UDC 517 . 9

THE EXISTENCE OF POSITIVE SOLUTION OF SYSTEMS OF VOLTERRA NONLINEAR DIFFERENCE EQUATIONS

ІСНУВАННЯ ДОДАТНИХ РОЗВ'ЯЗКІВ СИСТЕМ ВОЛЬТЕРРІВСЬКИХ РІЗНИЦЕВИХ РІВНЯНЬ

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In this paper, a classification scheme for the eventually positive solutions of a class of two-dimensional Volterra nonlinear difference equations is given in terms of asymptotic magnitudes. Some necessary as well as sufficient conditions for the existence of such solutions are provided, without any monotonicity conditions on the nonlinear term.

Для згодом додатних розв'язків деякого класу двовимірних вольтеррівських нелінійних рівнянь наведено класифікаційну схему в термінах умов на асимптотику. Наведено також деякі необхідні і достатні умови існування таких розв'язків без вимоги монотонності нелінійного члена.

1. Introduction. The study of difference equations has experienced a significant interest in the past years, as they arise naturally in the modelling of real word phenomena [1–4]. Volterra difference equation arise in the mathematical modelling of some real phenomena and also in various procedures of numerical solution of some differential and integral equations. For applications of the Volterra difference equation in combinatorics and in epidemics, see [5]. In the very recent paper [6], the authors gave a classification scheme for the eventually positive solutions and necessary as well as sufficient conditions for the existence of such solutions for the following of two-dimensional Volterra nonlinear difference equations:

$$\begin{aligned} \triangle x_n &= h_n x_n + \sum_{i=1}^{n} a_{n,i} f(y_i), \\ \triangle y_n &= p_n y_n + \sum_{i=1}^{n} b_{n,i} g(x_i), \quad n \ge 0, \end{aligned} \tag{1}$$

where $\{a_{n,k}\}$, $\{b_{n,k}\}$ are positive for $n, k \ge 0$. The functions f and g are real-valued continuous on the real line R. Also, the coefficients $\{h_n\}$ and $\{p_n\}$ are positive sequences for $n \ge 0$ and satisfy the condition

$$\sum_{n=0}^{\infty} h_n < \infty, \quad \sum_{n=0}^{\infty} p_n < \infty. \tag{2}$$

They improve many works contained in their references, such as [7, 8]. However, the main tool they used is Kanaster's fixed point theorem for increasing operators, which needs monotonicity of the functions f and g. In this paper, we will use Schauder's fixed point theorem to get the existence of the eventually positive solutions. We do not need any monotonicity for the nonlinear terms f and g. Some necessary as well as sufficient conditions for the existence of such solutions are presented. Our results improve the results of [6]. Throughout this paper we assume that $f(x) > 0$, $g(x) > 0$ for $x \neq 0$ and (2) holds.

Definition 1 [6]. *A pair of real-valued sequences $\{(x_n, y_n)\}$ is said to be*
1) *a solution of system (1) if it satisfies (1) for $n \geq 0$;*
2) *eventually positive if both $\{x_n\}$ and $\{y_n\}$ are eventually positive;*
3) *nonoscillatory if both $\{x_n\}$ and $\{y_n\}$ are either eventually positive or eventually negative.*

To simply notation, we use the notation of [6] and let

$$A_n = \sum_{i=0}^{n-1} \left(\sum_{j=0}^{i} a_{i,j} \right), \tag{3}$$

$$B_n = \sum_{i=0}^{n-1} \left(\sum_{j=0}^{i} b_{i,j} \right), \tag{4}$$

$$A_\infty = \lim_{n\to\infty} A_n, \quad B_\infty = \lim_{n\to\infty} B_n.$$

Let C be the set of all continuous functions and define

$$\Omega = \big\{\{(x_n, y_n)\} \in C | x_n, y_n > 0, n \geq 0\big\}.$$

By the equation (1) [6] gave the variation of parameters formula

$$x_{n+1} = (1 + h_n)x_n + \sum_{j=0}^{n} a_{n,j} f(y_j) = \tag{5}$$

$$= (1 + h_n) \left[(1 + h_{n-1})x_{n-1} + \sum_{j=0}^{n-1} a_{n,j} f(y_j) \right] + \sum_{j=0}^{n} a_{n,j} f(y_j) = \ldots$$

$$\ldots = \prod_{i=0}^{n} (1 + h_i)\, x_0 + \sum_{k=1}^{n+1} \prod_{i=k}^{n} (1 + h_i) \sum_{j=0}^{k-1} a_{k-1,j} f(y_j) \tag{6}$$

and

$$y_{n+1} = \prod_{i=0}^{n} (1 + p_i)\, y_0 + \sum_{k=1}^{n+1} \prod_{i=k}^{n} (1 + p_i) \sum_{j=0}^{k-1} b_{k-1,j} g(x_j), \tag{7}$$

where the notation $\prod_{k=n+1}^{n} = 1$ is used. It is clear from (5) and (7) that $\{x_n\}$ and $\{y_n\}$ are positive provided that $x_0,\ y_0 \geq 0$. Moreover, if $h_n,\ p_n > 0$, then from (1) we have $\triangle x_n$, $\triangle y_n > 0$. Now, for some positive constants α, β, we define the set

$$K(\alpha, \beta) = \big\{\{(x_n, y_n)\} \in \Omega | \lim_{n\to\infty} x_n = \alpha, \lim_{n\to\infty} y_n = \beta\big\},$$

where α and β maybe considered to be infinite.

2. Classification of positive solutions and existence. In this section, we should classify positive solutions of (1) according to their limiting behavior and then provide necessary and sufficient conditions for their existence in the cases (2), (3), and (4).

Theorem 1. *Any solution* $\{(x_n, y_n)\} \in \Omega$ *of (1) belongs to one of the following subsets:*

$$K(\alpha,\beta), \quad K(\alpha,\infty), \quad K(\infty,\beta), \quad K(\infty,\infty).$$

Proof. Since $\{(x_n, y_n)\} \in \Omega$, we have $\triangle x_n$, $\triangle y_n > 0$ for $n \geq 0$. Thus $\{x_n\}$ and $\{y_n\}$ are increasing. Hence, $\lim\limits_{n\to\infty} x_n = \alpha > 0$ or $\lim\limits_{n\to\infty} x_n = \infty$, and $\lim\limits_{n\to\infty} y_n = \beta > 0$ or $\lim\limits_{n\to\infty} y_n = \infty$. The proof is complete.

In the following we state several theorems; each of the theorems is related to one of the above mentioned cases.

Theorem 2. *Suppose that* $A_\infty = \infty$, $B_\infty = \infty$ *and* L_1, L_2 *are the lower bounds of the functions* f *and* g *on* R_+, *respectively. If* $L_1 > 0$, $L_2 > 0$, *then any solution* $\{(x_n, y_n)\} \in \Omega$ *of (1) belongs to the set* $K(\infty,\infty)$.

Proof. Let $\{(x_n, y_n)\} \in \Omega$ be a solution of system (1). Then $\triangle x_n$, $\triangle y_n > 0$ for $n \geq 0$. Thus $\{x_n\}$ and $\{y_n\}$ are increasing. As a consequence of this and (1), we arrive at

$$x_n = x_0 + \sum_{i=0}^{n-1} h_i x_i + \sum_{i=0}^{n-1}\left(\sum_{j=0}^{i} a_{i,j} f(y_j)\right) \geq$$

$$\geq L_1 \sum_{i=0}^{n-1}\left(\sum_{j=0}^{i} a_{i,j}\right) = L_1\, A_n \to \infty, \quad n \to \infty,$$

$$y_n = y_0 + \sum_{i=0}^{n-1} p_i y_i + \sum_{i=0}^{n-1}\left(\sum_{j=0}^{i} b_{i,j} g(x_j)\right) \geq$$

$$\geq L_2 \sum_{i=0}^{n-1}\left(\sum_{j=0}^{i} b_{i,j}\right) = L_1\, B_n \to \infty, \quad n \to \infty.$$

This shows that $x_n \to \infty$ and $y_n \to \infty$, as $n \to \infty$. The proof is complete.

If L_1 is the lower bound of the function f on R_+, for $c > 0$, we denote $M_1 = \sup\limits_{t\in[c/2,c]} f(t)$, then we have the following estimate formula for $x_0 \in R$ and c :

$$\widetilde{m_j} = \prod_{i=0}^{j-1}(1+h_i)x_0 + \sum_{k=1}^{j}\prod_{l=k}^{j-1}(1+h_i)\sum_{m=0}^{k-1} a_{k-1,m}L_1 \le$$

$$\le \prod_{i=0}^{j-1}(1+h_i)x_0 + \sum_{k=1}^{j}\prod_{l=k}^{j-1}(1+h_i)\sum_{m=0}^{k-1} a_{k-1,m}f(y_m) \le$$

$$\le \prod_{i=0}^{j-1}(1+h_i)x_0 + \sum_{k=1}^{j}\prod_{l=k}^{j-1}(1+h_i)\sum_{m=0}^{k-1} a_{k-1,m}M_1 = \widetilde{M_j}.$$

Then we can choose $\underline{c_j}, \overline{c_j} \in [\widetilde{m_j}, \widetilde{M_j}]$ such that

$$g(\underline{c_j}) = \min_{t\in[\widetilde{m_j},\widetilde{M_j}]} g(t), \quad g(\overline{c_j}) = \max_{t\in[\widetilde{m_j},\widetilde{M_j}]} g(t).$$

Theorem 3. *Suppose that $A_\infty = \infty$, $B_\infty < \infty$ hold. Then a necessary condition for (1) to have a solution $\{(x_n, y_n)\} \in \Omega$ which belongs to $K(\infty, \beta)$ is that*

$$\sum_{i=0}^{\infty}\left(\sum_{j=0}^{i} b_{i,j}g(\underline{c_j})\right) < \infty, \tag{8}$$

where $\underline{c_j}$ is defined corresponding to $c = \beta$ and the first term x_0 of $\{x_n\}$.

Proof. Let $\{(x_n, y_n)\} \in \Omega$, be a solution of system (1) which belongs to $K(\infty, \beta)$. Then $\triangle x_n, \triangle y_n > 0$ for $n \ge 0$. Thus $\{x_n\}$ and $\{y_n\}$ are increasing and $y_0 \le y_n \le \beta$ for $n \ge 0$. From (5) we have

$$x_n = \prod_{i=0}^{n-1}(1+h_i)x_0 + \sum_{k=1}^{n}\prod_{i=k}^{n-1}(1+h_i)\sum_{j=0}^{k-1} a_{k-1,j}f(y_j).$$

Since $\lim\limits_{n\to\infty} y_n = \beta$, without loss of generality, we can assume that $y_n \in [\beta/2, \beta]$ for any $n \ge 0$. Then we have

$$\beta \ge y_n = y_0 + \sum_{i=0}^{n-1} p_i y_i + \sum_{i=0}^{n-1}\left(\sum_{j=0}^{i} b_{i,j}g(x_j)\right) =$$

$$= y_0 + \sum_{i=0}^{n-1} p_i y_i + \sum_{i=0}^{n-1}\left(\sum_{j=0}^{i} b_{i,j}g\left(\prod_{l=0}^{j-1}(1+h_l)x_0 + \sum_{k=1}^{j}\prod_{l=k}^{j-1}(1+h_l)\sum_{l=0}^{k-1} a_{k-1,l}f(y_l)\right)\right) \ge$$

$$\ge \sum_{i=0}^{n-1}\left(\sum_{j=0}^{i} b_{i,j}g(\underline{c_j})\right).$$

By taking the limit at infinity in the above inequality we obtain (8). The proof is completed.

Theorem 4. *Suppose that (2) holds, and $A_\infty = \infty$ and $B_\infty < \infty$. Then a sufficient condition for (1) to have an eventually positive solution $\{(x_n, y_n)\} \in \Omega$ which belongs to $K(\infty, \beta)$ is that: there exists $c > 0$ such that*

$$\sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} b_{i,j} g(\overline{c_j}) \right) < \infty, \tag{9}$$

where the $\overline{c_j}$ is defined for c and $x_0 = 0$.

Proof. If (9) holds, we can choose an integer N large enough so that

$$\sum_{i=N}^{\infty} \left(\sum_{j=0}^{i} b_{i,j} g(\overline{c_j}) \right) \leq \frac{c}{4}, \tag{10}$$

and

$$\sum_{i=N}^{\infty} p_i \leq \frac{1}{4}. \tag{11}$$

Let X be the set of all bounded real-valued sequences $\{y_n\}$ equipped with the norm

$$\|y\| = \sup_{n \geq 0} |y_n|.$$

Then X is Banach space. For $c > 0$, define the subset Ω_c of X by

$$\Omega_c = \left\{ \{y_n\} \in X \,\middle|\, \frac{c}{2} \leq y_n \leq c, n \geq N, \{y_n\} \text{ is increasing} \right\}.$$

Then Ω_c is a bounded, convex and closed subset of X. Now define the operator $E : \Omega_c \to X$ by

$$(Ey)_n = \frac{c}{2} + \sum_{i=N}^{n-1} p_i y_i + \sum_{i=N}^{n-1} \left(\sum_{j=0}^{i} b_{i,j} g \left(\sum_{k=1}^{i} \prod_{l=k}^{j-1} (1 + h_l) \sum_{m=0}^{k-1} a_{k-1,m} f(y_m) \right) \right), \tag{12}$$

for $y \in \Omega_c$ and we take here $\sum\limits_{i=N}^{n-1} = 0$ for $n \leq N$. First, we note that E maps Ω_c into itself. Indeed, if $y \in \Omega_c$, then

$$\frac{c}{2} \leq (Ey)_n = \frac{c}{2} + \sum_{i=N}^{n-1} p_i y_i + \sum_{i=N}^{n-1} \left(\sum_{j=0}^{i} b_{i,j} g \left(\sum_{k=1}^{i} \prod_{l=1}^{j-1} (1 + h_l) \sum_{m=0}^{k-1} a_{k-1,m} f(y_m) \right) \right) \leq$$

$$\leq \frac{c}{2} + c \sum_{i=N}^{n-1} p_i + \sum_{i=N}^{n-1} \left(\sum_{j=0}^{i} b_{i,j} g \left(\sum_{k=1}^{i} \prod_{l=k}^{j-1} (1 + h_l) \sum_{m=0}^{k-1} a_{k-1,m} M_1 \right) \right) \leq$$

$$\leq \frac{c}{2} + \frac{c}{4} + \frac{c}{4} = c.$$

Next, we show that E is continuous. Let $\{y^{(l)}\}$ be a sequence in Ω_c such that

$$\lim_{l\to\infty} \|y^{(l)} - y\| = 0.$$

Since Ω_c is closed, $y \in \Omega_c$. Then by (12), we have

$$\left|(Ey^{(l)})_n - (Ey)_n\right| = \left|\sum_{i=N}^{n-1} p_i y_i^{(l)} + \sum_{i=N}^{n-1}\left(\sum_{j=0}^{i} b_{i,j} g\left(\sum_{k=1}^{i}\prod_{l=k}^{j-1}(1+h_l)\sum_{m=0}^{k-1} a_{k-1,m} f(y_m^{(l)})\right)\right) - \right.$$

$$\left. - \sum_{i=N}^{n-1} p_i y_i - \sum_{i=N}^{n-1}\left(\sum_{j=0}^{i} b_{i,j} g\left(\sum_{k=1}^{i}\prod_{l=k}^{j-1}(1+h_l)\sum_{m=0}^{k-1} a_{k-1,m} f(y_m)\right)\right)\right| \le$$

$$\le \sum_{i=N}^{\infty} p_i \left|y_i^{(l)} - y_i\right| + \sum_{i=N}^{\infty}\left(\sum_{j=0}^{i} b_{i,j}\left| g\left(\sum_{k=1}^{i}\prod_{l=k}^{j-1}(1+h_l)\sum_{m=0}^{k-1} a_{k-1,m} f(y_m^{(l)})\right) - \right.\right.$$

$$\left.\left. - g\left(\sum_{k=1}^{i}\prod_{l=k}^{j-1}(1+h_l)\sum_{m=0}^{k-1} a_{k-1,m} f(y_m)\right)\right|\right).$$

By the continuity of f and g and the Lebesgue dominated convergence theorem, it follows that

$$\lim_{l\to\infty}\sup_{n\ge 0}\left|(Ey^{(l)})_n - (Ey)_n\right| = 0,$$

or

$$\lim_{l\to\infty}\left\|Ey^{(l)} - Ey\right\| = 0.$$

This shows that E is continuous.

Finally, we show that $E\Omega_c$ is precompact. Let $\{y^{(l)}\}$ be a sequence in Ω_c, then for each n, $\{y_n^{(l)}\}$ is a bounded number sequence. This shows that $\{y_n^{(l)}\}$ has a convergent subsequence. By the diagonal process we can know that $\{y^{(l)}\}$ has a convergent subsequence in Ω_c. Since E is continuous, we know that $\{Ey^{(l)}\}$ has a convergent subsequence in Ω_c, this means that $E\Omega_c$ is precompact.

Now, by Schauder's fixed point theorem, we conclude that there exists $y \in \Omega_c$ such that $y = Ey$. Set

$$x_n = \sum_{k=1}^{n}\prod_{i=k}^{n-1}(1+h_i)\sum_{j=0}^{k-1} a_{k-1,j} f(y_j).$$

Then

$$\triangle x_n = h_n x_n + \sum_{i=0}^{n} a_{n,i} f(y_i),$$

$$x_n = \sum_{k=1}^{n}\prod_{i=k}^{n-1}(1+h_i)\sum_{j=0}^{k-1} a_{k-1,j} f(y_j) \ge \sum_{k=1}^{n}\prod_{i=k}^{n-1}(1+h_i)\sum_{j=0}^{k-1} a_{k-1,j} L_{c,},$$

where $L_c = \min\limits_{t\in[c/2,c]} f(t)$. Since $f(t) > 0$ for $t > 0$, we have $L_c > 0$. In view of $A_\infty = \infty$, we have

$$\lim_{n\to\infty} x_n = \infty.$$

On the other hand,

$$y_n = \frac{c}{2} + \sum_{i=N}^{n-1} p_i y_i + \sum_{i=N}^{n-1}\left(\sum_{j=0}^{i} b_{i,j} g\left(\prod_{l=0}^{j-1}(1+h_l)x_0 + \sum_{k=1}^{i}\prod_{l=k}^{j-1}(1+h_l)\sum_{m=0}^{k-1} a_{k-1,m} f(y_m)\right)\right)$$

from which we obtain

$$\lim_{n\to\infty} y_n = c_0,$$

where c_0 is a constant. Hence, (x_n, y_n) is an eventually positive solution of (1) which belongs to $K(\infty, \beta)$. This completes the proof.

If L_2 is a lower bound of the function g on R_+, for $c > 0$, denote $M_2' = \sup\limits_{t\in[c/2,c]} g(t)$. Then we have the following estimate formula for y_0 and c :

$$\widetilde{m_j}' = \prod_{i=0}^{j-1}(1+p_i)y_0 + \sum_{k=1}^{j}\prod_{l=k}^{j-1}(1+p_i)\sum_{m=0}^{k-1} b_{k-1,m} L_2 \le$$

$$\le \prod_{i=0}^{j-1}(1+p_i)y_0 + \sum_{k=1}^{j}\prod_{l=k}^{j-1}(1+p_i)\sum_{m=0}^{k-1} b_{k-1,m} g(x_m) \le$$

$$\le \prod_{i=0}^{j-1}(1+p_i)y_0 + \sum_{k=1}^{j}\prod_{l=k}^{j-1}(1+p_i)\sum_{m=0}^{k-1} b_{k-1,m} M_2' = \widetilde{M_j}'.$$

Then we can choose $\underline{c_j}', \overline{c_j}' \in [\widetilde{m_j}', \widetilde{M_j}']$ such that

$$f(\underline{c_j'}) = \min_{t\in[\widetilde{m_j}', \widetilde{M_j}']} f(t),$$

$$f(\overline{c_j'}) = \max_{t\in[\widetilde{m_j}', \widetilde{M_j}']} f(t).$$

The proof of the next theorems follow along the lines of the proof of Theorem 3 and Theorem 4, hence we omit them.

Theorem 5. *Assume that $A_\infty < \infty$, $B_\infty = \infty$. Then a necessary condition for (1) to have a solution $\{(x_n, y_n)\} \in \Omega$ which belongs to $K(\alpha, \infty)$ is that*

$$\sum_{i=0}^{\infty}\left(\sum_{j=0}^{i} a_{i,j} f(\underline{c_j}')\right) < \infty,$$

where $\underline{c_j}'$ is defined for $c = \alpha$ and the first term y_0 of $\{y_n\}$.

Theorem 6. *Assume that $A_\infty < \infty$, $B_\infty = \infty$. Then a sufficient condition for (1) to have an eventually positive solution $\{(x_n, y_n)\} \in \Omega$ which belongs to $K(\alpha, \infty)$ is that: there exists $c > 0$ such that*

$$\sum_{i=0}^{\infty}\left(\sum_{j=0}^{i} a_{i,j} f(\overline{c_j}')\right) < \infty,$$

where $\overline{c_j}'$ is defined for c and $y_0 = 0$.

Theorem 7. *Any solution $\{(x_n, y_n)\} \in \Omega$ of (1) belongs to the set $K(\alpha, \beta)$ if and only if $A_\infty < \infty$ and $B_\infty < \infty$.*

Proof. Let $\{(x_n, y_n)\}$ be a solution in Ω with $\lim\limits_{n\to\infty} x_n = \alpha > 0$ and $\lim\limits_{n\to\infty} y_n = \beta > 0$. Then there exists an integer $N \ge 0$ and two positive constants c_1 and c_2 such that $c_1 \le x_n \le \alpha$, $c_2 \le y_n \le \beta$ for $n \ge N$. From system (1) we have for $n \ge N$ that

$$x_n = x_N + \sum_{i=N}^{n-1} h_i x_i + \sum_{i=N}^{n-1}\left(\sum_{j=0}^{i} a_{i,j} f(y_j)\right),$$

$$y_n = y_N + \sum_{i=N}^{n-1} p_i y_i + \sum_{i=N}^{n-1}\left(\sum_{j=0}^{i} b_{i,j} g(x_j)\right).$$

Let $L_\alpha = \min\limits_{t\in[c_1,\alpha]} f(t)$ and $L_\beta = \min\limits_{t\in[c_2,\beta]} g(t)$. Then we have $L_\alpha > 0$ and $L_\beta > 0$. Without loss of generality, we can assume that $c_1 \le x_n \le \alpha$, $c_2 \le y_n \le \beta$ for $n \ge 0$. Thus,

$$\alpha \ge x_n = x_N + \sum_{i=N}^{n-1} h_i x_i + \sum_{i=N}^{n-1}\left(\sum_{j=0}^{i} a_{i,j} f(y_j)\right) \ge$$

$$\ge x_N + c_1\sum_{i=N}^{n-1} h_i + \sum_{i=N}^{n-1}\left(\sum_{j=0}^{i} a_{i,j} L_\alpha\right),$$

$$\beta \ge y_n = y_N + \sum_{i=N}^{n-1} p_i y_i + \sum_{i=N}^{n-1}\left(\sum_{j=0}^{i} b_{i,j} g(x_j)\right) \ge$$

$$\ge y_N + c_2\sum_{i=N}^{n-1} p_i + \sum_{i=N}^{n-1}\left(\sum_{j=0}^{i} b_{i,j} L_\beta\right).$$

So $A_\infty < \infty$, $B_\infty < \infty$.

Conversely, suppose that $A_\infty < \infty$ and $B_\infty < \infty$. First notice that for $n \ge 0$, the first equation of (1) can be written as

$$x_n = x_0 + \sum_{i=0}^{n-1} h_i x_i + \sum_{i=0}^{n-1}\left(\sum_{j=0}^{i} a_{i,j} f(y_j)\right).$$

In a similar fashion, we obtain from the second equation of (1) that

$$y_n = y_0 + \sum_{i=0}^{n-1} p_i y_i + \sum_{i=0}^{n-1} \left(\sum_{j=0}^{i} b_{i,j} g(x_j) \right).$$

Next, for $c > 0$, $d > 0$, let

$$M_3 = \sup_{t \in [c/2, c]} f(t), \quad M_3' = \sup_{t \in [d/2, d]} g(t).$$

We can choose an integer N large enough so that

$$\sum_{i=N}^{\infty} \left[\sum_{j=0}^{i} a_{i,j} \right] \le \frac{d}{4M_3}, \quad \sum_{i=N}^{\infty} \left[\sum_{j=0}^{i} b_{i,j} \right] \le \frac{c}{4M_3'}$$

and

$$\sum_{i=N}^{\infty} p_i \le \frac{1}{4}, \quad \sum_{i=N}^{\infty} h_i \le \frac{1}{4}.$$

Let X be the Banach space of all bounded real-valued sequences $\{(x_n, y_n)\}$ endowed with the norm

$$\|(x, y)\| = \max \left\{ \sup_{n \ge 0} |x_n|, \sup_{n \ge 0} |y_n| \right\}.$$

Define the subset $\Omega_{c,d}$ of X by

$$\Omega_{c,d} = \left\{ \{(x_n, y_n)\} \in X \,\middle|\, \frac{d}{2} \le x_n \le d, \frac{c}{2} \le y_n \le c, n \ge N, \right.$$

$$\left. \{x_n\} \text{ and } \{y_n\} \text{ are both increasing} \right\}.$$

Then $\Omega_{c,d}$ is a bounded, convex and closed subset of X. Now define the operator $E : \Omega_{c,d} \to X$ by

$$E \begin{pmatrix} x \\ y \end{pmatrix}_n = \begin{bmatrix} \dfrac{d}{2} \\ \dfrac{c}{2} \end{bmatrix} + \begin{bmatrix} \sum\limits_{i=N}^{n-1} h_i x_i + \sum\limits_{i=N}^{n-1} \left(\sum\limits_{j=0}^{i} a_{i,j} f(y_j) \right) \\ \sum\limits_{i=N}^{n-1} p_i y_i + \sum\limits_{i=N}^{n-1} \left(\sum\limits_{j=0}^{i} b_{i,j} g(x_j) \right) \end{bmatrix},$$

where $(x, y) \in \Omega_{c,d}$ and we take here $\sum_{i=N}^{n-1} = 0$ for $n \le N$. First, we note that E maps $\Omega_{c,d}$

into itself. Indeed, if $(x, y) \in \Omega_{c,d}$, then

$$\frac{d}{2} \le (Ex)_n = \frac{d}{2} + \sum_{i=N}^{n-1} h_i x_i + \sum_{i=N}^{n-1} \left(\sum_{j=0}^{i} a_{i,j} f(y_j) \right) \le$$

$$\le \frac{d}{2} + d \sum_{i=N}^{n-1} h_i + M_3 \sum_{i=N}^{n-1} \left(\sum_{j=0}^{i} a_{i,j} \right) \le$$

$$\le \frac{d}{2} + \frac{d}{4} + \frac{d}{4} = d.$$

This is similar to showing that $c/2 \le (Ey)_n \le c$.

Next, we show that E is continuous. Let $\{(x^{(l)}, y^{(l)})\}$ be a sequence in $\Omega_{c,d}$ such that

$$\lim_{l\to\infty} \left\| (x^{(l)}, y^{(l)}) - (x, y) \right\| = 0.$$

Since $\Omega_{c,d}$ is closed, $(x, y) \in \Omega_{c,d}$. Then by the definition of E, we have

$$\left| E(x^{(l)}, y^{(l)})_n - (E(x,y))_n \right| \le$$

$$\le \left| \sum_{i=N}^{n-1} h_i x_i^{(l)} + \sum_{i=N}^{n-1} \left(\sum_{j=0}^{i} a_{i,j} f(y_j^{(l)}) \right) - \sum_{i=N}^{n-1} h_i x_i - \sum_{i=N}^{n-1} \left(\sum_{j=0}^{i} a_{i,j} f(y_j) \right) \right| +$$

$$+ \left| \sum_{i=N}^{n-1} p_i y_i^{(l)} + \sum_{i=N}^{n-1} \left(\sum_{j=0}^{i} b_{i,j} g(x_j^{(l)}) \right) - \sum_{i=N}^{n-1} p_i y_i - \sum_{i=N}^{n-1} \left(\sum_{j=0}^{i} b_{i,j} g(x_j) \right) \right| \le$$

$$\le \sum_{i=N}^{\infty} h_i \left| x_i^{(l)} - x_i \right| + \sum_{i=N}^{\infty} \left(\sum_{j=0}^{i} a_{i,j} \left| f(y_j^{(l)}) - f(y_j) \right| \right) + \sum_{i=N}^{\infty} p_i \left| y_i^{(l)} - y_i \right| +$$

$$+ \sum_{i=N}^{\infty} \left(\sum_{j=0}^{i} b_{i,j} \left| g(x_j^{(l)}) - g(x_j) \right| \right).$$

By the continuity of f and g and the Lebesgue dominated convergence theorem, it follows that

$$\lim_{l\to\infty} \sup_{n\ge 0} \left| (E(x^{(l)}, y^{(l)}))_n - (E(x,y))_n \right| = 0,$$

or

$$\lim_{l\to\infty} \left\| E(x^{(l)}, y^{(l)}) - E(x, y) \right\| = 0.$$

This shows that E is continuous.

Finally, we show that $E\Omega_{c,d}$ is precompact. Let $\{(x^{(l)}, y^{(l)})\}$ be a sequence in $\Omega_{c,d}$, then for each n, $\{y_n^{(l)}\}$ is a bounded number sequence. This shows that $\{(x_n^{(l)}, y_n^{(l)})\}$ has a convergent subsequence. By the diagonal process we can know that $\{(x^{(l)}, y^{(l)})\}$ has a convergent subsequence in $\Omega_{c,d}$. Since E is continuous, we know that $E\{(x^{(l)}, y^{(l)})\}$ has a convergent subsequence in $E\Omega_{c,d}$. This means that $E\Omega_{c,d}$ is precompact.

Now, by Schauder's fixed point theorem, we conclude that there exists $\{(x, y)\} \in \Omega_{c,d}$ such that $(x, y) = E(x, y)$. That is

$$x_n = \frac{d}{2} + \sum_{i=N}^{n-1} h_i x_i + \sum_{i=N}^{n-1} \left(\sum_{j=0}^{i} a_{i,j} f(y_j) \right),$$

$$y_n = \frac{c}{2} + \sum_{i=N}^{n-1} p_i y_i + \sum_{i=N}^{n-1} \left(\sum_{j=0}^{i} b_{i,j} g(x_j) \right)$$

from which we obtain

$$\lim_{n\to\infty} x_n = d_0, \quad \lim_{n\to\infty} y_n = c_0,$$

where c_0, d_0 are positive constants. Hence $\{(x_n, y_n)\}$ is an eventually positive solution of (1) which belongs to $K(\alpha, \beta)$. The proof is completed.

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Received 01.09.2005