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GLOBAL ATTRACTOR FOR IMPULSIVE REACTION-DIFFUSION EQUATION

ГЛОБАЛЬНИЙ АТРАКТОР ДЛЯ ІМПУЛЬСНО ЗБУРЕНОГО РІВНЯННЯ РЕАКЦІЇ-ДИФУЗІЇ

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In this paper we consider a reaction-diffusion equation with nonsmooth nonlinearity, whose solutions have impulse effects at fixed moments of time. We show how this object generates a nonautonomous multi-valued dynamical system and prove the existence of a compact semiinvariant global attractor in the phase space.

Розглядається автономне рівняння реакції-дифузії з негладкою правою частиною, розв'язки якого зазнають імпульсного збурення в фіксовані моменти часу. Доведено, що такий об'єкт породжує неавтономну багатозначну динамічну систему, для якої в фазовому просторі існує компактний напівінваріантний глобальний атрактор.

Introduction. In this paper we study the asymptotic behaviour of solutions of an impulsive reaction-diffusion equation from the point of view of the theory of global attractors. In the literature there is a great number of results concerning the abstract theory of global attractors and its applications [1]. But in the considered case, in spite the fact that the equation is autonomous, the whole problem is nonautonomous because the moments of impulse effects are fixed. The key idea of this paper is to investigate such a problem by the methods of the theory of global attractors for nonautonomous dynamical systems.

The abstract classical (single-valued) theory of global attractors for nonautonomous dynamical systems and its applications to almost-periodic and cascad systems are developed in [2]. However, if we want to relax the restrictive conditions imposed on the nonlinearity in the classical approach, the uniqueness of the Cauchy problem is lost. In this case we deal with a set of solutions, so we need a generalization of the classical theory to the multivalued situation. On the other hand, an important reason for developing the theory of multivalued dynamical systems is justified by the well known models coming from the mathematical physics (Navier – Stokes equations, Ginzburg – Landau equation and others), for which the problem of uniqueness of the Cauchy problem is still open. In the recent years several approaches of multivalued analogous constructions of the global attractor theory both in the autonomous and nonautonomous cases were developed in [3 – 10]. Some applications of this theory to differential inclusions [6, 7], evolution equations without uniqueness [8 – 10] have been studied.

In this paper we develop abstract results from [7–10] and using methods, similar to [7], apply them for proving the existence of global attractor of reaction-diffusion equation with a nonsmooth nonlinearity and impulse effects occuring at fixed moments of time.

Setting of the problem. We consider the problem

$$\frac{\partial u(t,x)}{\partial t} = a\Delta u(t,x) - f(u(t,x)) + h(x), \qquad (t,x) \in (\tau, T) \times \Omega, \tag{1}$$

$$u(t,x)|_{x \in \partial\Omega} = 0,$$
$$u(t,x)|_{t=\tau} = u_\tau(x), \tag{2}$$

where $a > 0$, $\Omega \subset \mathbb{R}^n$ is a bounded open subset with smooth boundary, $\tau \geq 0$, $h \in L^2(\Omega)$, $f \in \mathbb{C}\left([0, +\infty)\right)$ and satisfies the following conditions:

$$\exists C_1\, C_2 > 0\ \exists p \geq 2\ \exists\alpha > 0\ \forall u \in \mathbb{R}:$$

$$|f(u)| \leq C_1(1 + |u|^{p-1}), \tag{3}$$

$$f(u)u \geq \alpha\, |u|^p - C_2.$$

Let $H = L^2(\Omega)$. By $\|\cdot\|$ and $(\cdot, \cdot)$ we denote the norm and the scalar product in H. We say that a function $u = u(t,x) \in L^2(\tau, T; H^1_0(\Omega)) \cap L^p(\tau, T; L^p(\Omega)) \cap C([\tau, T]\,; H)$ is a solution of (1), (2) on (τ, T), if for each $v \in H^1_0(\Omega) \cap L^p(\Omega)$,

$$\frac{d}{dt}(u,v) + a(u,v)_{H^1_0} + (f(t,u),v) - (h,v) = 0$$

in the sense of scalar distributions on (τ, T) and $u(\tau, x) = u_\tau(x)$.

Under conditions (3) it is known [9, 10] that for each $T > \tau$, $u_\tau \in H$ there exists at least one solution $u = u(t,x)$ of problems (1), (2), constructed by the Galerkin approximation method. So we can talk about globally resolvability of (1), (2), that is, $\forall\tau \geq 0\ \forall u_\tau \in H\ \exists u \in$ $\in W_\tau = L^2_{\text{loc}}(\tau, +\infty; H^1_0(\Omega)) \cap L^p_{\text{loc}}(\tau, +\infty; L^p(\Omega)) \cap \mathbb{C}([\tau, +\infty); H)$, a solution of (1) on each (τ, T), $u(\tau) = u_\tau$.

Now, we can correctly set the following impulsive problem. Let at the fixed moments of time $\{\tau_i\}_{i=1}^\infty$ every solution of (1), (2) in the phase space H have impulse effects of the form

$$u(\tau_i + 0) - u(\tau_i) \in \psi_i(u(\tau_i)), \qquad i \geq 1, \tag{4}$$

where $\psi_i : H \to 2^H$ is some multivalued map, $\tau_1 > 0$, $\tau_{i+1} - \tau_i \geq \gamma > 0\ \ \forall i \geq 1$.

Let us denote $\tau_0 := 0$, $\psi_0(u) \equiv 0$. Then $\forall\tau \in [\tau_i, \tau_{i+1}]$ the problem (1)–(4) is globally resolved in the class $W_\tau(\sigma_0)$, that is, $\forall u_\tau \in H\ \ \exists u \in L^2_{\text{loc}}(\tau, +\infty; H^1_0(\Omega)) \cap L^p_{\text{loc}}(\tau, +\infty; L^p(\Omega))$ is a solution of (1) on (τ, τ_{i+1}), (τ_{i+1}, τ_{i+2}), ..., $u(\tau) = u_\tau$ and at the points $\{\tau_i, \tau_{i+1}, \ldots\}$ the function $u(\cdot)$ has impulse effect (4) and is left-continuous.

Our aim is to investigate the qualitative behaviour, for $t \to \infty$, of the solutions of problems (1)–(4) in the phase space H using methods of the theory of global attractors of infinite dimensional multivalued dynamical systems. Since the moments $\{\tau_i\}_{i=1}^\infty$ are fixed, the problem (1)–(4) is nonautonomous, so we should use the theory of nonautonomous dynamical systems.

Elements of abstract theory of global attractors of nonautonomous multivalued dynamical systems. For some complete metric space (X, ρ) we denote by $P(X)$ $(\beta(X))$ the set of all nonempty (nonempty bounded) subsets of X, $\forall A, B \subset X$ $\mathrm{dist}\,(A, B) = \sup\limits_{x \in A} \inf\limits_{y \in B} \rho(x, y)$, $O_\delta(A) = \{x \in X | \mathrm{dist}\,(x, A) < \delta\}$, $B_r = \{x \in X | \rho(x, 0) \le r\}$, $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{R}_{+d} = \{(t, \tau) \in \mathbb{R}_+^2 | t \ge \tau\}$. Let Σ be some complete metric space.

Definition 1. *The family of multivalued maps $\{U_\sigma : \mathbb{R}_{+d} \times X \to P(X)\}_{\sigma \in \Sigma}$ is called a family of Multivalued SemiProcesses (MSP), if on Σ there acts a continuous semigroup $\{T(h) : \Sigma \to \Sigma\}_{h \ge 0}$ and $\forall \sigma \in \Sigma$ $\forall x \in X$:*

1) $U_\sigma(\tau, \tau, x) = x$ $\quad \forall \tau \ge 0$;
2) $U_\sigma(t, \tau, x) \subset U_\sigma(t, s, U_\sigma(s, \tau, x))$ $\quad \forall t \ge s \ge \tau$;
3) $U_\sigma(t + h, \tau + h, x) \subset U_{T(h)\sigma}(t, \tau, x)$ $\quad \forall t \ge \tau$ $\quad \forall h \ge 0$.

The family of MSP is called strict if in conditions $2), 3)$ there is equality.

We denote $U_\Sigma(t, \tau, x) = \bigcup\limits_{\sigma \in \Sigma} U_\sigma(t, \tau, x)$.

Definition 2. *A set $\theta_\Sigma \subset X$ is called a global attractor of the family of MSP $\{U_\sigma\}_{\sigma \in \Sigma}$, if $\theta_\Sigma \ne X$ and*

1) θ_Σ is a uniformly attracting set, that is, $\forall B \in \beta(X)$ $\forall \tau \ge 0$

$$\mathrm{dist}\,(U_\Sigma(t, \tau, B), \theta_\Sigma) \to 0, \ \ t \to \infty;$$

2) θ_Σ is a minimal uniformly attracting set, that is, for an arbitrary uniformly attracting set Y, we have $\theta_\Sigma \subset \overline{Y}$.

It is known [8–10] that if a family of MSP $\{U_\sigma\}_{\sigma \in \Sigma}$ satisfies the following conditions:

$$\forall B \in \beta(X) \quad \exists T = T(B) : \quad \bigcup\limits_{t \ge T} U_\Sigma(t, 0, B) \in \beta(X),$$

$$\forall B \in \beta(X) \quad \forall \{t_n | t_n \nearrow \infty\} \text{ arbitrary}, \tag{5}$$

the sequence $\{\xi_n | \xi_n \in U_\Sigma(t_n, 0, B)\}$ is precompact in X,

then the family of MSP $\{U_\sigma\}_{\sigma \in \Sigma}$ has a global attractor,

$$\theta_\Sigma = \bigcup\limits_{\tau \ge 0} \theta_\Sigma(\tau) = \theta_\Sigma(0), \tag{6}$$

where $\theta_\Sigma(\tau) = \bigcup\limits_{B \in \beta(X)} \omega_\Sigma(\tau, B)$, and $\omega_\Sigma(\tau, B) = \bigcap\limits_{s \ge \tau} \overline{\bigcup\limits_{t \ge s} U_\Sigma(t, \tau, B)}$ is compact in X. Moreover,

$$y \in \omega_\Sigma(\tau, B) \iff y = \lim\limits_{n \to \infty} y_n, \quad y_n \in U_\Sigma(t_n + \tau, \tau, B), \quad t_n \nearrow \infty.$$

Theorem 1. *Let a family of MSP $\{U_\sigma\}_{\sigma \in \Sigma}$ satisfy the conditions (5).*
1. If $\exists B_0 \in \beta(X)$ $\forall B \in \beta(X)$ $\mathrm{dist} : (U_\Sigma(t, 0, B), B_0) \to 0$, $t \to \infty$, then θ_Σ is compact in X.

2. If for some $t > 0$ and for each $\tau \geq 0$, we have the following property:

$$if\ \xi_n \in U_{T(t_n)\Sigma}(t + \tau, \tau, \eta_n),\ t_n \nearrow \infty,\ \xi_n \to \xi,\ \eta_n \to \eta, \tag{7}$$

$$then \quad \xi \in U_\Sigma(t + \tau, \tau, \eta),$$

then $\theta_\Sigma \subset U_\Sigma(t + \tau, \tau, \theta_\Sigma)$.

If, additionally, $\exists \gamma > 0$ such that (7) holds $\forall t \in (0, \gamma)$, and the family of MSP $\{U_\sigma\}_{\sigma \in \Sigma}$ is strict, then θ_Σ is semiinvariant, that is $\forall (t, \tau) \in \mathbb{R}_{+d}$

$$\theta_\Sigma \subset U_\Sigma(t, \tau, \theta_\Sigma).$$

Proof. 1. We have that $\forall \delta > 0 \quad \forall B \in \beta(X) \quad \exists T = T(\delta, B) \quad \forall t \geq T: \quad U_\Sigma(t, 0, B) \subset$ $\subset O_\delta(B_0)$. From (6) we obtain that $\omega_\Sigma(0, B) \subset O_\delta(B_0)$, so $\theta_\Sigma \subset O_\delta(B_0)$. Further, $\forall t \geq T$ $\forall p \geq 0: U_\Sigma(t + p, t, U_\Sigma(t, 0, B)) \subset U_\Sigma(t + p, t, O_\delta(B_0))$,

$$U_\Sigma(t + p, 0, B) \subset U_{T(t)\Sigma}(p, 0, O_\delta(B_0)) \subset U_\Sigma(p, 0, O_\delta(B_0)).$$

So $\forall s \geq T: \quad \bigcup_{t' \geq s+p} U_\Sigma(t', 0, B) \subset U_\Sigma(p, 0, O_\delta(B_0))$. Therefore, $\forall s' \geq 0$:

$$\bigcup_{p \geq s'} \bigcup_{t' \geq s+p} U_\Sigma(t', 0, B) \subset \bigcup_{p \geq s'} U_\Sigma(p, 0, O_\delta(B_0)),$$

$$\bigcup_{t' \geq s+s'} U_\Sigma(t', 0, B) \subset \bigcup_{p \geq s'} U_\Sigma(p, 0, O_\delta(B_0)),$$

and finally we obtain $\omega_\Sigma(0, B) \subset \omega_\Sigma(0, O_\delta(B_0))$. From this embedding and (6) we obtain $\theta_\Sigma =$ $= \omega_\Sigma(0, O_\delta(B_0))$ is compact in X.

2. According to (6), $\theta_\Sigma = \bigcup_{\tau \geq 0} \theta_\Sigma(\tau)$. Let $\xi \in \theta_\Sigma(\tau)$. So, from the structure of $\theta_\Sigma(\tau)$ there exists $B \in \beta(X)$, $\{\sigma_n\} \subset \Sigma$, $\{t_n \nearrow \infty\}$, $\xi_n \in U_{\sigma_n}(t_n + \tau, \tau, B)$ such that $\xi_n \to \xi$ in X.

So $\xi_n \in U_{\sigma_n}(t_n + t - t + \tau, \tau, B) \subset U_{\sigma_n}(t_n + t - t + \tau, t_n - t + \tau, U_{\sigma_n}(t_n - t + \tau, \tau, B)) \subset$ $\subset U_{T(t_n-t)\sigma_n}(t + \tau, \tau, \eta_n))$, where $\eta_n \in U_{\sigma_n}(t_n - t + \tau, \tau, B) \subset U_{T(\tau)\sigma_n}(t_n - \tau, \tau, \theta_\Sigma))$. From (5), (6) on some subsequence $\eta_n \to \eta \in \theta_\Sigma$. Therefore from (7) $\xi \in U_\Sigma(t + \tau, \tau, \eta) \subset U_\Sigma(t + \tau, \tau, \theta_\Sigma)$, so $\theta_\Sigma \subset U_\Sigma(t + \tau, \tau, \theta_\Sigma)$.

Now let $\{U_\sigma\}_{\sigma \in \Sigma}$ be strict and (7) take place $\forall t \in (0, \gamma)$. Then $\forall n \geq 1: U_\Sigma(nt + \tau, \tau, \theta_\Sigma) =$ $= U_\Sigma((n - 1)t + \tau + t, \tau, \theta_\Sigma) = U_\Sigma(t + \tau + (n - 1)t, \tau + (n - 1)t, U_\Sigma((n - 1)t + \tau, \tau, \theta_\Sigma))$. From this we immediately obtain that $\theta_\Sigma \subset U_\Sigma(nt + \tau, \tau, \theta_\Sigma) \ \forall n \geq 1 \ \forall t \in (0, \gamma)$. So $\forall t \geq 0$: $\theta_\Sigma \subset U_\Sigma(t + \tau, \tau, \theta_\Sigma)$ and the theorem is proved.

The family of MSP, generated by the problem (1) – (4). For using the abstract theory of MSP we need to embed the problem (1) – (4) into a family of specially constructed problems.

For any $h \in (\tau_{i-1}, \tau_i]$, $i \geq 1$, we denote by σ_h the problem (1), whose solutions have impulse effects of the form

$$u(\tau_j - h + 0) - u(\tau_j - h) \in \psi_j(u(\tau_j - h)), \quad j \geq i. \tag{4$_h$}$$

By σ_0 we denote the problem (1)–(4).

By σ_∞ we denote the problem (1), (2) without impulse effects.

Note that $\forall \tau \geq 0\ \forall h \in (\tau_{i-1}, \tau_i]$, $i \geq 1$, the problem (1)–(4)$_h$ is globally resolved in the same sense as problem (1)–(4). More exactly, we shall say that the problem (1)–(4)$_h$ is globally resolved in the class $W_\tau(\sigma_h)$, if $\forall u_\tau \in H\quad \exists u \in L^2_{\text{loc}}(\tau, +\infty; H^1_0(\Omega)) \cap L^p_{\text{loc}}(\tau, +\infty; L^p(\Omega))$ is a solution of (1) on $(\tau, \tau_j - h)$, $(\tau_j - h, \tau_{j+1} - h), \ldots, u(\tau) = u_\tau$, where $\tau_j - h$ is the nearest moment to τ, and at the points $\{\tau_j - h, \tau_{j+1} - h, \ldots\}$ the function $u(\cdot)$ has impulse effects and is left-continuous.

On the space $\Sigma := \{\sigma_h\}_{h \geq 0} \cup \{\sigma_\infty\}$ we define the function $\rho : \Sigma \times \Sigma \to \mathbb{R}$,

$$\rho(\sigma_{h_1}, \sigma_{h_2}) = \left| \frac{1}{h_1 + 1} - \frac{1}{h_2 + 1} \right|, \qquad \rho(\sigma_h, \sigma_\infty) = \frac{1}{h + 1},$$

and $\forall s \geq 0$ we define the map

$$T(s) : \Sigma \to \Sigma, \qquad T(s)\sigma_h = \sigma_{h+s}, \qquad T(s)\sigma_\infty = \sigma_\infty.$$

It is easy to show that (Σ, ρ) is a compact metric space and $\{T(s) : \Sigma \to \Sigma\}_{s \geq 0}$ is a continuous semigroup acting on Σ.

Now for any $\sigma_h \in \Sigma$, $0 \leq h \leq \infty$ we define the map

$$U_{\sigma_h}(t, \tau, u_\tau) = \{ u(t) \mid \quad u(\cdot) \in W_\tau(\sigma_h) \text{ is a solution of } (1)\text{–}(4)_h, \ u(\tau) = u_\tau \}. \qquad (8)$$

From noted above we have that formula (8) $\forall \sigma_h \in \Sigma$ correctly defines a multivalued map,

$$U_{\sigma_h} : \mathbb{R}_{+d} \times H \to P(H).$$

Lemma 1. *The family of maps, defined by (8) is a strict family of MSP.*

Proof. The problem (1)–(4)$_\infty$ is autonomous and the required result for U_{σ_∞} can be easily obtain from [10]. Let $0 \leq h < \infty$. From (8) $\forall \tau \geq 0$: $U_{\sigma_h}(\tau, \tau, u_T) = u(\tau) = u_\tau$, and we have condition 1) from Definition 1.

Let $\xi \in U_{\sigma_h}(t, \tau, x)$. Then $\xi = u(t)$, $u(\cdot) \in W_\tau(\sigma_h)$ is a solution of (1)–(4)$_h$, $u(\tau) = x$. From this, $\forall s \in (\tau, t)$: $u(s) \in U_{\sigma_h}(s, \tau, x)$. We put $\omega(p) = u(p)$, if $p \geq s$. Then $\omega(\cdot) \in W_s(\sigma_h)$ is a solution of (1)–(4)$_h$, $\omega(s) = u(s)$, so $\xi = u(t) = \omega(t) \in U_{\sigma_h}(t, s, u(s)) \subset U_{\sigma_h}(t, s, U_{\sigma_h}(s, \tau, x))$.

Let $\xi \in U_{\sigma_h}(t, s, U_{\sigma_h}(s, \tau, x))$. So $\xi = u(t)$, where $u(\cdot) \in W_s(\sigma_h)$ is a solution (1)–(4)$_h$, $u(s) = \eta$, $\eta = v(s)$, where $v(\cdot) \in W_s(\sigma_h)$ is a solution of (1)–(4)$_h$, $v(\tau) = x$. We put

$$\omega(p) = \begin{cases} v(p), & p \in [\tau, s], \\ u(p), & p > s. \end{cases}$$

Then $\omega(\cdot) \in W_\tau(\sigma_h)$ is solution of (1)–(4)$_h$, $\omega(\tau) = x$, so $\omega(t) = u(t) = \xi \in U_{\sigma_h}(t, \tau, x)$.

Let $\xi \in U_{\sigma_h}(t + s, \tau + s, x)$. Then $\xi = u(t + s)$, $u(\cdot) \in W_{\tau+s}(\sigma_h)$ is a solution of (1)–(4)$_h$, $u(\tau + s) = x$. We put $v(p) = u(p + s)$, $p \geq \tau$. If $\tau + s \in (\tau_{i-1} - h, \tau_i - h]$, then $u(\cdot)$ is a solution of (1) on $(\tau + s, \tau_i - h)$, $(\tau_i - h, \tau_{i+1} - h), \ldots$ which has impulse effect,

$$u(\tau_j - h + 0) - u(\tau_j - h) \in \psi_j(u(\tau_j - h)), \quad j \geq i.$$

So, $v(\cdot)$ is a solution of (1) on $(\tau, \tau_i - h - s), (\tau_i - h - s, \tau_{i+1} - h - s), \ldots$, which has impulse effect

$$v(\tau_j - h - s + 0) - v(\tau_j - h - s) \in \psi_j(v(\tau_j - h - s)), \quad j \geq i,$$

and $v(\tau) = u(\tau + s) = x$. Therefore, $\xi = u(t + s) = v(t) \in U_{\sigma_{h+s}}(t, \tau, x) = U_{T(s)\sigma_h}(t, \tau, x)$. Let $\xi \in U_{T(s)\sigma_h}(t, \tau, x) = U_{\sigma_{h+s}}(t, \tau, x)$. Then $\xi = u(t)$, $u(\cdot) \in W_\tau(\sigma_{h+s})$ is a solution of $(1)-(4)_{h+s}$, $u(\tau) = x$. We put $v(p) := u(p - s)$, $p \geq \tau + s$. Then analogously to the above arguments we obtain that $v(\cdot) \in W_{\tau+s}(\sigma_h)$ is a solution of $(1)-(4)_h$, $v(\tau + s) = u(\tau) = x$. So, $\xi = u(t) = v(t + s) \in U_{\sigma_h}(t + s, \tau + s, x)$.

The lemma is proved.

We will assume that the following dissipative property holds:

$$\exists R_1 > 0 \quad \forall \tau \geq 0 \quad \forall r \geq 0 \quad \forall u_\tau \in H : \quad \|u_\tau\| \leq r \text{ and, for an arbitrary solution,}$$

$$u(\cdot) \in W_\tau(\sigma_0) \text{ of the problem } (1)-(4), \ u(\tau) = u_\tau, \tag{9}$$

$$\exists T_1 = T_1(r) \quad \forall t \geq T_1 : \quad \|u(t + \tau)\| \leq R_1,$$

that is $\forall t \geq T_1 \colon U_{\sigma_0}(t + \tau, \tau, B_r) \subset B_{R_1}$.

We shall discuss sufficient condition for (9) in the terms of the initial data given at the end of this paper in Lemma 4.

Now we note that from (9) and Lemma 1, $\forall \sigma_h \in \Sigma, 0 \leq h < \infty$:

$$U_{\sigma_h}(t + \tau, \tau, B_r) = U_{T(h)\sigma_0}(t + \tau, \tau, B_r) = U_{\sigma_0}(t + \tau + h, \tau + h, B_r) \subset B_{R_1} \qquad \forall t \geq T_1,$$

where T_1 does not depend on τ and h. Using results from [9] we can write the estimate: $\forall u(\cdot) \in W_\tau$, a solution of (1), $\forall t \geq 0 \, \forall \tau \geq 0$:

$$\|u(t + \tau)\|^2 \leq \|u(\tau)\|^2 e^{-\delta t} + K, \tag{10}$$

where the constants $\delta > 0, K > 0$ do not depend on $u(\cdot), t, \tau$. From (10) we immediately obtain

$$\forall \tau \geq 0 \quad \forall r > 0 \quad \exists T_2 = T_2(r) \quad \forall t \geq T_2 : \quad U_{\sigma_\infty}(t + \tau, \tau, B_r) \subset B_{\sqrt{K+1}}.$$

So we have the following consequence of (9): $\exists R_0 > 0 \, \forall \tau \geq 0 \, \forall r \geq 0 \, \exists T = T(r) \, \forall t \geq T(r)$:

$$U_\Sigma(t + \tau, \tau, B_r) \subset B_{R_0}. \tag{11}$$

Also we need the following condition:

$$\forall r > 0 : \quad \sup_{u \in B_r} \|\psi_i(u)\| \to 0, \quad i \to \infty, \tag{12}$$

where $\|\psi_i(u)\| = \sup_{a \in \psi_i(u)} \|a\|$.

The main result.

Theorem 2. *Let the conditions (2), (9), (12) hold. Then the family of MSP $\{U_\sigma\}_{\sigma \in \Sigma}$ defined by (8), has a global attractor $\theta_\Sigma \subset H$, which is seminvariant and compact in H.*

Proof. Since embedding (11) takes place, for the existence of a global attractor we need to prove that for an arbitrary $r > 0$, $\{t_n \nearrow \infty\}$, the sequence $\{\xi_n | \xi_n \in U_\Sigma(t_n, 0, B_r)\}$ is precompact in H. To do this, we need the following result which is a particular case of the result obtained in [10].

Lemma 2. *Let $\{u_n(\cdot)\} \subset W_\tau$ be a sequence of solutions of (1) and $u_n(\tau) \to u_\tau$ weakly in H. Let us assume that $\forall T \geq \tau$ we have a sequence $\{t_n\} \subset (\tau, T)$ such, that $t_n \to t_0 \in (\tau, T)$. Then there exists $u(\cdot) \in W_\tau$, a solution of (1), such that $u(\tau) = u_\tau$ and, for some subsequence, $u_n(t_n) \to u(t_0)$ in H. If $u_n(\tau) \to u_\tau$ in H, then, moreover, for $t_n \to \tau$ on some subsequence, $u_n(t_n) \to u_\tau$ in H.*

Now let $\xi_n \in U_{\sigma_{h_n}}(t_n, 0, B_r)$. Then $\forall t^* \in (0, \gamma)$ $\forall n \geq N(r)$ we have $\xi_n \in U_{\sigma_{h_n}}(t_n - t^* + t^*, t_n - t^*, B_{R_0}) = U_{T(t_n - t^*)\sigma_{h_n}}(t^*, 0, B_{R_0})$. So there exists $\{\eta_n\} \subset B_{R_0}$ such that $\xi_n \in U_{T(t_n - t^*)\sigma_{h_n}}(t^*, 0, \eta_n)$ and $\eta_n \to \eta$ weakly in H (we always can choose a suitable subsequence).

Let $h_n \in (t_{i_{n-1}}, t_{i_n}]$. Since $t_n \nearrow \infty$, we always have $T(t_n - t^*)\sigma_{h_n} \to \sigma_\infty$ in Σ. Note that for every $n \geq 1$ we know the elements $\{\tau_i - h_n - t_n\}_{i \geq 1}$. Then $\forall n \geq 1$ $\exists! m(n) \geq 1$ such that $\lambda_n := \tau_{m(n)} - h_n - t_n < 0$, $\theta_n := \tau_{m(n)+1} - h_n - t_n \geq 0$. Moreover, $m(n) \to \infty$, $n \to \infty$ and $\forall n \geq 1$: $\theta_n - \lambda_n \geq \gamma > 0$. Since we can always consider a subsequence, we should investigate only two situations:

1. $\exists \varepsilon \in (0, \gamma)$ $\forall n \geq N(r)$: $\lambda_n < -\varepsilon$. In this case we choose $t^* \in (0, \varepsilon)$ and obtain that $\lambda_n + t^* < 0$, $\theta_n + t^* \geq t^*$. Since $\xi_n \in U_{\sigma_{h_n + t_n - t^*}}(t^*, 0, \eta_n)$, we have $\xi_n = u_n(t^*)$, where $u_n(\cdot) \in W_0(\sigma_{h_n + t_n - t^*})$ is a solution of $(1) - (4)_{h_n + t_n - t^*}$, $u_n(0) = \eta_n$. Therefore, $u_n(\cdot)$ has the first impulse effect at the moment $\theta_n + t^* \geq t^*$, so it has no impulse on $[0, t^*)$. From this we can use Lemma 2 and obtain that on some subsequence $\xi_n = u_n(t^*) \to u(t^*)$, where $u(\cdot) \in W_0$ is a solution of (1), $u(0) = \eta$. So, in this case the sequence $\{\xi_n\}$ is precompact in H.

2. $\forall n \geq N(r)$: $\lambda_n < 0$, but $\lambda_n \nearrow 0$, $n \to \infty$. In this case we take arbitrary $t^* \in (0, \gamma)$ and for sufficiently large $n \geq N(r)$ we have $\lambda_n + t^* \in (0, t^*)$, $\theta_n + t^* > t^*$. So for $\xi_n = u_n(t^*)$ the solution $u_n(\cdot)$ has a unique impulse effect at the moment of time $s_n = \lambda_n + t^* = \tau_{m(n)} - h_n - t_n + t^*$, $s_n \nearrow t^*$, which is characterized by the inclusion

$$u_n(s_n + 0) - u_n(s_n) \in \psi_{m(n)}(u_n(s_n)).$$

According to Lemma 2, $u_n(s_n) \to u(t^*)$, where $u(\cdot)$ is a solution of (1), $u(0) = \eta$. In particular, $\exists r > 0 : \|u_n(s_n)\| \leq r$. So $\|u_n(s_n + 0) - u_n(s_n)\| \leq \|\psi_{m(n)}(u_n(s_n))\| \leq \sup_{u \in B_r}\|\psi_{m(n)}(u)\| \to 0$, $n \to \infty$. Therefore, $u_n(s_n + 0) \to u(t^*)$, $n \to \infty$. Because $\xi_n = u_n(t^*) \in U_{\sigma_\infty}(t^*, s_n, u_n(s_n + 0)) = U_{T(s_n)\sigma_\infty}(t^* - s_n, 0, u_n(s_n + 0))$, we have $\xi_n = v_n(t^* - s_n)$, where $v_n(\cdot) \in W_0$ is a solution of (1), $v_n(0) = u_n(s_n + 0) \to u(t^*)$. So from Lemma 2 on some subsequence, $\xi_n = u_n(t^*) = v_n(t^* - s_n) \to u(t^*)$, and we obtain precompactness of $\{\xi_n\}$ in H.

Further, from embedding (11) and Theorem 1 we immediately obtain that the family of MSP $\{U_\sigma\}_{\sigma \in \Sigma}$ has a compact global attractor $\theta_\Sigma \subset H$.

Now we shall prove semiinvariance of θ_Σ. We need the following result.

Lemma 3. *If $\xi_n \in U_{T(t_n)\Sigma}(t, 0, \eta_n)$, where $t \in (0, \gamma)$, $t_n \nearrow \infty$, $\eta_n \to \eta$, then on some subsequence $\xi_n \to \xi \in U_{\sigma_\infty}(t, 0, \eta)$.*

Proof. Our arguments in this case will be similar to the one described above, but not analogous, because in this lemma we cannot choose $t \in (0, \gamma)$, but we have a strong convergence $\eta_n \to \eta$ in H.

Let $\xi_n \in U_{T(t_n)\sigma_{h_n}}(t, 0, \eta_n)$, $\sigma_{h_n} \in \Sigma$. For arbitrary $n > 1$ we use $\lambda_n < 0$ and $\theta_n \geq 0$ which are introduced above. We have $\xi_n = u_n(t)$, where $u_n(\cdot) \in W_0(\sigma_{t_n+h_n})$ is a solution of $(1)-(4)_{t_n+h_n}$, $u_n(0) = \eta_n \to \eta$. As $t \in (0, \gamma)$, we have that $u_n(\cdot)$ on $[0, t]$ is not bigger than one impulse effect at the moment θ_n. If $\forall n \geq 1: \quad \theta_n \geq t$, then $u_n(\cdot)$ on $[0, t)$ has no impulse effect and, according to Lemma 2, $\xi_n = u_n(t) \to u(t) \in U_{\sigma_\infty}(t, 0, \eta)$.

Let $\theta_n \in [0, t)$. Note that from estimate (10) $\exists r > 0 \quad \forall n \geq 1: \quad \|u_n(\theta_n)\| \leq r$, so $\|u_n(\theta_n + 0) - u_n(\theta_n)\| \leq \|\psi_{m(n)}(u_n(\theta_n))\| \leq \sup_{u \in B_r} \|\psi_{m(n)}(u)\| \to 0, n \to \infty$.

Now we should consider all possible cases for θ_n. If $\theta_n \searrow 0$ (or $\theta_n = 0$), then from Lemma 2 we obtain $u_n(\theta_n) \to \eta$ in H. Therefore $u_n(\theta_n + 0) \to \eta$ and $\xi_n = u_n(t) \in U_{\sigma_\infty}(t, \theta_n, u_n(\theta_n + 0)) = U_{\sigma_\infty}(t - \theta_n, 0, u_n(\theta_n + 0))$, where $t - \theta_n \nearrow t$. Using again Lemma 2, we have $\xi_n \to \xi \in U_{\sigma_\infty}(t, 0, \eta)$.

If $\theta_n \nearrow t$, then $\xi_n = u_n(t) \in U_{\sigma_\infty}(t, \theta_n, u_n(\theta_n + 0)) = U_{\sigma_\infty}(t - \theta_n, 0, u_n(\theta_n + 0))$, where $t - \theta_n \searrow 0$. As $u_n(\theta_n + 0) \to u(t) \in U_{\sigma_\infty}(t, 0, \eta)$, we can use Lemma 2 and obtain that $\xi_n \to \xi = u(t) \in U_{\sigma_\infty}(t, 0, \eta)$.

If, finally, $\theta_n \to \theta \in (0, t)$, then $\xi_n = u_n(t) \in U_{\sigma_\infty}(t, \theta_n, u_n(\theta_n + 0)) = U_{\sigma_\infty}(t - \theta_n, 0, u_n(\theta_n + 0))$, where $t - \theta_n \to t - \theta$. Since, by Lemma 2, $u_n(\theta_n) \to u(\theta) \in U_{\sigma_\infty}(\theta, 0, \eta)$, so $u_n(\theta_n + 0) \to u(\theta)$. However $\xi_n = v_n(t - \theta_n)$, where $v_n(\cdot) \in W_0$ is a solution of (1), $v_n(0) = u_n(\theta_n + 0)$. Therefore it follows from Lemma 2 that $\xi_n = v_n(t - \theta_n) \to \xi = v(t - \theta) \in U_{\sigma_\infty}(t - \theta, 0, u(\theta)) \subset U_{\sigma_\infty}(t, \theta, U_{\sigma_\infty}(\theta, 0, \eta)) = U_{\sigma_\infty}(t, 0, \eta)$ and the lemma is proved.

Now according to Theorem 1, let $\xi_n \in U_{T(t_n)\Sigma}(t + \tau, \tau, \eta_n)$, where $t \in (0, \gamma)$, $\tau \geq 0$, $t_n \nearrow \infty$, $\xi_n \to \xi$, $\eta_n \to \eta$. Then $\xi_n \in U_{T(t_n+t)\Sigma}(t, 0, \eta_n)$ and, from Lemma 3, we obtain

$$\xi_n \to \xi \in U_{\sigma_\infty}(t, 0, \eta) = U_{T(\tau)\sigma_\infty}(t, 0, \eta) = U_{\sigma_\infty}(t + \tau, \tau, \eta) \subset U_{\Sigma}(t + \tau, \tau, \eta).$$

So from Theorem 1, it follows that θ_Σ is seminvariant.

The theorem is proved.

Lemma 4. *If* $\forall i \geq 1 \quad \forall u \in H: \quad \|\psi_i(u)\| \leq a\|u\| + b$, *where* $a, b \geq 0$ *and*

$$-\delta + \frac{1}{\gamma}\ln(1 + (a + 1)^2) < 0, \tag{13}$$

where $\delta > 0$ *is the constant from estimate (10), then (9) holds.*

Proof. From [10] we have, that if $u(\cdot) \in W_\tau$ is a solution of (1), (2), then the scalar function $t \to \|u(t)\|^2$ is absolutely continuous and for almost all $t > \tau$ it satisfies the inequality

$$\frac{d}{dt}\|u(t)\|^2 + \delta\|u(t)\|^2 \leq \widetilde{C}, \tag{14}$$

when the constant $\widetilde{C} > 0$ depends only on the constants if problem (1). Moreover, from (4) we have $\left|\|u(\tau_i + 0)\|^2 - \|u(\tau_i)\|^2\right| = \|\|u(\tau_i + 0)\| - \|u(\tau_i)\|\|\|u(\tau_i + 0)\| + \|u(\tau_i)\|\| \leq (a\|u(\tau_i)\| +$

$+b)\left((2+a)\left\|u\left(\tau_{i}\right)\right\|+b\right) \leq\left((a+1)\left\|u\left(\tau_{i}\right)\right\|+b\right)^{2} \leq\left(\varepsilon+(a+1)^{2}\right)\left\|u\left(\tau_{i}\right)\right\|^{2}+C_{\varepsilon} b^{2} \ \forall \varepsilon>0.$ So for the function $t \rightarrow\|u(t)\|^{2}$ we have the following impulse problem:

$$
\frac{d}{dt}\|u(t)\|^{2}+\delta\|u(t)\|^{2} \leq \widetilde{C},
$$

$$
\|u(\tau_{i}+0)\|^{2}-\|u(\tau_{i})\|^{2} \leq\left(\varepsilon+(a+1)^{2}\right)\|u\left(\tau_{i}\right)\|^{2}+C_{\varepsilon} b^{2}.
$$

$$(15)$$

Obviously, every solution of (15) is bounded by the solution of following problem:

$$
\frac{d}{dt}x(t)+\delta x(t)=\widetilde{C},
$$

$$
x(\tau_{i}+0)-x(\tau_{i})=\left(\varepsilon+(a+1)^{2}\right) x\left(\tau_{i}\right)+C_{\varepsilon} b^{2}.
$$

$$(16)$$

For each $\tau \geq 0, x_{\tau} \in \mathbb{R}$ the solution $x(\cdot)$ of (16), $x(\tau)=x_{\tau}$, is given by the formula

$$
x(t)=e^{-\delta(t-\tau)}(1+\varepsilon+(a+1)^{2})^{i(t,\tau)} x_{\tau}+\widetilde{C} \int_{\tau}^{t} e^{-\delta(t-p)}(1+\varepsilon+(a+1)^{2})^{i(t,p)} dp+
$$

$$
+C_{\varepsilon} b^{2} \sum_{\tau \leq \tau_{i}<t} e^{-\delta(t-\tau_{i})}(1+\varepsilon+(a+1)^{2})^{i(t,\tau_{i})},
$$

where $i(t, s)$ is a number of moments τ_{i}, which belong to $[s, t)$.

From (13), $\exists \varepsilon>0 \ \exists \mu>0$ such that

$$
-\delta+\frac{1}{\gamma} \ln(1+\varepsilon+(a+1)^{2}) \leq-\mu<0.
$$

Then for $x_{\tau}=\|u(\tau)\|^{2}$ we can easily obtain the following estimate:

$$
\|u(t+\tau)\|^{2} \leq x(t+\tau) \leq e^{-\mu t}\|u(\tau)\|^{2}+M
$$

from what (9) follows.

The lemma is proved.

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