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ON THE COMPLETENESS OF OSCILLATION SPACES*

ПРО ПОВНОТУ ПРОСТОРІВ КОЛИВАНЬ

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The oscillation spaces $\mathcal{O}_p^{s,s'}(\mathbb{R}^d)$, introduced by Jaffard, are a variation on the definition of Besov spaces for either $s \geq 0$ or $s \leq -d/p$. On the contrary the spaces $\mathcal{O}_p^{s,s'}(\mathbb{R}^d)$ for $-d/p < s < 0$ cannot be sharply imbedded between Besov spaces with almost the same exponents, and thus are new spaces of really different nature. Their norms take into account correlations between the positions of large wavelet coefficients through the scales. Several numerical studies have uncovered such correlations in several settings including turbulence, image processing, traffic, finance,... These spaces allow to capture oscillatory behaviors which are left undetected by Sobolev or Besov spaces. Unlike Sobolev spaces (resp. Besov spaces $B_p^{s,q}(\mathbb{R}^d)$) which are expressed by simple conditions on wavelet coefficients as ℓ^p norms (resp. mixed $\ell^p - \ell^q$ norms), oscillation spaces are written as ℓ^p averages of local $C^{s'}$ norms. In this paper, we prove the completeness of oscillation spaces in spite of such a mixture of two norms of different kinds.

Простори коливань $\mathcal{O}_p^{s,s'}(\mathbb{R}^d)$, введені Джаффаром, є варіаціями означення просторів Бєсова для $s \geq 0$ або $s \leq -d/p$. Але, якщо $-d/p < s < 0$, простори $\mathcal{O}_p^{s,s'}(\mathbb{R}^d)$ не можуть бути строго включені між просторами Бєсова з майже такими ж показниками, і тому є новими просторами, що мають дійсно іншу природу. Значення норми в цих просторах залежить від кореляції положення коефіцієнтів при великих вейвлетах у послідовності просторів. Декілька чисельних досліджень відкрили таку кореляцію в кількох випадках, що охоплюють турбулентність, обробку зображень, рух машин, фінанси і т. д. Ці простори дозволяють помітити коливання, які залишаються непомітними у просторах Соболєва та Бєсова. На відміну від просторів Соболєва (відповідно Бєсова, $B_p^{s,q}(\mathbb{R}^d)$), які визначаються простими умовами на коефіцієнти при вейвлетах у термінах норм ℓ^p (відповідно $\ell^p - \ell^q$), простори коливань визначаються ℓ^p-середніми локальних норм $C^{s'}$. У статті доведено повноту просторів коливань, незважаючи на таке поєднання норм різних типів.

1. Introduction. We will use a family of $2^d - 1$ smooth wavelets $\Psi^{(i)}$, such that the $\Psi^{(i)}$ and their partial derivatives have fast decay. The $2^{dj/2}\Psi^{(i)}(2^j x - k)$ $(i = 1, \ldots, 2^d - 1, j \in \mathbb{Z}, k \in \mathbb{Z}^d)$ form an orthonormal basis of $L^2(\mathbb{R}^d)$. We will use a L^∞ normalization for wavelets, so that we write

$$f(x) = \sum_{i,j,k} C_{j,k}^{(i)} \Psi^{(i)}(2^j x - k), \tag{1}$$

where

$$C_{j,k}^{(i)} = C_{j,k}^{(i)}(f) = 2^{dj} \int f(t)\Psi^{(i)}(2^j t - k)dt.$$

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We will use the following simpler notations; λ and λ' will denote respectively the cubes $\lambda_{j,k} = k2^{-j} + [0, 2^{-j}[^d$ and $\lambda_{j',k'} = k'2^{-j'} + [0, 2^{-j'}[^d$, C_λ will denote the coefficient $C^{(i)}_{j,k}$, and Ψ_λ will denote the wavelet $\Psi^{(i)}(2^j x - k)$ (note that we forget the index i of the wavelet which is of no consequence).

Recall that f belongs to the Besov space $B^{s,q}_p(\mathbb{R}^d)$ with $p > 0$ and $q > 0$ if

$$2^{sj} 2^{-dj/p} \left(\sum_k |C_\lambda|^p \right)^{1/p} := \varepsilon_j \quad \text{with} \quad \varepsilon_j \in l^q \tag{2}$$

(which follows directly from [1, p. 50, 197] and [2, p. 45]). Note that $B^{s,2}_2$ is the Sobolev space H^s and that $B^{s,\infty}_\infty$ is the Hölder space C^s. Recall that $f \in C^s(\mathbb{R}^d)$ for $s > 0$ if there exist a polynomial P of degree smaller than s and a constant C such that

$$\forall x \in \mathbb{R}^d \quad \forall x_0 \in \mathbb{R}^d : |f(x) - P(x - x_0)| \leq C|x - x_0|^s. \tag{3}$$

The spaces $\mathcal{O}^{s,s'}_p(\mathbb{R}^d)$ are function spaces that have been introduced by Jaffard in [3] in order to quantify the degree of correlations between positions of large wavelet coefficients through the scales. Several numerical studies have uncovered such correlations in several settings including turbulence [4], image processing, traffic [5], finance [6], ... Oscillation spaces allow to capture oscillatory behaviors which are left undetected by Sobolev or Besov spaces.

Definition 1. *Let $p > 0$, and $s, s' \in \mathbb{R}$; a function f belongs to the oscillation space $\mathcal{O}^{s,s'}_p(\mathbb{R}^d)$ if its wavelet coefficients satisfy*

$$\sup_j \left[2^{sj} \left(\sum_k \sup_{\lambda' \subset \lambda} |C_{\lambda'} 2^{s'j'}|^p \right)^{1/p} \right] < \infty \tag{4}$$

(*modified if* $p = +\infty$).

The left-hand side defines the $\mathcal{O}^{s,s'}_p(\mathbb{R}^d)$ quasinorm. Note that this definition is independent on the wavelet basis which is chosen (see [3]).

In [7], Jaffard proved that, for either $s \geq 0$ or $s \leq -d/p$, the $\mathcal{O}^{s,s'}_p(\mathbb{R}^d)$ are a variation on the definition of Besov spaces ($\mathcal{O}^{s,s'}_p = B^{s+s'+d/p,\infty}_p$ if $s > 0$ and $C^{s'}$ if $s \leq -d/p$, and $B^{s'+d/p,p}_p \hookrightarrow \mathcal{O}^{0,s'}_p \hookrightarrow B^{s'+d/p,\infty}_p$). On the contrary the spaces $\mathcal{O}^{s,s'}_p$ for $-d/p < s < 0$ cannot be sharply imbedded between Besov spaces with almost the same exponents (in fact $B^{s'+d/p,p}_\infty \hookrightarrow \mathcal{O}^{s,s'}_p \hookrightarrow C^{s'}$ and $C^{s+s'+d/p} \hookrightarrow \mathcal{O}^{s,s'}_p$ and the imbeddings are optimal), and thus are new spaces of really different nature.

In [8], Jaffard proved several related results concerning the genericity (in the sense of Baire's categories) of multifractal functions. One result asserts that, if $s > \dfrac{d}{p}$, generically, functions of the Besov space $B^{s,q}_p(\mathbb{R}^d)$ are multifractal. The completeness of Besov spaces was a key topological property in his proof. (Note that the validity of the multifractal formalism never holds in complete generality, but it has also been checked under an additional self-similarity assumption in [9–11].)

In the next section, we will first briefly recall the proof of the completeness of Besov spaces. We will then see that oscillation spaces can be written as ℓ^p averages of *local* $C^{s'}$ norms. We will prove the completeness of these spaces in spite of such a mixture of two norms of different kinds. So, the main statement of the this paper is the following theorem.

Theorem 1. *The oscillating spaces* $\mathcal{O}_p^{s,s'}$ *for* $p > 0$, *and* $s, s' \in \mathbb{R}$; *are quasi-Banach spaces* (*Banach spaces if* $p \geq 1$).

2. The completeness. ***2.1. Completeness of Besov spaces.*** Recall that if A is a (real or complex) linear vector space, $\|\cdot\|$ is said to be a quasinorm if $\|\cdot\|$ satisfies the usual conditions of a norm with the exception of the triangle inequality, which will be replaced by

$$\exists L \geq 1 \quad \forall (a_1, a_2) \in A^2 : \|a_1 + a_2\| \leq L(\|a_1\| + \|a_2\|). \tag{5}$$

(If $L = 1$, then A is a normed space). A quasinormed space is said to be a quasi-Banach space if it is complete (i.e., any Cauchy sequence in A with respect to $\|\cdot\|$ converges).

It is well known that $L^p = L^p(\mathbb{R}^d)$ (the set of all Borel measurable complex valued functions on $\mathbb{R}^d$ such that $\int |f(x)|^p \, dx < \infty$) is a quasi-Banach space (a Banach space if $p \geq 1$).

The vector space ℓ_p of all sequences $b = (b_k)_{k \in \mathbb{N}}$ of complex numbers such that

$$\|b\|_{\ell_p} := \|(b_k)_k\|_{\ell_p} = \left(\sum_{k=0}^{\infty} |b_k|^p \right)^{1/p} < \infty$$

(modified if $p = \infty$) is a quasi-Banach space (a Banach space if $p \geq 1$). Using the normalization we choose, the wavelet characterization of Besov spaces $B_p^{s,q} = B_p^{s,q}(\mathbb{R}^d)$ for ($s \in \mathbb{R}$, $p > 0$, $q > 0$) can be written

$$\|f\|_{B_p^{s,q}} = \left\| \left(2^{(s - \frac{d}{p})j} \, \|(C_{j,k})_k\|_{\ell^p} \right)_j \right\|_{\ell^q} < \infty. \tag{6}$$

The completeness of Besov spaces $B_p^{s,q}$ can be deduced from the continuous isometry that relates it to quasi-Banach spaces $\ell^q(\ell^p)$ (a mixed $\ell^p - \ell^q$ norm) (see [2, p. 14, 48]).

2.2. Completeness of oscillation spaces. Let us begin by some remarks.

Remark 1. If $T(f) := \sum\limits_{\lambda} \omega(\lambda)\Psi_\lambda$, with $\omega(\lambda) = \sup\limits_{\lambda' \subset \lambda} |C_{\lambda'}(f)|$, then we can easily check that

$$f \in \mathcal{O}_p^{s,0} \Leftrightarrow T(f) \in B_p^{s+d/p,\infty}.$$

Nevertheless, the mapping T is not linear.

Remark 2. In [3], Jaffard proved that $\|(-\Delta)^{\alpha/2} f\|_{\mathcal{O}_p^{s,0}}$ and $\|f\|_{\mathcal{O}_p^{s,\alpha}}$ are equivalent norms. It follows that $\mathcal{O}_p^{s,s'}$ is complete if and only if $\mathcal{O}_p^{s,0}$ is complete. Therefore, we can restrict the proof of the completeness to the space $\mathcal{O}_p^{s,0}$.

Remark 3. The left-hand side of (4) is an ℓ^∞ norm (on the scale j) of the sequence 2^{sj} multiplied by a ℓ^p norm of the quantities

$$\sup_{\lambda' \subset \lambda} |C_{\lambda'}| \, 2^{s'j'} \tag{7}$$

and these quantities clearly look like a local Hölder $C^{s'}$ norm. Indeed a function belongs to $C^{s'}$ if its wavelet coefficients satisfy $\sup_{\lambda'} |C_{\lambda'}| 2^{s'j'} < \infty$, but in (7) the supremum is restricted to the subcubes of λ. Hence $\mathcal{O}_p^{s,s'}$ can be written as a ℓ^p average of local $C^{s'}$ norms. Such a mixture of two norms of different kinds does not allow us to find an isometry similar to the above one for Besov spaces, in order to check the completeness of $\mathcal{O}_p^{s,s'}$. We will instead use the following result which can be deduced from [12, p. 58].

Proposition 1. *Let A be a quasinormed vector space. Denote L a constant which appears in the generalized triangle inequality (5). The space A is complete if and only if any sequence $(a_n)_n$ of elements of A satisfies the following property:*
"if there exists a constant $D > L$ such that for any n $\|a_n\| \le D^{-n}$ then the series $\sum_n a_n$ converges in A."

Proof. Assume that A is a complete vector space. Let L be a constant which appears in the generalized triangle inequality (5). Let $(a_n)_n$ be a sequence of A. Assume that there exists a constant $D > L$ such that $\|a_n\| \le D^{-n}$ $\forall n$. Let $S_N = \sum_{n=1}^{N} a_n$. For $M > N$

$$\|S_M - S_N\| = \Big\| \sum_{n=N+1}^{M} a_n \Big\|.$$

If A is normed then

$$\|S_M - S_N\| \le \sum_{n=N+1}^{M} \|a_n\| \le \sum_{n=N+1}^{M} D^{-n} \le \frac{D^{-(N+1)}}{D-1}.$$

Since $D > L \ge 1$, (S_N) is a Cauchy sequence of A. So, (S_N) converges. If A is quasinormed then

$$\begin{aligned}
\|S_M - S_N\| = \Big\| \sum_{n=N+1}^{M} a_n \Big\| &\le \\
&\le L\|a_{N+1}\| + L\Big\| \sum_{n=N+2}^{M} a_n \Big\| \le \\
&\le L\|a_{N+1}\| + L^2\|a_{N+2}\| + \ldots + L^{M-N}\|a_M\| \le \\
&\le LD^{-(N+1)} + L^2 D^{-(N+2)} + \ldots + L^{M-N} D^{-M} \le \\
&\le \frac{D^{-N}}{1 - \frac{L}{D}} \quad \text{(because } D > L\text{)}.
\end{aligned}$$

So, (S_N) converges.

Now, for the converse part of Proposition 1, assume that A is a quasinormed vector space, and that any sequence $(a_n)_n$ of elements of A satisfies the following property: "if there exists a constant $D > L$ such that $\|a_n\| \leq D^{-n}$ $\forall n$, then the series $\sum\limits_n a_n$ converges in A". We will prove that A is complete; let (b_n) be a Cauchy sequence of A. It suffices to show that there exists a subsequence $(b_{n_k})_k$ which converges in A; we extract a subsequence $(b_{n_k})_k$ such that

$$\forall k \geq 1 : \|b_{n_{k+1}} - b_{n_k}\| \leq D^{-k}. \tag{8}$$

Denote $a_k = b_{n_{k+1}} - b_{n_k}$. Relation (8) implies that $\forall k \geq 1 : \|a_k\| \leq D^{-k}$. Hence, by the assumption, the series $\sum\limits_k a_k$ converges in A. Denote by S its limit. Denote $S_K = \sum\limits_{k=1}^{K} a_k$. We have $S_K = b_{n_{K+1}} - b_{n_1}$. So, $(b_{n_k})_k$ converges to $S + b_{n_1}$.

The proof of Proposition 1 is now finished.

We will now pursue the proof of completeness of the oscillation spaces using Proposition 1. As mentioned in Remark 2, we only have to do this for $\mathcal{O}_p^{s,0}(\mathbb{R}^d)$. Recall that $f \in \mathcal{O}_p^{s,0}(\mathbb{R}^d)$ if there exists a constant $C > 0$ such that

$$\left(\sum_k (\omega(\lambda))^p\right)^{1/p} \leq C 2^{-sj} \quad \forall j, \tag{9}$$

where $\omega(\lambda) = \sup\limits_{\lambda' \subset \lambda} |C_{\lambda'}(f)|$. We will use Proposition 1; we take $A = \mathcal{O}_p^{s,0}(\mathbb{R}^d)$, $D = 2$ if $p \geq 1$, and $D = L + 1$ if $0 < p < 1$ (with L a constant which appears in the generalized triangle inequality (5)). Let $(f_n)_n$ be a sequence in $\mathcal{O}_p^{s,0}$ such that

$$\|f_n\|_{\mathcal{O}_p^{s,0}} \leq D^{-n} \quad \forall n. \tag{10}$$

We will prove that the series $\sum\limits_n f_n$ converges in $\mathcal{O}_p^{s,0}$. We will divide the proof into three steps.

First step. In this step we will prove that for any cube λ the series $\sum\limits_n C_\lambda(f_n)$ converges; relation (10) is equivalent to

$$\left(\sum_k (\omega_n(\lambda))^p\right)^{1/p} \leq D^{-n} 2^{-js} \quad \forall j \tag{11}$$

with

$$\omega_n(\lambda) = \sup_{\lambda' \subset \lambda} |C_{\lambda'}(f_n)|. \tag{12}$$

It follows that

$$|C_\lambda(f_n)| \leq D^{-n} 2^{-js} \quad \forall j \ \forall k. \tag{13}$$

Since $D > 1$ we deduce that the series $\sum\limits_n C_\lambda(f_n)$ converges.

Second step. For any cube λ we write

$$\sum_{n=0}^{\infty} C_\lambda(f_n) := \tilde{C}_\lambda. \tag{14}$$

We denote $\tilde{f} = \sum_\lambda \tilde{C}_\lambda \Psi_\lambda$. We will prove that $\tilde{f} \in \mathcal{O}_p^{s,0}$ (and in the third step, we will prove that the series $\sum_n f_n$ converges to $\tilde{f}$ in $\mathcal{O}_p^{s,0}$). For that we will first show that $\left(\tilde{f}_N := \sum_{n=0}^{N} f_n\right)_N$ is a bounded sequence in $\mathcal{O}_p^{s,0}$.

If $p \geq 1$ then

$$\|\tilde{f}_N\| \leq \sum_{n=0}^{N} \|f_n\| \leq \sum_{n=0}^{N} 2^{-n} \leq 2 \quad \forall N, \tag{15}$$

if $0 < p < 1$ then

$$\|\tilde{f}_N\| \leq L\|f_0\| + L^2\|f_1\| + \ldots + L^{N+1}\|f_N\| \leq$$
$$\leq L(L+1)^{-0} + L^2(L+1)^{-1} + \ldots + L^{N+1}(L+1)^{-N} \quad \forall N.$$

Hence

$$\|\tilde{f}_N\| \leq L(L+1) \quad \forall N. \tag{16}$$

Consequently, $(\tilde{f}_N)_N$ is a bounded sequence in $\mathcal{O}_p^{s,0}$. This property, together with (14), will allow us to show that $\tilde{f} = \sum_\lambda \tilde{C}_\lambda \Psi_\lambda \in \mathcal{O}_p^{s,0}$. Let $(\tilde{C}_N(\lambda))_\lambda$ denote the wavelet coefficients of $\tilde{f}_N$, i.e.,

$$\tilde{C}_N(\lambda) = \sum_{n=0}^{N} C_\lambda(f_n).$$

Relation (14) can be written

$$\lim_{N\to+\infty} \tilde{C}_N(\lambda) = \tilde{C}_\lambda \quad \forall \lambda. \tag{17}$$

Let $\tilde{w}_N(\lambda)$ denote $\sup_{\lambda'\subset\lambda} |\tilde{C}_N(\lambda')|$, and $\tilde{w}(\lambda)$ denote $\sup_{\lambda'\subset\lambda} |\tilde{C}_{\lambda'}|$. We have

$$\tilde{w}(\lambda) = \sup_{\lambda'\subset\lambda} \Big| \sum_{n=0}^{\infty} C_{\lambda'}(f_n) \Big| \leq \sum_{n=0}^{\infty} \sup_{\lambda'\subset\lambda} |C_{\lambda'}(f_n)| = \sum_{n=0}^{\infty} w_n(\lambda).$$

Relation (11) implies that $w_n(\lambda) \leq D^{-n} 2^{-js}$. Whence $\tilde{w}(\lambda)$ is finite.

For $\varepsilon > 0$ there exists $\lambda'_\varepsilon \subset \lambda$ such that

$$\tilde{w}(\lambda) \le |\tilde{C}_{\lambda'_\varepsilon}| + \varepsilon.$$

It follows from (17) that for N large enough,

$$\tilde{w}(\lambda) \le |\tilde{C}_N(\lambda'_\varepsilon)| + 2\varepsilon \le \tilde{w}_N(\lambda) + 2\varepsilon.$$

Therefore

$$\tilde{w}(\lambda) \le \liminf_{N\to+\infty} \tilde{w}_N(\lambda) \quad \forall\lambda. \tag{18}$$

The previous relation implies that

$$\sum_k (\tilde{w}(\lambda))^p \le \liminf_{N\to+\infty} \sum_k (\tilde{w}_N(\lambda))^p \quad \forall j.$$

From (9) and the fact that $(\tilde{f}_N)_N$ is bounded in $\mathcal{O}_p^{s,0}$, we deduce that

$$\left(\sum_k (\tilde{w}(\lambda)^p\right)^{1/p} \le C2^{-sj} \quad \forall j.$$

Whence $\tilde{f} \in \mathcal{O}_p^{s,0}$.

Third step. We will prove that the series $\sum\limits_n f_n$ converges to $\tilde{f}$ in $\mathcal{O}_p^{s,0}$:

if $p \ge 1$ then, for $M > N$,

$$\|\tilde{f}_M - \tilde{f}_N\| = \left\| \sum_{n=N+1}^{M} f_n \right\| \le \sum_{n=N+1}^{M} \|f_n\| \le \sum_{n=N+1}^{M} 2^{-n}$$

(where $\|\cdot\| = \|\cdot\|_{\mathcal{O}_p^{s,0}}$), hence

$$\left\|\tilde{f}_M - \tilde{f}_N\right\| \le 2^{-N}; \tag{19}$$

if $0 < p < 1$ then, for $M > N$,

$$\begin{aligned} \|\tilde{f}_M - \tilde{f}_N\| = \left\| \sum_{n=N+1}^{M} f_n \right\| \le \\ \le L\|f_{N+1}\| + L\left\| \sum_{n=N+2}^{M} f_n \right\| \le \\ \le L\|f_{N+1}\| + L^2\|f_{N+2}\| + \ldots + L^{M-N}\|f_M\| \le \\ \le L(L+1)^{-(N+1)} + L^2(L+1)^{-(N+2)} + \ldots + L^{M-N}(L+1)^{-M}. \end{aligned}$$

Hence

$$\|\tilde f_M - \tilde f_N\| \le L(L+1)^{-N}. \tag{20}$$

Using similar arguments to those of the previous step, properties (19), (20) and (17) will imply that $(\tilde f_N)_N$ converges to $\tilde f$ in $\mathcal{O}_p^{s,0}$; let $N \ge 1$. For $\varepsilon > 0$ there exists $\lambda'_{\varepsilon,N} \subset \lambda$ such that

$$\sup_{\lambda'\subset\lambda} |\tilde C_{\lambda'} - \tilde C_N(\lambda')| \le \varepsilon + |\tilde C_{\lambda'_{\varepsilon,N}} - \tilde C_N(\lambda'_{\varepsilon,N})|.$$

It follows from (17) that for M large enough

$$\sup_{\lambda'\subset\lambda} |\tilde C_{\lambda'} - \tilde C_N(\lambda')| \le 2\varepsilon + |\tilde C_M(\lambda'_{\varepsilon,N}) - \tilde C_N(\lambda'_{\varepsilon,N})| \le 2\varepsilon + \sup_{\lambda'\subset\lambda} |\tilde C_M(\lambda') - \tilde C_N(\lambda')|.$$

Therefore

$$\sup_{\lambda'\subset\lambda} |\tilde C_{\lambda'} - \tilde C_N(\lambda')| \le \liminf_{M\to+\infty} \left[\sup_{\lambda'\subset\lambda} |\tilde C_M(\lambda') - \tilde C_N(\lambda')|\right] \quad \forall\lambda. \tag{21}$$

The previous relation implies that

$$\sum_k (\sup_{\lambda'\subset\lambda} |\tilde C_{\lambda'} - \tilde C_N(\lambda')|)^p \le \liminf_{M\to+\infty} \left[\sum_k (\sup_{\lambda'\subset\lambda} |\tilde C_M(\lambda') - \tilde C_N(\lambda')|)^p\right] \quad \forall j.$$

Hence $\sup_j \left[2^{sj}\left(\sum_k (\sup_{\lambda'\subset\lambda} |\tilde C_{\lambda'} - \tilde C_N(\lambda')|)^p\right)^{1/p}\right]$ is smaller than

$$\liminf_{M\to+\infty} \left[\sup_j \left[2^{sj}\left(\sum_k (\sup_{\lambda'\subset\lambda} |\tilde C_M(\lambda') - \tilde C_N(\lambda')|)^p\right)^{1/p}\right]\right].$$

From properties (19) and (20), we deduce that $\|\tilde f_N - \tilde f\| \le 2^{-N}$ if $p \ge 1$, and $L(L+1)^{-N}$ if $0 < p < 1$. Whence $(\tilde f_N)_N$ converges to $\tilde f$ in $\mathcal{O}_p^{s,0}$.

The proof of Theorem 1 is now achieved.

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