be a process of duration* d *applied at time* $t = 0$ *and related to* $\mathbf{E}_0(t)$, $\mathbf{H}_0(t)$, $\vartheta_0(t)$, $(\mathbf{E}_P * \bar{\mathbf{0}}^\dagger)^t$, *and* $(\mathbf{g}_P * \bar{\mathbf{0}}^\dagger)^t$, *given by (5.5), (5.6), if the work done on* P *is finite, then* P *is said to be a finite work process.*

Such a work is given by (5.3), which, using (5.5), (5.6), assumes the form

$$W(\sigma_0(0), P) = \tilde{W}(\mathbf{0}, \mathbf{0}, 0, \bar{\mathbf{0}}^\dagger, \bar{\mathbf{0}}^\dagger; \dot{\mathbf{E}}_P, \dot{\mathbf{H}}_P, \dot{\vartheta}_P, \mathbf{g}_P) =$$

$$= \int_0^d \left[\frac{c}{\Theta_0} \dot{\vartheta}_0(t)\vartheta_0(t) + + \frac{1}{\varepsilon} \dot{\mathbf{D}}(\mathbf{E}_0(t), \vartheta_0(t)) \cdot \mathbf{D}(\mathbf{E}_0(t), \vartheta_0(t)) + \right.$$

$$+ \frac{1}{\mu} \dot{\mathbf{B}}(\mathbf{H}_0(t)) \cdot \mathbf{B}(\mathbf{H}_0(t)) + \tilde{\mathbf{J}}((\bar{\mathbf{E}} * \bar{\mathbf{0}}^\dagger)^t) \cdot \mathbf{E}_0(t) -$$

$$\left. - \frac{1}{\Theta_0} \tilde{\mathbf{q}}((\mathbf{g}_P * \bar{\mathbf{0}}^\dagger)^t) \cdot \mathbf{g}_P(t) \right] dt, \tag{5.7}$$

which, taking account of (2.6) with (5.6), can be written as follows:

$$W(\sigma_0(0), P) = \frac{1}{2} \left[\frac{c}{\Theta_0} \vartheta_0^2(d) + \frac{1}{\varepsilon} \mathbf{D}_0^2(d) + \frac{1}{\mu} \mathbf{B}_0^2(d) \right] -$$

$$- \int_0^d \left[\int_0^t \alpha'(s) \bar{\mathbf{E}}_0^t(s) ds + \int_t^{+\infty} \alpha'(s) \bar{\mathbf{E}}_0^t(t) ds \right] \cdot \mathbf{E}_0(t) dt -$$

$$- \frac{1}{\Theta_0} \int_0^d \left[\int_0^t k'(s) \bar{\mathbf{g}}_P^t(s) ds + \int_t^{+\infty} k'(s) \bar{\mathbf{g}}_P^t(t) ds \right] \cdot \mathbf{g}_P(t) dt. \tag{5.8}$$

Lemma 5.1. *The work obtained by any process, which is a finite work process in the sense of Definition 5.1, is positive.*

Proof. Let P be a finite work process of duration d; such a process, starting from the state $\sigma_0(0)$ at time $t = 0$, yields the work (5.8), which, integrating by parts the integrals over $[0,t]$ and evaluating the other two, becomes

$$W(\sigma_0(0), P) = \frac{1}{2}\left[\frac{c}{\Theta_0}\vartheta_0^2(d) + \frac{1}{\varepsilon}\mathbf{D}_0^2(d) + \frac{1}{\mu}\mathbf{B}_0^2(d)\right] +$$

$$+ \int_0^d \left[\int_0^t \alpha(s)\mathbf{E}_0^t(s)ds \cdot \mathbf{E}_0(t) + \frac{1}{\Theta_0}\int_0^t k(s)\mathbf{g}_P^t(s)ds \cdot \mathbf{g}_P(t)\right] dt. \tag{5.9}$$

Assuming that the functions in (5.9) are equal to zero for any $t > d$, the integral in (5.9) being extended on $\mathbf{R}^+$, and using * to denote the complex conjugate, by means of Plancherel's theorem assumes the form

$$\int_0^{+\infty} \left[\int_0^t \alpha(s)\mathbf{E}_0^t(s)ds \cdot \mathbf{E}_0(t) + \frac{1}{\Theta_0}\int_0^t k(s)\mathbf{g}_P^t(s)ds \cdot \mathbf{g}_P(t)\right] dt =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left[\alpha_F(\omega)\mathbf{E}_{0_F}(\omega) \cdot \mathbf{E}_{0_F}^*(\omega) + \frac{1}{\Theta_0}k_F(\omega)\mathbf{g}_{P_F}(\omega) \cdot \mathbf{g}_{P_F}^*(\omega)\right] d\omega =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left\{\alpha_c(\omega)[\mathbf{E}_{0_c}^2(\omega) + \mathbf{E}_{0_s}^2(\omega)] + \frac{1}{\Theta_0}k_c(\omega)[\mathbf{g}_{P_c}^2(\omega) + \mathbf{g}_{P_s}^2(\omega)]\right\} d\omega > 0. \tag{5.10}$$

The last result follows easily from (2.21) and from the consideration that the Fourier transforms of the functions, defined on $\mathbf{R}^+$ and equal to zero on $\mathbf{R}^{--}$, by means of (2.18) can be expressed in terms of their cosine and sine transforms, which are even and odd functions, respectively.

The lemma is proved.

Usually, the duration of a process P is finite, $d < +\infty$; however, as we have already done in the proof of the previous lemma, we can define P on $\mathbf{R}^+$ by putting $P(\tau) = (\dot{\mathbf{E}}_P(\tau), \dot{\mathbf{H}}_P(\tau), \dot{\vartheta}_P(\tau), \mathbf{g}_P(\tau)) = (\mathbf{0}, \mathbf{0}, 0, \mathbf{0})$ for any $\tau \geq d$. If we assume that $\mathbf{E}_P(\tau) = \mathbf{0}$, $\mathbf{H}_P(\tau) = \mathbf{0}$, $\vartheta_P(\tau) = 0$ for any $\tau > d$ too, (5.7) can be written as follows:

$$W(\sigma_0(0), P) = \int_0^d \left[\frac{c}{\Theta_0}\dot{\vartheta}_0(\xi)\vartheta_0(\xi) + \frac{1}{\varepsilon}\dot{\mathbf{D}}_0(\xi) \cdot \mathbf{D}_0(\xi) + \frac{1}{\mu}\dot{\mathbf{B}}_0(\xi) \cdot \mathbf{B}_0(\xi)\right] d\xi -$$

$$-\int\limits_0^{+\infty}\left[\int\limits_0^{\eta}\alpha'(s)\bar{\mathbf{E}}_P^{\eta}(s)ds+\int_{\eta}^{+\infty}\alpha'(s)\bar{\mathbf{E}}_P^{\eta}(\eta)ds\right]\cdot\mathbf{E}_P(\eta)d\eta-$$

$$-\frac{1}{\Theta_0}\int\limits_0^{+\infty}\left[\int\limits_0^{\eta}k'(s)\bar{\mathbf{g}}_P^{\eta}(s)ds+\int_{\eta}^{+\infty}k'(s)\bar{\mathbf{g}}_P^{\eta}(\eta)ds\right]\cdot\mathbf{g}_P(\eta)d\eta, \tag{5.11}$$

which, taking into account that the initial state is $\sigma_0(0)$, given by (5.4), and integrating by parts, reduces to

$$W(\sigma_0(0),P)=\frac{1}{2}\left[\frac{c}{\Theta_0}\vartheta^2(d)+\frac{1}{\varepsilon}\mathbf{D}^2(d)+\frac{1}{\mu}\mathbf{B}^2(d)\right]+$$

$$+\int\limits_0^{+\infty}\left[\int\limits_0^{\eta}\alpha(s)\mathbf{E}_P^{\eta}(s)ds\cdot\mathbf{E}_P(\eta)+\frac{1}{\Theta_0}\int\limits_0^{\eta}k(s)\mathbf{g}_P^{\eta}(s)ds\cdot\mathbf{g}_P(\eta)\right]d\eta=$$

$$=\frac{1}{2}\left[\frac{c}{\Theta_0}\vartheta^2(d)+\frac{1}{\varepsilon}\mathbf{D}^2(d)+\frac{1}{\mu}\mathbf{B}^2(d)\right]+$$

$$+\frac{1}{2}\int\limits_0^{+\infty}\left[\int\limits_0^{+\infty}\alpha(|\ \eta-\rho\ |)\mathbf{E}_P(\rho)d\rho\cdot\mathbf{E}_P(\eta)+\right.$$

$$\left.+\frac{1}{\Theta_0}\int\limits_0^{+\infty}k(|\ \eta-\rho\ |)\mathbf{g}_P(\rho)d\rho\cdot\mathbf{g}_P(\eta)\right]d\eta. \tag{5.12}$$

In the last relation we have the even functions $\alpha(|\ \eta-\rho\ |)$ and $k(|\ \eta-\rho\ |)$, whose Fourier transforms can be written in terms of their Fourier cosine transforms; therefore, using $(2.18)_2$, (5.12) becomes

$$W(\sigma_0(0),P)=\frac{1}{2}\left[\frac{c}{\Theta_0}\vartheta^2(d)+\frac{1}{\varepsilon}\mathbf{D}^2(d)+\frac{1}{\mu}\mathbf{B}^2(d)\right]+$$

$$+\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\left[\alpha_c(\omega)\mathbf{E}_{P+}(\omega)\cdot\mathbf{E}_{P+}^*(\omega)+\frac{1}{\Theta_0}k_c(\omega)\mathbf{g}_{P+}(\omega)\cdot\mathbf{g}_{P+}^*(\omega)\right]d\omega. \tag{5.13}$$

This result allows us to introduce the functional spaces

$$\tilde{H}_\alpha(\mathbf{R}^+,\mathbf{R}^3) = \left\{ \mathbf{E} : \mathbf{R}^+ \to \mathbf{R}^3;\ \int\limits_{-\infty}^{+\infty} \alpha_c(\omega)\mathbf{E}_+(\omega)\cdot\mathbf{E}_+^*(\omega)d\omega < +\infty \right\}, \tag{5.14}$$

$$\tilde{H}_k(\mathbf{R}^+,\mathbf{R}^3) = \left\{ \mathbf{g} : \mathbf{R}^+ \to \mathbf{R}^3;\ \int\limits_{-\infty}^{+\infty} k_c(\omega)\mathbf{g}_+(\omega)\cdot\mathbf{g}_+^*(\omega)d\omega < +\infty \right\}, \tag{5.15}$$

which characterize the finite work processes. The completions of these spaces, with respect to the norms corresponding to the inner products defined by $(\mathbf{E}_1,\mathbf{E}_2)_\alpha = \int_{-\infty}^{+\infty} \alpha_c(\omega)\mathbf{E}_{1+}(\omega)\cdot$ $\cdot\mathbf{E}_{2+}^*(\omega)d\omega$ and $(\mathbf{g}_1,\mathbf{g}_2)_k = \int_{-\infty}^{+\infty} k_c(\omega)\mathbf{g}_{1+}(\omega)\cdot\mathbf{g}_{2+}^*(\omega)d\omega$, respectively, yield two Hilbert's spaces, which are denoted by $H_\alpha(\mathbf{R}^+,\mathbf{R}^3)$ and $H_k(\mathbf{R}^+,\mathbf{R}^3)$ and characterize the spaces of the processes.

Let us now assume $\sigma(t) = (\mathbf{E}(t),\mathbf{H}(t),\vartheta(t),\bar{\mathbf{E}}^t,\bar{\mathbf{g}}^t)$ as the initial state of $\mathcal{B}$ such that its integrated histories $\bar{\mathbf{E}}^t \in \Gamma_\alpha$ and $\bar{\mathbf{g}}^t \in \Gamma_k$, see (3.6), (3.7), are admissible histories which yield a finite work during any process, characterized by $\mathbf{g}_P \in H_k(\mathbf{R}^+,\mathbf{R}^3)$ and related to $\mathbf{E}_P \in$ $\in H_\alpha(\mathbf{R}^+,\mathbf{R}^3)$. If $P = (\dot{\mathbf{E}}_P,\dot{\mathbf{H}}_P,\dot{\vartheta}_P,\mathbf{g}_P)$ is one of these processes with duration $d < +\infty$, it may be extended on $\mathbf{R}^+$ on supposing that $P(\tau) = (\mathbf{0},\mathbf{0},0,\mathbf{0})$ for every $\tau \ge d$ and that it is related to $\mathbf{E}_P(\tau) = \mathbf{0}$, $\mathbf{H}_P(\tau) = \mathbf{0}$, $\vartheta_P(\tau) = 0\ \forall\tau > d$. The work done on such a process is given by (5.1), which, using (3.20), (3.21), now assumes the following form:

$$\begin{aligned}
W(\sigma(t),P) = \int\limits_0^d \Big[& \frac{c}{\Theta_0}\dot{\vartheta}_P(\tau)\vartheta_P(\tau) + \frac{1}{\varepsilon}\dot{\mathbf{D}}(\mathbf{E}_P(\tau),\vartheta_P(\tau))\cdot\mathbf{D}(\mathbf{E}_P(\tau),\vartheta_P(\tau)) + \\
& + \frac{1}{\mu}\dot{\mathbf{B}}(\mathbf{H}_P(\tau))\cdot\mathbf{B}(\mathbf{H}_P(\tau))\Big] d\tau + \int\limits_0^{+\infty}\Big[\alpha(\tau)\bar{\mathbf{E}}_P^\tau(\tau) - \int\limits_0^\tau \alpha'(s)\bar{\mathbf{E}}_P^\tau(s)ds - \\
& - \int\limits_0^{+\infty}\alpha'(\tau+\xi)\bar{\mathbf{E}}^t(\xi)d\xi\Big]\cdot\mathbf{E}_P(\tau)d\tau + \frac{1}{\Theta_0}\int\limits_0^{+\infty}\Big[k(\tau)\bar{\mathbf{g}}_P^\tau(\tau) - \int\limits_0^\tau k'(s)\bar{\mathbf{g}}_P^\tau(s)ds - \\
& - \int\limits_0^{+\infty} k'(\tau+\xi)\bar{\mathbf{g}}^t(\xi)d\xi\Big]\cdot\mathbf{g}_P(\tau)d\tau.
\end{aligned} \tag{5.16}$$

This expression, by evaluating the first integral and integrating by parts in the other two, can be changed as follows:

$$W(\sigma(t),P)=\frac{1}{2}\left[\frac{c}{\Theta_0}\vartheta_P^2(d)+\frac{1}{\varepsilon}\mathbf{D}^2(\mathbf{E}_P(d),\vartheta_P(d))+\frac{1}{\mu}\mathbf{B}^2(\mathbf{H}_P(d)\right]-$$

$$-\frac{1}{2}\left[\frac{c}{\Theta_0}\vartheta_P^2(0)+\frac{1}{\varepsilon}\mathbf{D}^2(\mathbf{E}_P(0),\vartheta_P(0))+\frac{1}{\mu}\mathbf{B}^2(\mathbf{H}_P(0))\right]+$$

$$+\int\limits_0^{+\infty}\left[\frac{1}{2}\int\limits_0^{+\infty}\alpha(\mid\tau-\eta\mid)\mathbf{E}_P(\eta)d\eta-\mathbf{I}_{(\alpha)}(\tau,\bar{\mathbf{E}}^t)\right]\cdot\mathbf{E}_P(\tau)d\tau+$$

$$+\frac{1}{\Theta_0}\int\limits_0^{+\infty}\left[\frac{1}{2}\int\limits_0^{+\infty}k(\mid\tau-\eta\mid)\mathbf{g}_P(\eta)d\eta-\mathbf{I}_{(k)}(\tau,\bar{\mathbf{g}}^t)\right]\cdot\mathbf{g}_P(\tau)d\tau, \tag{5.17}$$

where we have put

$$\mathbf{I}_{(\alpha)}(\tau,\bar{\mathbf{E}}^t)=\int\limits_0^{+\infty}\alpha'(\tau+\xi)\bar{\mathbf{E}}^t(\xi)d\xi,\ \ \mathbf{I}_{(k)}(\tau,\bar{\mathbf{g}}^t)=\int\limits_0^{+\infty}k'(\tau+\xi)\bar{\mathbf{g}}^t(\xi)d\xi,\ \ \tau\geq 0. \tag{5.18}$$

These two quantities, defined on $\mathbf{R}^+$, are present in (2.30), (2.31) and hence are related to the static continuations having the duration τ; their induced regularities allow us to evaluate the Fourier transforms,

$$\mathbf{I}_{(\alpha)+}(\omega,\bar{\mathbf{E}}^t)=\int\limits_0^{+\infty}e^{-i\omega\tau}\mathbf{I}_{(\alpha)}(\tau,\bar{\mathbf{E}}^t)d\tau,\ \ \mathbf{I}_{(k)+}(\omega,\bar{\mathbf{g}}^t)=\int\limits_0^{+\infty}e^{-i\omega\tau}\mathbf{I}_{(k)}(\tau,\bar{\mathbf{g}}^t)d\tau.$$

Thus, by means of Plancherel's theorem, (5.17) becomes

$$W(\sigma(t),P)=\frac{1}{2}\left[\frac{c}{\Theta_0}\vartheta_P^2(d)+\frac{1}{\varepsilon}\mathbf{D}_P^2(\mathbf{E}_P(d),\vartheta_P(d))+\frac{1}{\mu}\mathbf{B}_P^2(\mathbf{H}_P(d))\right]-$$

$$-\frac{1}{2}\left[\frac{c}{\Theta_0}\vartheta_P^2(0)+\frac{1}{\varepsilon}\mathbf{D}_P^2(\mathbf{E}_P(0),\vartheta_P(0))+\frac{1}{\mu}\mathbf{B}_P^2(\mathbf{H}_P(0))\right]+$$

$$+\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\left[\alpha_c(\omega)\mathbf{E}_{P+}(\omega)\cdot\mathbf{E}_{P+}^*(\omega)d\omega+\frac{1}{\Theta_0}k_c(\omega)\mathbf{g}_{P+}(\omega)\cdot\mathbf{g}_{P+}^*(\omega)\right]d\omega-$$

$$-\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\left[\mathbf{I}_{(\alpha)+}(\omega,\bar{\mathbf{E}}^t)\cdot\mathbf{E}_{P+}^*(\omega)+\frac{1}{\Theta_0}\mathbf{I}_{(k)+}(\omega,\bar{\mathbf{g}}^t)\cdot\mathbf{g}_{P+}^*(\omega)\right]d\omega.$$

6. The equivalence relation done in terms of the thermoelectromagnetic work. In Definition 4.3 we have called equivalent two couples of integrated histories, $\bar{\mathbf{E}}_j^t$ and $\bar{\mathbf{g}}_j^t$, $j = 1, 2$, if they give the same electric current density and the same heat flux when the body is subjected to the same process; moreover, as we have already observed, this equivalence relation yields the same response of the material because of the equality of the values of $\mathbf{E}$, $\mathbf{H}$ and ϑ at the initial instant of the process. An analogous equivalence relation may be done by means of the termolectromagnetic work as follows.

Definition 6.1. *Let* $\sigma_j(t) = (\mathbf{E}_j(t), \mathbf{H}_j(t), \vartheta_j(t), \bar{\mathbf{E}}_j^t, \bar{\mathbf{g}}_j^t)$, $j = 1, 2$, *be two states of* $\mathcal{B}$. *Two couples of integrated histories* $\bar{\mathbf{E}}_1^t$, $\bar{\mathbf{g}}_1$ *and* $\bar{\mathbf{E}}_2^t$, $\bar{\mathbf{g}}_2^t$ *are said to be w-equivalent if and only if the equality*

$$\tilde{W}(\mathbf{E}_1(t), \mathbf{H}_1(t), \vartheta_1(t), \bar{\mathbf{E}}_1^t, \bar{\mathbf{g}}_1^t; \dot{\mathbf{E}}_P, \dot{\mathbf{H}}_P, \dot{\vartheta}_P, \mathbf{g}_P) =$$

$$= \tilde{W}(\mathbf{E}_2(t), \mathbf{H}_2(t), \vartheta_2(t), \bar{\mathbf{E}}_2^t, \bar{\mathbf{g}}_2^t; \dot{\mathbf{E}}_P, \dot{\mathbf{H}}_P, \dot{\vartheta}_P, \mathbf{g}_P) \tag{6.1}$$

is satisfied for every $\dot{\mathbf{E}}_P : [0, \tau) \to \mathbf{R}^3$, $\dot{\mathbf{H}}_P : [0, \tau) \to \mathbf{R}^3$, $\dot{\vartheta}_P : [0, \tau) \to \mathbf{R}$, $\mathbf{g}_P : [0, \tau) \to \mathbf{R}^3$ *and for every* $\tau > 0$.

The two definitions of equivalence are equivalent as the following theorem states.

Theorem 6.1. *For the thermoelectromagnetic body* $\mathcal{B}$, *two couples of integrated histories of the electric field and of the temperature gradient are w-equivalent if and only if they are equivalent in the sense of Definition 4.3.*

Proof. It is obvious that if two couples of integrated histories, $\bar{\mathbf{E}}_1^t$, $\bar{\mathbf{g}}_1$ and $\bar{\mathbf{E}}_2^t$, $\bar{\mathbf{g}}_2^t$, are equivalent in the sense of Definition 4.3, then for every $\mathbf{E}_P : (0, \tau] \to \mathbf{R}^3$, $\mathbf{H}_P : (0, \tau] \to \mathbf{R}^3$, $\vartheta_P : (0, \tau] \to \mathbf{R}$, $\mathbf{g}_P : [0, \tau) \to \mathbf{R}^3$ and for every $\tau > 0$ we have

$$\int_0^d \left[\frac{c}{\Theta_0} \dot{\vartheta}_P(\tau)\vartheta_P(\tau) + \frac{1}{\varepsilon} \dot{\mathbf{D}}(\mathbf{E}_P(\tau), \vartheta_P(\tau)) \cdot \mathbf{D}(\mathbf{E}_P(\tau), \vartheta_P(\tau)) + \frac{1}{\mu} \dot{\mathbf{B}}(\mathbf{H}_P(\tau)) \cdot \right.$$

$$\left. \cdot \mathbf{B}(\mathbf{H}_P(\tau)) + \tilde{\mathbf{J}}((\mathbf{E}_P * \bar{\mathbf{E}}_1)^{t+\tau}) \cdot \mathbf{E}_P(\tau) - \frac{1}{\Theta_0} \tilde{\mathbf{q}}((\mathbf{g}_P * \bar{\mathbf{g}}_1)^{t+\tau}) \cdot \mathbf{g}_P(\tau) \right] d\tau =$$

$$= \int_0^d \left[\frac{c}{\Theta_0} \dot{\vartheta}_P(\tau)\vartheta_P(\tau) + \frac{1}{\varepsilon} \dot{\mathbf{D}}(\mathbf{E}_P(\tau), \vartheta_P(\tau)) \cdot \mathbf{D}(\mathbf{E}_P(\tau), \vartheta_P(\tau)) + \frac{1}{\mu} \dot{\mathbf{B}}(\mathbf{H}_P(\tau)) \cdot \right.$$

$$\left. \cdot \mathbf{B}(\mathbf{H}_P(\tau)) + \tilde{\mathbf{J}}((\mathbf{E}_P * \bar{\mathbf{E}}_2)^{t+\tau}) \cdot \mathbf{E}_P(\tau) - \frac{1}{\Theta_0} \tilde{\mathbf{q}}((\mathbf{g}_P * \bar{\mathbf{g}}_2)^{t+\tau}) \cdot \mathbf{g}_P(\tau) \right] d\tau, \tag{6.2}$$

because of (3.12) – (3.15) and of the conditions imposed by Definition 4.3. This relation yields the equality of the two works done on the same process of duration d, applied to the states $(\mathbf{E}_j(t), \mathbf{H}_j(t), \vartheta_j(t), \bar{\mathbf{E}}_j^t, \bar{\mathbf{g}}_j^t)$, $j = 1, 2$, whose instantaneous values coincide.

On the other hand, on supposing that the two couples of integrated histories, $\bar{\mathbf{E}}_1^t$, $\bar{\mathbf{g}}_1$ and $\bar{\mathbf{E}}_2^t$, $\bar{\mathbf{g}}_2^t$, are w-equivalent, (6.1) holds for any P with an arbitrary duration d. Using the expression (5.17) of the work, (6.1) yields

$$\frac{c}{2\Theta_0}\left\{\vartheta_{P_1}^2(d)-\vartheta_{P_2}^2(d)-[\vartheta_{P_1}^2(0)-\vartheta_{P_2}^2(0)]\right\}+\frac{1}{2\varepsilon}\left\{\mathbf{D}_{P_1}^2(d)-\mathbf{D}_{P_2}^2(d)-\right.$$

$$\left.-[\mathbf{D}_{P_1}^2(0)-\mathbf{D}_{P_2}^2(0)]\right\}+\frac{1}{2\mu}\left\{\mathbf{B}_{P_1}^2(d)-\mathbf{B}_{P_2}^2(d)-[\mathbf{B}_{P_1}^2(0)-\mathbf{B}_{P_2}^2(0)]\right\}+$$

$$+\frac{1}{2}\int_0^{+\infty}\int_0^{+\infty}\alpha(\mid\tau-\eta\mid)[\mathbf{E}_{P_1}(\eta)\cdot\mathbf{E}_{P_1}(\tau)-\mathbf{E}_{P_2}(\eta)\cdot\mathbf{E}_{P_2}(\tau)]d\eta d\tau-$$

$$-\int_0^{+\infty}[\mathbf{I}_{(\alpha)}(\tau,\bar{\mathbf{E}}_1^t)\cdot\mathbf{E}_{P_1}(\tau)-\mathbf{I}_{(\alpha)}(\tau,\bar{\mathbf{E}}_2^t)\cdot\mathbf{E}_{P_2}(\tau)]d\tau-$$

$$-\frac{1}{\Theta_0}\int_0^{+\infty}[\mathbf{I}_{(k)}(\tau,\bar{\mathbf{g}}_1^t)-\mathbf{I}_{(k)}(\tau,\bar{\mathbf{g}}_2^t)]\cdot\mathbf{g}_P(\tau)d\tau=0, \tag{6.3}$$

where the quantities evaluated in d and 0 are given by (3.14) and (2.1) taking into account (3.12)–(3.14). In particular, we have $\vartheta_{P_j}(d)=\vartheta_j(t)+\int_0^d\dot{\vartheta}_P(s)ds$ and $\vartheta_{P_j}(0)=\vartheta_j(t)$, $j=1,2$, and analogous relations for $\mathbf{E}_{P_j}$ and $\mathbf{H}_{P_j}$, $j=1,2$. Obviously, since the same $\mathbf{g}_P$ appears on both sides of (6.1) the integral with $k(\mid\tau-\xi\mid)$ has been eliminated. Substituting all these relations into (6.3), we get

$$\left\{\left(\frac{c}{\Theta_0}+\frac{\mathbf{a}^2}{\varepsilon}\right)[\vartheta_1(t)-\vartheta_2(t)]+\mathbf{a}\cdot[\mathbf{E}_1(t)-\mathbf{E}_2(t)]\right\}\int_0^d\dot{\vartheta}_P(s)ds+\{\mathbf{a}[\vartheta_1(t)-$$

$$-\vartheta_2(t)]+\varepsilon[\mathbf{E}_1(t)-\mathbf{E}_2(t)]\}\cdot\int_0^d\dot{\mathbf{E}}_P(s)ds+\mu[\mathbf{H}_1(t)-\mathbf{H}_2(t)]\cdot\int_0^d\dot{\mathbf{H}}_P(s)ds+$$

$$+\frac{1}{2}[\mathbf{E}_1^2(t)-\mathbf{E}_2^2(t)]\int_0^{+\infty}\int_0^{+\infty}\alpha(\mid\tau-\eta\mid)d\eta d\tau+\frac{1}{2}[\mathbf{E}_1(t)-\mathbf{E}_2(t)]\cdot$$

$$\cdot\int_0^{+\infty}\int_0^{+\infty}\alpha(\mid\tau-\eta\mid)\left[\int_0^{\tau}\dot{\mathbf{E}}_P(s)ds+\int_0^{\eta}\dot{\mathbf{E}}_P(s)ds\right]d\eta d\tau-\int_0^{+\infty}[\mathbf{I}_{(\alpha)}(\tau,\bar{\mathbf{E}}_1^t)\cdot$$

$$\cdot\,\mathbf{E}_1(t) - \mathbf{I}_{(\alpha)}(\tau, \bar{\mathbf{E}}_2^t) \cdot \mathbf{E}_2(t)]d\tau - \int\limits_0^{+\infty} [\mathbf{I}_{(\alpha)}(\tau, \bar{\mathbf{E}}_1^t) - \mathbf{I}_{(\alpha)}(\tau, \bar{\mathbf{E}}_2^t)] \cdot$$

$$\cdot \int\limits_0^{\tau} \dot{\mathbf{E}}_P(s) ds d\tau - \frac{1}{\Theta_0} \int\limits_0^{+\infty} [\mathbf{I}_{(k)}(\tau, \bar{\mathbf{g}}_1^t) - \mathbf{I}_{(k)}(\tau, \bar{\mathbf{g}}_2^t]) \cdot \mathbf{g}_P(\tau) d\tau = 0, \qquad (6.4)$$

which must hold for any P and any $d > 0$. The arbitrariness of $\dot{\vartheta}_P$ and the one of $\dot{\mathbf{E}}_P$ yield the following system:

$$\left(\frac{c}{\Theta_0} + \frac{\mathbf{a}^2}{\varepsilon}\right) [\vartheta_1(t) - \vartheta_2(t)] + \mathbf{a} \cdot [\mathbf{E}_1(t) - \mathbf{E}_2(t)] = 0,$$
$$\mathbf{a}[\vartheta_1(t) - \vartheta_2(t)] + \varepsilon[\mathbf{E}_1(t) - \mathbf{E}_2(t)] = \mathbf{0}, \qquad (6.5)$$

whence it follows that

$$\vartheta_1(t) = \vartheta_2(t), \qquad \mathbf{E}_1(t) = \mathbf{E}_2(t); \qquad (6.6)$$

moreover, since $\dot{\mathbf{H}}_P$ and $\mathbf{g}_P$ are also arbitrary, we get

$$\mathbf{H}_1(t) = \mathbf{H}_2(t), \quad \mathbf{I}_{(\alpha)}(\tau, \bar{\mathbf{E}}_1^t) = \mathbf{I}_{(\alpha)}(\tau, \bar{\mathbf{E}}_2^t), \quad \mathbf{I}_{(k)}(\tau, \bar{\mathbf{g}}_1^t) = \mathbf{I}_{(k)}(\tau, \bar{\mathbf{g}}_2^t). \qquad (6.7)$$

From $(6.7)_{2,3}$ and (5.18) we obtain

$$\int\limits_0^{+\infty} \alpha'(\tau + \xi)[\bar{\mathbf{E}}_1^t(\xi) - \bar{\mathbf{E}}_2^t(\xi)])d\xi = \mathbf{0}, \quad \int\limits_0^{+\infty} k'(\tau + \xi)[\bar{\mathbf{g}}_1^t(\xi) - \bar{\mathbf{g}}_2^t(\xi)]d\xi = \mathbf{0}, \qquad (6.8)$$

which, together with the conditions (6.6) and $(6.7)_1$, expresses the equivalence of the two couples of integrated histories $\bar{\mathbf{E}}_j^t, \bar{\mathbf{g}}_j^t$, $j = 1, 2$, since the differences $\bar{\mathbf{E}}^t = \bar{\mathbf{E}}_1^t - \bar{\mathbf{E}}_2^t$, $\bar{\mathbf{g}}^t = \bar{\mathbf{g}}_1^t - \bar{\mathbf{g}}_2^t$ satisfy $(4.5)_2$ and $(4.6)_2$.

7. Maximum recoverable work. The maximum recoverable work is the maximum work obtainable from the material at a given state. It is defined as follows.

Definition 7.1. *Given a state σ of $\mathcal{B}$, the maximum work obtained by starting from σ is*

$$W_R(\sigma) = \sup\{-W(\sigma, P) : \ P \in \Pi\}, \qquad (7.1)$$

where Π denotes the set of finite work processes.

We observe that from thermodynamic considerations we have $W_R(\sigma) < +\infty$; moreover, $W_R(\sigma)$ is a nonnegative function of the state, since the null process, which belongs to Π, yields a null work. In many works [2, 8, 12] it has been shown that such a work (7.1) coincides with the minimum free energy, that is,

$$\psi_m(\sigma) = W_R(\sigma). \qquad (7.2)$$

In order to derive an expression for these quantities we consider an initial state $\sigma(t) = (\mathbf{E}(t), \mathbf{H}(t), \vartheta(t), \bar{\mathbf{E}}^t, \bar{\mathbf{g}}^t)$ at a fixed time t when we apply a process $P \in \Pi$ with a finite duration d but extended on $[d, +\infty)$, where $P = 0$ and we assume the following values: $\mathbf{E}_P(d) = \mathbf{0}$, $\mathbf{H}_P(d) = \mathbf{0}$, $\vartheta_P(d) = 0$. Then, we consider the corresponding work expressed by (5.17), which now reduces to

$$W(\sigma, P) = -\frac{1}{2}\left\{\frac{c}{\Theta_0}\vartheta^2(t) + \frac{1}{\varepsilon}[\varepsilon\mathbf{E}(t) + \vartheta(t)\mathbf{a}]^2 + \mu\mathbf{H}^2(t)\right\} +$$

$$+\frac{1}{2}\int_0^{+\infty}\int_0^{+\infty}\left[\alpha(|\ \tau-\xi\ |)\mathbf{E}_P(\eta)\cdot\mathbf{E}_P(\tau) + \frac{1}{\Theta_0}k(|\ \tau-\eta\ |)\mathbf{g}_P(\eta)\cdot\mathbf{g}_P(\tau)\right]d\eta d\tau -$$

$$-\int_0^{+\infty}\left[\mathbf{I}_{(\alpha)}(\tau, \bar{\mathbf{E}}^t)\cdot\mathbf{E}_P(\tau) + \frac{1}{\Theta_0}\mathbf{I}_{(k)}(\tau, \bar{\mathbf{g}}^t)\cdot\mathbf{g}_P(\tau)\right]d\tau. \tag{7.3}$$

The required maximum recoverable work will be obtained by an opportune process $P^{(m)}$ related to $\mathbf{E}^{(m)}$ and $\mathbf{g}^{(m)}$; therefore, we consider the set of processes related to

$$\mathbf{E}_P(\tau) = \mathbf{E}^{(m)}(\tau) + \gamma\mathbf{e}(\tau), \quad \mathbf{g}_P(\tau) = \mathbf{g}^{(m)}(\tau) + \delta\mathbf{v}(\tau) \quad \tau \in \mathbf{R}^+, \tag{7.4}$$

with γ and δ real parameters, $\mathbf{e}$ and $\mathbf{v}$ arbitrary smooth functions with $\mathbf{e}(0) = \mathbf{0}$ and $\mathbf{v}(0) = \mathbf{0}$, and we study the maximum of $-W(\sigma, P)$ by substituting (7.4) into (7.3) and evaluating

$$\begin{gathered}\frac{\partial}{\partial\gamma}[-W(\sigma, P)]\ |_{\gamma=0} = -\int_0^{+\infty}\left[\int_0^{+\infty}\alpha(|\ \tau-\eta\ |)\mathbf{E}^{(m)}(\eta)d\eta - \mathbf{I}_{(\alpha)}(\tau, \bar{\mathbf{E}}^t)\right]\cdot\mathbf{E}(\tau)d\tau = 0,\\ \frac{\partial}{\partial\delta}[-W(\sigma, P)]\ |_{\delta=0} = -\frac{1}{\Theta_0}\int_0^{+\infty}\left[\int_0^{+\infty}k(|\ \tau-\eta\ |)\mathbf{g}^{(m)}(\eta)d\eta - \mathbf{I}_{(k)}(\tau, \bar{\mathbf{g}}^t)\right]\cdot\mathbf{v}(\tau)d\tau = 0,\end{gathered} \tag{7.5}$$

whence it follows that

$$\begin{gathered}\int_0^{+\infty}\alpha(|\ \tau-\eta\ |)\mathbf{E}^{(m)}(\eta)d\eta = \mathbf{I}_{(\alpha)}(\tau, \bar{\mathbf{E}}^t),\\ \int_0^{+\infty}k(|\ \tau-\eta\ |)\mathbf{g}^{(m)}(\eta)d\eta = \mathbf{I}_{(k)}(\tau, \bar{\mathbf{g}}^t)\end{gathered} \tag{7.6}$$

for all $\tau \in \mathbf{R}^+$. These relations turn out to be two integral equations of the Wiener–Hopf type and of the first kind, which are not solvable in the general case. Nevertheless, the thermodynamic properties of the kernels α and k and some theorems on factorization allow us to determine the solutions $\mathbf{E}^{(m)}$ and $\mathbf{g}^{(m)}$ of (7.6), which corresponds to the maximum recoverable work, whose expression, derived from (7.1), (7.3) with (7.6), is

$$W_R(\sigma) = \frac{1}{2}\left\{\frac{c}{\Theta_0}\vartheta^2(t) + \frac{1}{\varepsilon}[\varepsilon\mathbf{E}(t) + \vartheta(t)\mathbf{a}]^2 + \mu\mathbf{H}^2(t)\right\} +$$

$$+\frac{1}{2}\int_0^{+\infty}\int_0^{+\infty}\Big[\alpha(\mid \tau - \xi \mid)\mathbf{E}^{(m)}(\eta)\cdot\mathbf{E}^{(m)}(\tau) +$$

$$+\frac{1}{\Theta_0}k(\mid \tau - \eta \mid)\mathbf{g}^{(m)}(\eta)\cdot\mathbf{g}^{(m)}(\tau)\Big]\, d\eta d\tau \tag{7.7}$$

which, applying Plancherel's theorem, becomes

$$W_R(\sigma) = \frac{1}{2}\left\{\frac{c}{\Theta_0}\vartheta^2(t) + \frac{1}{\varepsilon}[\varepsilon\mathbf{E}(t) + \vartheta(t)\mathbf{a}]^2 + \mu\mathbf{H}^2(t)\right\} +$$

$$+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\alpha_c(\omega)\mathbf{E}_+^{(m)}(\omega)\cdot(\mathbf{E}_+^{(m)}(\omega))^* + \frac{1}{\Theta_0}k_c(\omega)\mathbf{g}_+^{(m)}(\omega)\cdot(\mathbf{g}_+^{(m)}(\omega))^*\right] d\omega. \tag{7.8}$$

It remains to solve the Wiener–Hopf equations (7.6). For this purpose let

$$\mathbf{r}^{(\alpha)}(\tau) = \int_{-\infty}^{+\infty}\alpha(\mid \tau - s \mid)\mathbf{E}^{(m)}(s)ds,\ \ \mathbf{r}^{(k)}(\tau) = \int_{-\infty}^{+\infty}k(\mid \tau - s \mid)\mathbf{g}^{(m)}(s)ds\ \ \forall\tau\in\mathbf{R}^-, \tag{7.9}$$

be equal to zero on $\mathbf{R}^{++}$. We observe that $\mathrm{supp}(\mathbf{r}^{(\alpha)}) \subseteq \mathbf{R}^-$, $\mathrm{supp}(\mathbf{r}^{(k)}) \subseteq \mathbf{R}^-$, $\mathrm{supp}(\mathbf{E}^{(m)}) \subseteq$ $\subseteq \mathbf{R}^+$, $\mathrm{supp}(\mathbf{g}^{(m)}) \subseteq \mathbf{R}^+$, $\mathrm{supp}(\mathbf{I}_{(\alpha)}(\cdot,\bar{\mathbf{E}}^t)) \subseteq \mathbf{R}^+$, $\mathrm{supp}(\mathbf{I}_{(k)}(\cdot,\bar{\mathbf{g}}^t)) \subseteq \mathbf{R}^+$; therefore, (7.6) can be rewritten as follows:

$$\begin{gathered}\int_0^{+\infty}\alpha(\mid \tau - \eta \mid)\mathbf{E}^{(m)}(\eta)d\eta = \mathbf{I}_{(\alpha)}(\tau,\bar{\mathbf{E}}^t) + \mathbf{r}^{(\alpha)}(\tau),\\ \int_0^{+\infty}k(\mid \tau - \eta \mid)\mathbf{g}^{(m)}(\eta)d\eta = \mathbf{I}_{(k)}(\tau,\bar{\mathbf{g}}^t) + \mathbf{r}^{(k)}(\tau)\end{gathered} \tag{7.10}$$

for all $\tau\in\mathbf{R}$, whence Fourier's transform yields

$$2\alpha_c(\omega)\mathbf{E}_+^{(m)}(\omega) = \mathbf{I}_{(\alpha)+}(\omega,\bar{\mathbf{E}}^t) + \mathbf{r}_-^{(\alpha)}(\omega),\ \ 2k_c(\omega)\mathbf{g}_+^{(m)}(\omega) = \mathbf{I}_{(k)+}(\omega,\bar{\mathbf{g}}) + \mathbf{r}_-^{(k)}(\omega). \tag{7.11}$$

Let us introduce

$$K^{(\alpha)}(\omega) = (1+\omega^2)\alpha_c(\omega),\quad K^{(k)}(\omega) = (1+\omega^2)k_c(\omega), \tag{7.12}$$

which are two functions without zeros for any real $\omega \in \mathbf{R}$, and at infinity, because of the properties (2.22)–(2.25). They can be factorized as well as $\alpha_c(\omega)$ and $k_c(\omega)$ and, therefore, we have

$$K^{(\alpha)}(\omega) = K^{(\alpha)}_{(+)}(\omega)K^{(\alpha)}_{(-)}(\omega), \quad K^{(k)}(\omega) = K^{(k)}_{(+)}(\omega)K^{(k)}_{(-)}(\omega), \tag{7.13}$$

$$\alpha_c(\omega) = \alpha_{(+)}(\omega)\alpha_{(-)}(\omega), \quad k_c(\omega) = k_{(+)}(\omega)k_{(-)}(\omega), \tag{7.14}$$

whence, from (7.12) it follows that

$$\alpha_{(\pm)}(\omega) = \frac{1}{1 \pm i\omega}K^{(\alpha)}_{(\pm)}(\omega), \quad k_{(\pm)}(\omega) = \frac{1}{1 \pm i\omega}K^{(k)}_{(\pm)}(\omega). \tag{7.15}$$

Thus, from (7.11) we obtain

$$\alpha_{(+)}(\omega)\mathbf{E}^{(m)}_{+}(\omega) = \frac{1}{2\alpha_{(-)}(\omega)}[\mathbf{I}_{(\alpha)+}(\omega, \bar{\mathbf{E}}^t) + \mathbf{r}^{(\alpha)}_{-}(\omega)], \tag{7.16}$$

$$k_{(+)}(\omega)\mathbf{g}^{(m)}_{+}(\omega) = \frac{1}{2k_{(-)}(\omega)}[\mathbf{I}_{(k)+}(\omega, \bar{\mathbf{g}}^t) + \mathbf{r}^{(k)}_{-}(\omega)]. \tag{7.17}$$

Let us consider

$$\mathbf{P}^t_{(\alpha)}(z) = \frac{1}{4\pi i}\int_{-\infty}^{+\infty}\frac{\mathbf{I}_{(\alpha)+}(\omega, \bar{\mathbf{E}}^t)/\alpha_{(-)}(\omega)}{\omega - z}d\omega, \;\; \mathbf{P}^t_{(\alpha)(\pm)}(\omega) = \lim_{\beta\to 0^{\mp}} \lim\mathbf{P}^t_{(\alpha)}(\omega + i\beta), \tag{7.18}$$

$$\mathbf{P}^t_{(k)}(z) = \frac{1}{4\pi i}\int_{-\infty}^{+\infty}\frac{\mathbf{I}_{(k)+}(\omega, \bar{\mathbf{g}}^t)/k_{(-)}(\omega)}{\omega - z}d\omega, \;\; \mathbf{P}^t_{(k)(\pm)}(\omega) = \lim_{\beta\to 0^{\mp}} \lim\mathbf{P}^t_{(k)}(\omega + i\beta), \tag{7.19}$$

which, by using the Plemelj formulae [18], yield

$$\frac{\mathbf{I}_{(\alpha)+}(\omega, \bar{\mathbf{E}}^t)}{2\alpha_{(-)}(\omega)} = \mathbf{P}^t_{(\alpha)(-)}(\omega) - \mathbf{P}^t_{(\alpha)(+)}(\omega), \tag{7.20}$$

$$\frac{\mathbf{I}_{(k)+}(\omega, \bar{\mathbf{g}}^t)}{2k_{(-)}(\omega)} = \mathbf{P}^t_{(k)(-)}(\omega) - \mathbf{P}^t_{(k)(+)}(\omega). \tag{7.21}$$

We observe that both $\mathbf{P}^t_{(\alpha)(\pm)}(z)$ and $\mathbf{P}^t_{(k)(\pm)}(z)$ have zeros and singularities in $z \in \mathbf{C}^{\pm}$, and hence they are analytic in $\mathbf{C}^{(\mp)}$ and, by the hypothesis on the Fourier transforms [8], also on $\mathbf{R}$.

From (7.16), (7.17), taking into account (7.20), (7.21), we have

$$\alpha_{(+)}(\omega)\mathbf{E}_{+}^{(m)}(\omega) + \mathbf{P}_{(\alpha)(+)}^{t}(\omega) = \mathbf{P}_{(\alpha)(-)}^{t}(\omega) + \frac{\mathbf{r}_{-}^{(\alpha)}(\omega)}{2\alpha_{(-)}(\omega)}, \tag{7.22}$$

$$k_{(+)}(\omega)\mathbf{g}_{+}^{(m)}(\omega) + \mathbf{P}_{(k)(+)}^{t}(\omega) = \mathbf{P}_{(k)(-)}^{t}(\omega) + \frac{\mathbf{r}_{-}^{(k)}(\omega)}{2k_{(-)}(\omega)}, \tag{7.23}$$

where the quantities in the left-hand sides, considered as functions of z, are analytic on $\mathbf{C}^{-}$, while the others in the right-hand sides are analytic on $\mathbf{C}^{+}$; consequently, the functions in the left-hand sides have analytic extensions on $\mathbf{C}$ and vanish at infinity, therefore are equal to zero and hence we get

$$\mathbf{E}_{+}^{(m)}(\omega) = -\frac{\mathbf{P}_{(\alpha)(+)}^{t}(\omega)}{\alpha_{(+)}(\omega)}, \quad \mathbf{g}_{+}^{(m)}(\omega) = -\frac{\mathbf{P}_{(k)(+)}^{t}(\omega)}{k_{(+)}(\omega)}; \tag{7.24}$$

analogous relations may be derived by putting the right-hand sides of (7.22), (7.23) equal to zero.

These last relations (7.24) substituted into (7.8) yield the required expression of the minimum free energy

$$\psi_m(\sigma(t)) = \frac{1}{2}\left\{\frac{c}{\Theta_0}\vartheta^2(t) + \frac{1}{\varepsilon}[\varepsilon\mathbf{E}(t) + \vartheta(t)\mathbf{a}]^2 + \mu\mathbf{H}^2(t)\right\} +$$

$$+ \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\mid \mathbf{P}_{(\alpha)(+)}^{t}(\omega) \mid^2 + \frac{1}{\Theta_0} \mid \mathbf{P}_{(k)(+)}^{t}(\omega) \mid^2\right] d\omega. \tag{7.25}$$

8. An equivalent formulation of ψ_m. The minimum free energy ψ_m, we have now derived, is expressed in terms of $\mathbf{P}_{(\alpha)(+)}^{t}(\omega)$ and $\mathbf{P}_{(k)(+)}^{t}(\omega)$, which are related to $\bar{\mathbf{E}}^t$ and $\bar{\mathbf{g}}^t$. In order to obtain new relations in function of these last quantities, we extend the kernels $\alpha'(s)$ and $k'(s)$ on $\mathbf{R}^{--}$ with odd functions, denoted by $\alpha'^{(o)}(s)$ and $k'^{(o)}(s)$, such that $\alpha'^{(o)}(s) = \alpha'(s)$, $k'^{(o)}(s) = k'(s)\ \forall s \geq 0$; moreover, we take the usual extensions on $\mathbf{R}^{--}$ for $\bar{\mathbf{E}}^t$ and $\bar{\mathbf{g}}^t$, i.e., $\bar{\mathbf{E}}^t(s) = \mathbf{0}$, $\bar{\mathbf{g}}^t(s) = \mathbf{0}\ \forall s < 0$. Identifying these functions with their extensions, (5.18) become

$$\mathbf{I}_{(\alpha)}(\tau, \bar{\mathbf{E}}^t) = \int_{-\infty}^{+\infty}\alpha'^{(o)}(\tau+\xi)\bar{\mathbf{E}}^t(\xi)d\xi,\ \mathbf{I}_{(k)}(\tau, \bar{\mathbf{g}}^t) = \int_{-\infty}^{+\infty}k'^{(o)}(\tau+\xi)\bar{\mathbf{g}}^t(\xi)d\xi,\ \tau \geq 0, \tag{8.1}$$

which, by defining

$$\mathbf{I}_{(\alpha)}^{(N)}(\tau, \bar{\mathbf{E}}^t) = \int_{-\infty}^{+\infty}\alpha'^{(o)}(\tau+\xi)\bar{\mathbf{E}}^t(\xi)d\xi,\ \mathbf{I}_{(k)}^{(N)}(\tau, \bar{\mathbf{g}}^t) = \int_{-\infty}^{+\infty}k'^{(o)}(\tau+\xi)\bar{\mathbf{g}}^t(\xi)d\xi,\ \tau < 0, \tag{8.2}$$

can be extended on $\mathbf{R}$ as follows:

$$\mathbf{I}^{(\mathbf{R})}_{(\alpha)}(\tau,\bar{\mathbf{E}}^t) = \int\limits_{-\infty}^{+\infty} \alpha'^{(o)}(\tau+\xi)\bar{\mathbf{E}}^t(\xi)d\xi = \begin{cases} \mathbf{I}_{(\alpha)}(\tau,\bar{\mathbf{E}}^t) & \forall\tau \geq 0, \\ \mathbf{I}^{(N)}_{(\alpha)}(\tau,\bar{\mathbf{E}}^t) & \forall\tau < 0, \end{cases} \tag{8.3}$$

$$\mathbf{I}^{(\mathbf{R})}_{(k)}(\tau,\bar{\mathbf{g}}^t) = \int\limits_{-\infty}^{+\infty} k'^{(o)}(\tau+\xi)\bar{\mathbf{g}}^t(\xi)d\xi = \begin{cases} \mathbf{I}_{(k)}(\tau,\bar{\mathbf{g}}^t) & \forall\tau \geq 0, \\ \mathbf{I}^{(N)}_{(k)}(\tau,\bar{\mathbf{g}}^t) & \forall\tau < 0. \end{cases} \tag{8.4}$$

Introducing $\bar{\mathbf{E}}^t_N(s) = \bar{\mathbf{E}}^t(-s)$, $\bar{\mathbf{g}}^t_N(s) = \bar{\mathbf{g}}^t(-s)\ \forall s \leq 0$ with their extensions $\bar{\mathbf{E}}^t_N(s) = \mathbf{0}$, $\bar{\mathbf{g}}^t_N(s) = \mathbf{0}\ \forall s > 0$, their Fourier's transforms are

$$\bar{\mathbf{E}}^t_{N_F}(\omega) = \bar{\mathbf{E}}^t_{N_-}(\omega) = \left(\bar{\mathbf{E}}^t_+(\omega)\right)^*, \quad \bar{\mathbf{g}}^t_{N_F}(\omega) = \bar{\mathbf{g}}^t_{N_-}(\omega) = \left(\bar{\mathbf{g}}^t_+(\omega)\right)^*. \tag{8.5}$$

Thus, (8.3) and (8.4) can be written as

$$\mathbf{I}^{(\mathbf{R})}_{(\alpha)}(\tau,\bar{\mathbf{E}}^t) = \int\limits_{-\infty}^{+\infty} \alpha'^{(o)}(\tau-s)\bar{\mathbf{E}}^t_N(s)ds,\ \mathbf{I}^{(\mathbf{R})}_{(k)}(\tau,\bar{\mathbf{g}}^t) = \int\limits_{-\infty}^{+\infty} k'^{(o)}(\tau-s)\bar{\mathbf{g}}^t_N(s)ds, \tag{8.6}$$

whose Fourier's transforms, taking into account $(2.18)_3$, are given by

$$\mathbf{I}^{(\mathbf{R})}_{(\alpha)_F}(\omega,\bar{\mathbf{E}}^t) = -2i\alpha'_s(\omega)\left(\bar{\mathbf{E}}^t_+(\omega)\right)^*, \quad \mathbf{I}^{(\mathbf{R})}_{(k)_F}(\omega,\bar{\mathbf{g}}^t) = -2ik'_s(\omega)\left(\bar{\mathbf{g}}^t_+(\omega)\right)^*. \tag{8.7}$$

Hence, using (2.22) and (7.14), it follows that

$$\frac{1}{2\alpha_{(-)}(\omega)}\mathbf{I}^{(\mathbf{R})}_{(\alpha)_F}(\omega,\bar{\mathbf{E}}^t) = i\omega\alpha_{(+)}(\omega)\left(\bar{\mathbf{E}}^t_+(\omega)\right)^*, \tag{8.8}$$

$$\frac{1}{2k_{(-)}(\omega)}\mathbf{I}^{(\mathbf{R})}_{(k)_F}(\omega,\bar{\mathbf{g}}^t) = i\omega k_{(+)}(\omega)\left(\bar{\mathbf{g}}^t_+(\omega)\right)^*. \tag{8.9}$$

Directly from the definitions (8.3), (8.4) we obtain

$$\mathbf{I}^{(\mathbf{R})}_{(\alpha)_F}(\omega,\bar{\mathbf{E}}^t) = \mathbf{I}^{(N)}_{(\alpha)-}(\omega,\bar{\mathbf{E}}^t) + \mathbf{I}_{(\alpha)+}(\omega,\bar{\mathbf{E}}^t), \tag{8.10}$$

$$\mathbf{I}^{(\mathbf{R})}_{(k)_F}(\omega,\bar{\mathbf{g}}^t) = \mathbf{I}^{(N)}_{(k)-}(\omega,\bar{\mathbf{g}}^t) + \mathbf{I}_{(k)+}(\omega,\bar{\mathbf{g}}^t), \tag{8.11}$$

whence, taking into account (7.20), (7.21), we get

$$\frac{1}{2\alpha_{(-)}(\omega)}\mathbf{I}^{(\mathbf{R})}_{(\alpha)_F}(\omega,\bar{\mathbf{E}}^t) = \frac{1}{2\alpha_{(-)}(\omega)}\mathbf{I}^{(N)}_{(\alpha)-}(\omega,\bar{\mathbf{E}}^t) + \mathbf{P}^t_{(\alpha)(-)}(\omega) - \mathbf{P}^t_{(\alpha)(+)}(\omega), \tag{8.12}$$

$$\frac{1}{2k_{(-)}(\omega)}\mathbf{I}^{(\mathbf{R})}_{(k)_F}(\omega,\bar{\mathbf{g}}^t) = \frac{1}{2k_{(-)}(\omega)}\mathbf{I}^{(N)}_{(k)-}(\omega,\bar{\mathbf{g}}^t) + \mathbf{P}^t_{(k)(-)}(\omega) - \mathbf{P}^t_{(k)(+)}(\omega), \tag{8.13}$$

where, using the Plemelj formulae, we also have

$$\frac{1}{2\alpha_{(-)}(\omega)}\mathbf{I}^{(\mathbf{R})}_{(\alpha)_F}(\omega,\bar{\mathbf{E}}^t) = \mathbf{P}^{(1)^t}_{(\alpha)(-)}(\omega) - \mathbf{P}^{(1)^t}_{(\alpha)(+)}(\omega), \tag{8.14}$$

$$\frac{1}{2k_{(-)}(\omega)}\mathbf{I}^{(\mathbf{R})}_{(k)_F}(\omega,\bar{\mathbf{g}}^t) = \mathbf{P}^{(1)^t}_{(k)(-)}(\omega) - \mathbf{P}^{(1)^t}_{(k)(+)}(\omega), \tag{8.15}$$

$\mathbf{P}^{(1)^t}_{(\alpha)(\pm)}(\omega)$ and $\mathbf{P}^{(1)^t}_{(k)(\pm)}(\omega)$ being defined as in (7.18) and (7.19).

Thus, (8.12) – (8.15) yield two relations, which define the functions

$$\mathbf{V}_{(\alpha)}(\omega) \equiv \mathbf{P}^t_{(\alpha)(+)}(\omega) - \mathbf{P}^{(1)^t}_{(\alpha)(+)}(\omega) = \mathbf{P}^t_{(\alpha)(-)}(\omega) - \mathbf{P}^{(1)^t}_{(\alpha)(-)}(\omega) + \frac{1}{2\alpha_{(-)}(\omega)}\mathbf{I}^{(N)}_{(\alpha)-}(\omega,\bar{\mathbf{E}}^t), \tag{8.16}$$

$$\mathbf{V}_{(k)}(\omega) \equiv \mathbf{P}^t_{(k)(+)}(\omega) - \mathbf{P}^{(1)^t}_{(k)(+)}(\omega) = \mathbf{P}^t_{(k)(-)}(\omega) - \mathbf{P}^{(1)^t}_{(k)(-)}(\omega) + \frac{1}{2k_{(-)}(\omega)}\mathbf{I}^{(N)}_{(k)-}(\omega,\bar{\mathbf{g}}^t), \tag{8.17}$$

with two different expressions, which assure the analyticity on $\mathbf{C}^+$ and on $\mathbf{C}^-$, respectively, and vanish at infinity; therefore, we have $\mathbf{V}_{(\alpha)}(\omega) = \mathbf{0}$ and $\mathbf{V}_{(k)}(\omega) = \mathbf{0}$. Hence, it follows that

$$\mathbf{P}^t_{(\alpha)(+)}(\omega) = \mathbf{P}^{(1)^t}_{(\alpha)(+)}(\omega), \quad \mathbf{P}^t_{(\alpha)(-)}(\omega) = \mathbf{P}^{(1)^t}_{(\alpha)(-)}(\omega) - \frac{1}{2\alpha_{(-)}(\omega)}\mathbf{I}^{(N)}_{(\alpha)-}(\omega,\bar{\mathbf{E}}^t), \tag{8.18}$$

$$\mathbf{P}^t_{(k)(+)}(\omega) = \mathbf{P}^{(1)^t}_{(k)(+)}(\omega), \quad \mathbf{P}^t_{(k)(-)}(\omega) = \mathbf{P}^{(1)^t}_{(k)(-)}(\omega) - \frac{1}{2k_{(-)}(\omega)}\mathbf{I}^{(N)}_{(k)-}(\omega,\bar{\mathbf{g}}^t). \tag{8.19}$$

Hence, taking account of (7.18), (7.19), (8.18), (8.19) and (8.8), (8.9), we get

$$\mathbf{P}^t_{(\alpha)(+)}(\omega) = \mathbf{P}^{(1)^t}_{(\alpha)(+)}(\omega) = \lim_{z\to\omega^-}\lim\frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty}\frac{i\omega'\alpha_{(+)}(\omega')\left(\bar{\mathbf{E}}^t_+(\omega')\right)^*}{\omega'-z}d\omega', \tag{8.20}$$

$$\mathbf{P}^t_{(k)(+)}(\omega) = \mathbf{P}^{(1)^t}_{(k)(+)}(\omega) = \lim_{z\to\omega^-}\lim\frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty}\frac{i\omega' k_{(+)}(\omega')\left(\bar{\mathbf{g}}^t_+(\omega')\right)^*}{\omega'-z}d\omega', \tag{8.21}$$

from which we have

$$\left(\mathbf{P}^t_{(\alpha)(+)}(\omega)\right)^* = i\lim_{\eta\to\omega^+}\lim\frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty}\frac{\omega'\alpha_{(-)}(\omega')\bar{\mathbf{E}}^t_+(\omega')}{\omega'-\eta}d\omega', \tag{8.22}$$

$$\left(\mathbf{P}^t_{(k)(+)}(\omega)\right)^* = i\lim_{\eta\to\omega^+}\lim\frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty}\frac{\omega' k_{(-)}(\omega')\bar{\mathbf{g}}^t_+(\omega')}{\omega'-\eta}d\omega'. \tag{8.23}$$

Application of the Plemelj formulae to the last two relations yields

$$\omega\alpha_{(-)}(\omega)\bar{\mathbf{E}}^t_+(\omega) = \mathbf{Q}^t_{(\alpha)(-)}(\omega) - \mathbf{Q}^t_{(\alpha)(+)}(\omega), \tag{8.24}$$

$$\omega k_{(-)}(\omega)\bar{\mathbf{g}}^t_+(\omega) = \mathbf{Q}^t_{(k)(-)}(\omega) - \mathbf{Q}^t_{(k)(+)}(\omega), \tag{8.25}$$

where $\mathbf{Q}^t_{(\alpha)(\pm)}(z)$, $\mathbf{Q}^t_{(k)(\pm)}(z)$ have zeros and singularities for $z \in \mathbf{C}^\pm$, being

$$\mathbf{Q}^t_{(\alpha)(\pm)}(\omega) = \lim_{z\to\omega^\mp} \lim \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\omega'\alpha_{(-)}(\omega')\bar{\mathbf{E}}^t_+(\omega')}{\omega'-z} d\omega', \tag{8.26}$$

$$\mathbf{Q}^t_{(k)(\pm)}(\omega) = \lim_{z\to\omega^\mp} \lim \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\omega' k_{(-)}(\omega')\bar{\mathbf{g}}^t_+(\omega')}{\omega'-z} d\omega'. \tag{8.27}$$

Comparison of (8.22), (8.23) with (8.26), (8.27) yields

$$\left(\mathbf{P}^t_{(\alpha)(+)}(\omega)\right)^* = i\mathbf{Q}^t_{(\alpha)(-)}(\omega), \qquad \left(\mathbf{P}^t_{(k)(+)}(\omega)\right)^* = i\mathbf{Q}^t_{(k)(-)}(\omega), \tag{8.28}$$

which, substituting into (7.25), gives the required new expression

$$\psi_m(t) = \frac{1}{2}\left\{\frac{c}{\Theta_0}\vartheta^2(t) + \frac{1}{\varepsilon}[\varepsilon\mathbf{E}(t) + \vartheta(t)\mathbf{a}]^2 + \mu\mathbf{H}^2(t)\right\} +$$

$$+ \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\mid \mathbf{Q}^t_{(\alpha)(-)}(\omega) \mid^2 + \frac{1}{\Theta_0} \mid \mathbf{Q}^t_{(k)(-)}(\omega) \mid^2\right] d\omega. \tag{8.29}$$

We note that both the current density and the heat flux can be expressed in terms of the last quantities we have derived. For this purpose we apply the Plancherel theorem to (3.5); using $(2.18)_3$, since α' and k' are considered as two odd functions, (2.22), where α_c and k_c are factorized by means of (7.14), and (8.24), (8.25), we get

$$\tilde{\mathbf{J}}(\bar{\mathbf{E}}^t) = \frac{i}{\pi}\int_{-\infty}^{+\infty} \alpha_{(+)}(\omega)\left[\mathbf{Q}^t_{(\alpha)(-)}(\omega) - \mathbf{Q}^t_{(\alpha)(+)}(\omega)\right] d\omega,$$

$$\tilde{\mathbf{q}}(\bar{\mathbf{g}}^t) = -\frac{i}{\pi}\int_{-\infty}^{+\infty} k_{(+)}(\omega)\left[\mathbf{Q}^t_{(k)(-)}(\omega) - \mathbf{Q}^t_{(k)(+)}(\omega)\right] d\omega.$$

These expressions, for the analyticity of $\alpha_{(+)}(\omega)\mathbf{Q}^t_{(\alpha)(+)}(\omega)$ and $k_{(+)}(\omega)\mathbf{Q}^t_{(k)(+)}(\omega)$ in $\mathbf{C}^-$, reduce to

$$\tilde{\mathbf{J}}(\bar{\mathbf{E}}^t) = \frac{i}{\pi}\int_{-\infty}^{+\infty} \alpha_{(+)}(\omega)\mathbf{Q}^t_{(\alpha)(-)}(\omega)d\omega, \;\; \tilde{\mathbf{q}}(\bar{\mathbf{g}}^t) = -\frac{i}{\pi}\int_{-\infty}^{+\infty} k_{(+)}(\omega)\mathbf{Q}^t_{(k)(-)}(\omega)d\omega,$$

which must be real and give the required relations.

9. A discrete spectrum model. We now apply the results of Section 8 to the particular class of response functions that characterize the discrete spectrum model. The relaxation functions α and k have the form

$$\alpha(t) = \begin{cases} \sum_{i=1}^{n} g_i e^{-\alpha_i t}, & t \geq 0, \\ 0, & t < 0, \end{cases} \qquad k(t) = \begin{cases} \sum_{i=1}^{n} h_i e^{-k_i t}, & t \geq 0, \\ 0, & t < 0, \end{cases} \tag{9.1}$$

where $g_i, \alpha_i, h_i, k_i \in \mathbf{R}^{++}$, $i = 1, 2, \ldots, n$, $n \in \mathbf{N}$ and $\alpha_j < \alpha_{j+1}$, $k_j < k_{j+1}$, $j = 1, 2, \ldots, n-$ -1. These hypotheses assure that $\alpha(0) = \sum_{i=1}^{n} g_i > 0$, $k(0) = \sum_{i=1}^{n} h_i > 0$, which are two conditions derived in [1] on account of (2.21) by using the inverse Fourier transforms of α_c and k_c.

The Fourier transforms of (9.1) are

$$\alpha_F(\omega) = \sum_{i=1}^{n} \frac{g_i}{\alpha_i + i\omega}, \quad k_F(\omega) = \sum_{i=1}^{n} \frac{h_i}{k_i + i\omega}, \quad \omega \in \mathbf{R},$$

whence, taking account of $(2.18)_1$, it follows that

$$\alpha_c(\omega) = \sum_{i=1}^{n} \frac{\alpha_i g_i}{\omega^2 + \alpha_i^2}, \quad k_c(\omega) = \sum_{i=1}^{n} \frac{k_i h_i}{\omega^2 + k_i^2}, \quad \omega \in \mathbf{R},$$

and we write (7.12) as

$$K^{(\alpha)}(\omega) = \sum_{i=1}^{n} \alpha_i g_i \frac{\omega^2 + 1}{\omega^2 + \alpha_i^2}, \quad K^{(k)}(\omega) = \sum_{i=1}^{n} k_i h_i \frac{\omega^2 + 1}{\omega^2 + k_i^2}, \quad \omega \in \mathbf{R}. \tag{9.2}$$

The last two expressions coincide with the ones derived in [1]. Thus, we recall the results of the study of the function $f^{(\alpha)}(z) = K^{(\alpha)}(z)$ and $f^{(k)}(z) = K^{(k)}(z)$, where $z = -\omega^2$.

Let $n \neq 1$. If we suppose that $\alpha_i^2, k_i^2 \neq 1$', $i = 1, 2, ..., n$, the functions $f^{(\alpha)}(z)$ and $f^{(k)}(z)$ have n simple poles at α_i^2 and k_i^2, $i = 1, 2, ..., n$. The number of simple zeros is n if $1 < \alpha_1^2$ and $1 < k_1^2$ or $\alpha_n^2 < 1$ and $k_n^2 < 1$, which are denoted by $\gamma_1^2 = 1$, γ_j^2, $j = 2, 3, ..., n$, and $\delta_1^2 = 1$, δ_j^2, $j = 2, 3, ..., n$; if $\alpha_p^2 < 1 < \alpha_{p+1}^2$ and $k_{p'}^2 < 1 < k_{p'+1}^2$, where p and p' are two integer numbers, which may also coincide but in any case they must assume only one of the values $1, 2, ..., n-1$, then the zeros γ_{p+1}^2 and $\delta_{p'+1}^2$ are such that $\gamma_{p+1}^2 \lim \gtrless 1 = \gamma_1^2$ and $\delta_{p'+1}^2 \lim \gtrless 1 = \delta_1^2$ and hence they can be equal to 1, therefore, they have multiplicity 2, and the number of the distinct zeros reduces to $n-1$. In any case the zeros different from 1 are such that

$$\begin{aligned} &\alpha_1^2 < \gamma_2^2 < \alpha_2^2 < \ldots < \alpha_p^2 < \gamma_{p+1}^2 < \alpha_{p+1}^2 < \ldots < \alpha_{n-1}^2 < \gamma_n^2 < \alpha_n^2, \\ &k_1^2 < \delta_2^2 < k_2^2 < \ldots < k_{p'}^2 < \delta_{p'+1}^2 < k_{p'+1}^2 < \ldots < k_{n-1}^2 < \delta_n^2 < k_n^2, \end{aligned} \tag{9.3}$$

moreover, (9.2) can be written as

$$K^{(\alpha)}(\omega) = K_{\infty}^{(\alpha)} \prod_{i=1}^{n} \left\{ \frac{\omega^2 + \gamma_i^2}{\omega^2 + \alpha_i^2} \right\}, \qquad K^{(k)}(\omega) = K_{\infty}^{(k)} \prod_{i=1}^{n} \left\{ \frac{\omega^2 + \delta_i^2}{\omega^2 + k_i^2} \right\}, \tag{9.4}$$

where

$$K_{\infty}^{(\alpha)} = \lim_{\omega \to \pm\infty} \lim K^{(\alpha)}(\omega) = \sum_{i=1}^{n} \alpha_i g_i > 0, \quad K_{\infty}^{(k)} = \lim_{\omega \to \pm\infty} \lim K^{(k)}(\omega) = \sum_{i=1}^{n} k_i h_i > 0$$

and $\gamma_1 = 1$, $\delta_1 = 1$, and only one of the other zeros, say γ_{p+1}^2 and $\delta_{p'+1}^2$, can be equal to 1, which thus becomes a zero of multiplicity 2.

The factorizations (7.13), taking account of (9.4), yield, in particular,

$$K_{(-)}^{(\alpha)}(\omega) = k_{\infty}^{(\alpha)} \prod_{i=1}^{n} \left\{ \frac{\omega + i\gamma_i}{\omega + i\alpha_i} \right\}, \qquad K_{(-)}^{(k)}(\omega) = k_{\infty}^{(k)} \prod_{i=1}^{n} \left\{ \frac{\omega + i\delta_i}{\omega + ik_i} \right\}, \tag{9.5}$$

where

$$k_{\infty}^{(\alpha)} = \sqrt{K_{\infty}^{(\alpha)}}, \qquad k_{\infty}^{(k)} = \sqrt{K_{\infty}^{(k)}}.$$

We must consider (8.26), (8.27), which give $\mathbf{Q}_{(\alpha)(-)}^{t}(\omega)$ and $\mathbf{Q}_{(k)(-)}^{t}(\omega)$ present in (8.29); in them $\alpha_{(-)}(\omega)$ and $k_{(-)}(\omega)$, contrary to what occurs in [1], are multiplied by ω, therefore the new zeros $\gamma_0 = 0$ and $\delta_0 = 0$ are introduced and (7.15), taking account of (9.5), yield

$$\omega\alpha_{(-)}(\omega) = ik_{\infty}^{(\alpha)} \frac{\omega}{\omega + i} \prod_{i=1}^{n} \left\{ \frac{\omega + i\gamma_i}{\omega + i\alpha_i} \right\}, \ \omega k_{(-)}(\omega) = ik_{\infty}^{(k)} \frac{\omega}{\omega + i} \prod_{i=1}^{n} \left\{ \frac{\omega + i\delta_i}{\omega + ik_i} \right\}. \tag{9.6}$$

These expressions, taking into account that $\gamma_1^2 = 1$, $\delta_1^2 = 1$ and putting $\rho_1 = \gamma_0 = 0$ and $\rho_j = \gamma_j$, $j = 2, 3, ..., n$, $\phi_1 = \delta_0 = 0$ and $\phi_j = \delta_j$, $j = 2, 3, ..., n$, can be written as follows:

$$\omega\alpha_{(-)}(\omega) = ik_{\infty}^{(\alpha)} \prod_{i=1}^{n} \left\{ \frac{\omega + i\rho_i}{\omega + i\alpha_i} \right\} = ik_{\infty}^{(\alpha)} \left(1 + i \sum_{r=1}^{n} \frac{A_r}{\omega + i\alpha_r} \right), \tag{9.7}$$

$$\omega k_{(-)}(\omega) = ik_{\infty}^{(k)} \prod_{i=1}^{n} \left\{ \frac{\omega + i\phi_i}{\omega + ik_i} \right\} = ik_{\infty}^{(k)} \left(1 + i \sum_{r=1}^{n} \frac{B_r}{\omega + ik_r} \right), \tag{9.8}$$

with

$$A_r = (\rho_r - \alpha_r) \prod_{i=1, i\neq r}^{n} \left\{ \frac{\rho_i - \alpha_r}{\alpha_i - \alpha_r} \right\}, \ B_r = (\phi_r - k_r) \prod_{i=1, i\neq r}^{n} \left\{ \frac{\phi_i - k_r)}{k_i - k_r} \right\}, \tag{9.9}$$

where we can have $\rho_{p+1} = \gamma_{p+1} = 1$ and $\phi_{p'+1} = \delta_{p'+1} = 1$.

We now consider the case where one of the inverse decay times α_i and of k_i, $i = 1, 2, ..., n$, is equal to 1. If $\alpha_1^2 = k_1^2 = 1$ or $\alpha_n^2 = k_n^2 = 1$ then $f^{(\alpha)}(z)$ and $f^{(k)}(z)$ have $n-1$ zeros γ_j^2, δ_j^2, $j = 2, 3, ..., n$, and $n-1$ poles $\alpha_j^2, k_j^2, j = 2, 3, ..., n$, or $\alpha_i^2, k_i^2, i = 1, 2, ..., n-1$, in the two cases; the presence of the factor $\dfrac{\omega}{\omega + i}$ in (9.6) yields the zeros $\rho_1 = 0$, $\phi_1 = 0$ and thus (9.7), (9.8) and (9.9), where $\alpha_1^2 = k_1^2 = 1$ or $\alpha_n^2 = k_n^2 = 1$, still hold. It may be that $\alpha_p^2 = 1, 1 < p < n$, $k_{p'}^2 = 1, 1 < p' < n$, therefore we have $n-1$ zeros γ_j^2, δ_j^2, $j = 2, 3, ..., n$, and $n-1$ poles α_i^2, $i = 1, 2, ..., p-1, p+1, ..., n, k_i^2, i = 1, 2, ..., p'-1, p'+1, ..., n$, all ordered as in (9.3) and also by $\alpha_{p-1}^2 < \gamma_p^2 < 1 < \gamma_{p+1}^2 < \alpha_{p+1}^2$, $k_{p'-1}^2 < \delta_{p'}^2 < 1 < \delta_{p'+1}^2 < k_{p'+1}^2$; however, (9.7), (9.8) and (9.9) hold with $\alpha_p^2 = k_{p'}^2 = 1$.

Let $n = 1$. This particular case must be considered separately by writing (7.15) by means of (9.5) with $n = 1$; we get

$$\omega\alpha_{(-)}(\omega) = \frac{ik_\infty^{(\alpha)}\omega}{\omega + i\alpha_1} = ik_\infty^{(\alpha)}\left(1 + \frac{iA_1}{\omega + i\alpha_1}\right), \;\; A_1 = -\alpha_1, \;\; k_\infty^{(\alpha)} = \sqrt{\alpha_1 g_1}, \tag{9.10}$$

$$\omega k_{(-)}(\omega) = \frac{ik_\infty^{(k)}\omega}{\omega + ik_1} = ik_\infty^{(k)}\left(1 + \frac{iB_1}{\omega + ik_1}\right), \;\; B_1 = -k_1, \;\; k_\infty^{(k)} = \sqrt{k_1 h_1}. \tag{9.11}$$

When $n \neq 1$, that is in the general case, from (8.26), (8.27) with $(9.7)_2$, $(9.8)_2$ we have

$$\mathbf{Q}^t_{(\alpha)(-)}(\omega) = i\frac{k_\infty^{(\alpha)}}{2\pi i}\int_{-\infty}^{+\infty}\frac{\bar{\mathbf{E}}^t_+(\omega')}{\omega' - \omega^+}d\omega' - \sum_{r=1}^{n}\frac{k_\infty^{(\alpha)}A_r}{2\pi i}\int_{-\infty}^{+\infty}\frac{\bar{\mathbf{E}}^t_+(\omega')/(\omega' - \omega^+)}{\omega' - (-i\alpha_r)}d\omega',$$

$$\mathbf{Q}^t_{(k)(-)}(\omega) = i\frac{k_\infty^{(k)}}{2\pi i}\int_{-\infty}^{+\infty}\frac{\bar{\mathbf{g}}^t_+(\omega')}{\omega' - \omega^+}d\omega' - \sum_{r=1}^{n}\frac{k_\infty^{(k)}B_r}{2\pi i}\int_{-\infty}^{+\infty}\frac{\bar{\mathbf{g}}^t_+(\omega')/(\omega' - \omega^+)}{\omega' - (-ik_r)}d\omega',$$

where the first integrals of the two expressions vanish because $\bar{\mathbf{E}}^t_+$, $\bar{\mathbf{g}}^t_+$, considered as functions of $z \in \mathbf{C}$, are analytic on $\mathbf{C}^{(-)}$, and the same integrals can be extended on an infinite contour on $\mathbf{C}^{(-)}$ without changing their values, which are zero; then, it remains to evaluate the second integrals by closing again on $\mathbf{C}^{(-)}$ and taking account of the sense of the integrations, thus obtaining

$$\mathbf{Q}^t_{(\alpha)(-)}(\omega) = -k_\infty^{(\alpha)}\sum_{r=1}^{n}\frac{A_r}{\omega + i\alpha_r}\bar{\mathbf{E}}^t_+(-i\alpha_r), \tag{9.12}$$

$$\mathbf{Q}^t_{(k)(-)}(\omega) = -k_\infty^{(k)}\sum_{r=1}^{n}\frac{B_r}{\omega + ik_r}\bar{\mathbf{g}}^t_+(-ik_r). \tag{9.13}$$

We observe that $(2.16)_1$ gives

$$\bar{\mathbf{E}}^t_+(-i\alpha_r) = \int\limits_0^{+\infty} e^{-\alpha_r s}\bar{\mathbf{E}}^t(s)ds = \left(\bar{\mathbf{E}}^t_+(-i\alpha_r)\right)^*, \tag{9.14}$$

$$\bar{\mathbf{g}}^t_+(-ik_r) = \int\limits_0^{+\infty} e^{-k_r s}\bar{\mathbf{g}}^t(s)ds = \left(\bar{\mathbf{g}}^t_+(-ik_r)\right)^*, \tag{9.15}$$

which allow us to obtain from (9.12), (9.13) that

$$\left(\mathbf{Q}^t_{(\alpha)(-)}(\omega)\right)^* = -k^{(\alpha)}_\infty \sum_{r=1}^n \frac{A_r}{\omega - i\alpha_r}\bar{\mathbf{E}}^t_+(-i\alpha_r),$$

$$\left(\mathbf{Q}^t_{(k)(-)}(\omega)\right)^* = -k^{(k)}_\infty \sum_{r=1}^n \frac{B_r}{\omega - ik_r}\bar{\mathbf{g}}^t_+(-i\alpha_r).$$

Finally, we can evaluate the integrals

$$\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} \left|\mathbf{Q}^t_{(\alpha)(-)}(\omega)\right|^2 d\omega =$$

$$= \left(k^{(\alpha)}_\infty\right)^2 \sum_{r,l=1}^n A_r A_l \bar{\mathbf{E}}^t_+(-i\alpha_r)\cdot\bar{\mathbf{E}}^t_+(-i\alpha_l)\frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty}\frac{i/(\omega+i\alpha_r)}{\omega - i\alpha_l}d\omega =$$

$$= K^{(\alpha)}_\infty \sum_{r,l=1}^n \frac{A_r A_l}{\alpha_r+\alpha_l}\bar{\mathbf{E}}^t_+(-i\alpha_r)\cdot\bar{\mathbf{E}}^t_+(-i\alpha_l),$$

$$\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} \left|\mathbf{Q}^t_{(k)(-)}(\omega)\right|^2 d\omega =$$

$$= \left(k^{(k)}_\infty\right)^2 \sum_{r,l=1}^n B_r B_l \bar{\mathbf{g}}^t_+(-ik_r)\cdot\bar{\mathbf{g}}^t_+(-ik_l)\frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty}\frac{i/(\omega+ik_r)}{\omega - ik_l}d\omega =$$

$$= K^{(k)}_\infty \sum_{r,l=1}^n \frac{B_r B_l}{k_r+k_l}\bar{\mathbf{g}}^t_+(-ik_r)\cdot\bar{\mathbf{g}}^t_+(-ik_l),$$

which, after using $(9.14)_2$, $(9.15)_2$, can be substituted into (8.29) and give

$$\psi_m(t) = \frac{1}{2}\left\{\frac{c}{\Theta_0}\vartheta^2(t) + \frac{1}{\varepsilon}[\varepsilon\mathbf{E}(t) + \vartheta(t)\mathbf{a}]^2 + \mu\mathbf{H}^2(t)\right\} +$$

$$+\frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} 2\left[\sum_{r,l=1}^{n} K_\infty^{(\alpha)}\frac{A_rA_l}{\alpha_r+\alpha_l}e^{-(\alpha_r s_1+\alpha_l s_2)}\bar{\mathbf{E}}^t(s_1)\cdot\bar{\mathbf{E}}^t(s_2) + \right.$$

$$\left. +\frac{1}{\Theta_0}\sum_{r,l=1}^{n} K_\infty^{(k)}\frac{B_rB_l}{k_r+k_l}e^{-(k_r s_1+k_l s_2)}\bar{\mathbf{g}}^t(s_1)\cdot\bar{\mathbf{g}}^t(s_2)\right] ds_1 ds_2.$$

This expression in the case where $n = 1$, taking account of $(9.10)_{2,3}$, $(9.11)_{2,3}$, assumes the simpler form

$$\psi_m(t) = \frac{1}{2}\left\{\frac{c}{\Theta_0}\vartheta^2(t) + \frac{1}{\varepsilon}[\varepsilon\mathbf{E}(t) + \vartheta(t)\mathbf{a}]^2 + \mu\mathbf{H}^2(t)\right\} +$$

$$+\frac{1}{2}\left\{\alpha_1^2 g_1\left[\int_0^{+\infty} e^{-\alpha_1 s}\bar{\mathbf{E}}^t(s)ds\right]^2 + \frac{1}{\Theta_0}k_1^2 h_1\left[\int_0^{+\infty} e^{-k_1 s}\bar{\mathbf{g}}^t(s)ds\right]^2\right\}.$$

In this last case it is easy to derive by means of integrations by parts the results obtained in [1], where the histories of $\mathbf{E}$ and $\mathbf{g}$ are considered instead of their integrated histories.

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