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ENTIRE BIVARIATE FUNCTIONS OF UNBOUNDED INDEX IN EACH DIRECTION

ЦІЛІ ФУНКЦІЇ ВІД ДВОХ ЗМІННИХ НЕОБМЕЖЕНОГО ІНДЕКСУ ЗА КОЖНИМ НАПРЯМКОМ

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We investigate a class of entire functions $f(z_1, z_2)$ with property $\forall \mathbf{b} = (b_1, b_2) \in \mathbb{C}^2 \setminus \{0\}$ $\forall\, z_1^0,\ z_2^0 \in \mathbb{C}$, the function $f(z_1^0 + tb_1, z_2^0 + tb_2)$, as a function of one variable $t \in \mathbb{C}$, has a bounded index but the function $f(z_1, z_2)$ has an unbounded index in every direction $\mathbf{b}$. In particular, we prove that, for an arbitrary even entire function $f(t)$ that has an infinite sequence of complex zeros, the corresponding function $f\left(\sqrt{z_1 z_2}\right)$ has an unbounded index in every direction $\mathbf{b}$. It improves our similar result [*Bandura A. I.* A class of entire functions of unbounded index in each direction // Mat. Stud. – 2015. – **44**, № 1. – P. 107 – 112] proved for even entire functions $f(t)$ with complex zeros c_k such that $c_k^2 \in \mathbb{R}$.

Досліджується клас цілих функцій $f(z_1, z_2)$ з такою властивістю: $\forall \mathbf{b} = (b_1, b_2) \in \mathbb{C}^2 \setminus \{0\}$ $\forall\, z_1^0$, $z_2^0 \in \mathbb{C}$ функція $f(z_1^0 + tb_1, z_2^0 + tb_2)$ має обмежений індекс як функція від однієї змінної $t \in \mathbb{C}$, але функція $f(z_1, z_2)$ є необмеженого індексу за кожним напрямком $\mathbf{b}$. Зокрема, доведено, що для довільної парної цілої функції $f(t)$, яка має нескінченну послідовність комплексних нулів, відповідна функція $f(\sqrt{z_1 z_2})$ є необмеженого індексу за кожним напрямком $\mathbf{b}$. Це покращує подібний результат з [*Bandura A. I.* A class of entire functions of unbounded index in each direction // Mat. Stud. – 2015. – **44**, № 1. – P. 107 – 112], доведений для парних цілих функцій $f(t)$ з комплексними нулями c_k такими, що $c_k^2 \in \mathbb{R}$.

1. Introduction. To state a problem and a main result, we need some denotations. Let $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$ be a direction, $L\colon \mathbb{C}^n \to \mathbb{R}_+$ be a continuous function, $F\colon \mathbb{C}^n \to \mathbb{C}$ be an entire function, $g_{z^0}(t) := F\left(z^0 + t\mathbf{b}\right)$, $l_{z^0}(t) := L\left(z^0 + t\mathbf{b}\right)$, $t \in \mathbb{C}$.

Definition 1 [1, 2]**.** *An entire function $F(z)$, $z \in \mathbb{C}^n$, is said to be of bounded L-index in the direction $\mathbf{b}$, if there exists $m_0 \in \mathbb{Z}_+$ such that for all $m \in \mathbb{Z}_+$ and every $z \in \mathbb{C}^n$ the next inequality is true:*

$$\frac{1}{m!L^m(z)}\left|\frac{\partial^m F(z)}{\partial \mathbf{b}^m}\right| \le \max\left\{\frac{1}{k!L^k(z)}\left|\frac{\partial^k F(z)}{\partial \mathbf{b}^k}\right| : 0 \le k \le m_0\right\}, \tag{1}$$

where

$$\frac{\partial^0 F(z)}{\partial \mathbf{b}^0} = F(z), \qquad \frac{\partial F(z)}{\partial \mathbf{b}} = \sum\nolimits_{j=1}^{n} \frac{\partial F(z)}{\partial z_j}\, b_j, \qquad \frac{\partial^k F(z)}{\partial \mathbf{b}^k} = \frac{\partial}{\partial \mathbf{b}}\left(\frac{\partial^{k-1} F(z)}{\partial \mathbf{b}^{k-1}}\right), \quad k \ge 2.$$

The least such integer m_0 is called the L-index in the direction $\mathbf{b}$ of $F(z)$ and is denoted by $N_{\mathbf{b}}(F, L)$. If such m_0 does not exist, then we put $N_{\mathbf{b}}(F, L) = \infty$ and F is said to be of unbounded L-index in the direction.

If $L(z) \equiv 1$ then the function F is called of bounded index in the direction $\mathbf{b}$ and $N_{\mathbf{b}}(F) \equiv \equiv N_{\mathbf{b}}(F, 1)$ is called the index in the direction $\mathbf{b}$. If $n = 1$, $\mathbf{b} = 1$ and $L(z) = l(z)$, $z \in \mathbb{C}$, inequality (1) defines a bounded l-index with the l-index $N(F, l) \equiv N_1(F, l)$ [3]. And in the case $L(z) \equiv 1$ we get a notion of bounded index with the index $N(F) \equiv N_1(F, 1)$ [4].

These functions have been used in the theory value distribution and differential equations ([see bibliography in [5]). In particular, every entire function is a function of bounded value distribution if and only if its derivative is a function of bounded index [6], and every entire solution of the differential equation $f^{(n)}(t) + \sum_{j=0}^{n-1} a_j f^{(j)}(t) = 0$ is a function of bounded index [7]. More general results for PDE's are obtained by ours [1, 2, 8]. There are sufficient conditions that every entire function satisfying some PDE is of bounded L-index in direction.

Another definition of bounded index in $\mathbb{C}^2$ is considered in the paper of F. Nuray and R. F. Patterson [9]. Using this notion they presented a series of sufficient conditions that bivariate entire function is of exponential type. The presented conditions are weaker than known necessary and sufficient conditions of bounded index in joint variables. Besides, they [10] established the relationship between the concept of bounded index and the radius of univalence, respectively p-valence, of entire bivariate functions and their partial derivatives at arbitrary points in $\mathbb{C}^2$.

Recently we published the paper [11], which is devoted to interesting and important open problems in the theory of entire functions of bounded index. In particular, there was formulated the following

Problem 1 ([11], Problem 17). *What are conditions on zero sets and growth of entire functions providing the bounded index of $F(z_1^0+b_1t, z_2^0+b_2t)$ for every $(z_1^0, z_2^0) \in \mathbb{C}^2$ and the unbounded index of $F(z_1, z_2)$ in the direction $\mathbf{b} = (b_1, b_2)$?*

For example, $f(z_1, z_2) = \cos\sqrt{z_1 z_2}$ has described properties [12]. It was proved by a construction of some PDE and investigation of properties of its entire solutions.

We solved the mentioned problem [13], supposing $c_k^2 \in \mathbb{R}$, where c_k, $k \in \mathbb{N}$, are zeros of entire function $f(t)$. We got the next theorem as our decision.

Theorem 1 [13]. *Let $f(t)$, $t \in \mathbb{C}$, be an even entire transcendental function of bounded index. Then:*

(i) *for each direction $\mathbf{b} = (b_1, b_2) \in \mathbb{C}^2 \setminus \{0\}$ and for every fixed $z_1^0, z_2^0 \in \mathbb{C}$ the function $g(t) = f\left(\sqrt{\left(z_1^0 + b_1 t\right)\left(z_2^0 + b_2 t\right)}\right)$ is an entire function of bounded index $t \in \mathbb{C}$;*

(ii) *if $f(t)$ has no zeros or has a finite number of zeros, then $f\left(\sqrt{z_1 z_2}\right)$ is of unbounded index in each direction $\mathbf{b}$;*

(iii) *if $\{c_k\}$ is an infinite sequence of zeros of $f(t)$, $|c_1| < |c_2| < \ldots < |c_k| < \ldots$ and for every $k \in \mathbb{N}$, $c_k^2 \in \mathbb{R}$, then the function $f\left(\sqrt{z_1 z_2}\right)$ is of unbounded index in each direction $\mathbf{b}$.*

The following remark will be useful in this paper.

Remark 1 [13]. The condition $c_k^2 \in \mathbb{R}$ can be replaced by the condition that there exists an infinite subsequence of zeros of the form $c'_k = |c'_k| \cdot e^{i\theta}$, i.e., all c'_k lay on a some ray. Then in the case $b_1 \neq 0$, $b_2 \neq 0$ we choose $\varphi = 2\theta + arg(b_1 b_2)$ and in the case $b_1 \neq 0$, $b_2 = 0$ we choose $z_1^0 = e^{i(2\theta+\varphi)}$. Other considerations are remained without changes in the proof of Theorem 1.

In 2015 at the Lviv seminar on the theory of analytic functions Prof. O. Skaskiv assumed that condition "for every $k \in \mathbb{N}$, $c_k^2 \in \mathbb{R}$" is excessive. It leads to a new next problem.

Problem 2. *If $f(t),\ t\in\mathbb{C}$, is an even entire transcendental function of bounded index, which has an infinite number of zeros, then a function $f\left(\sqrt{z_1z_2}\right)$ is of unbounded index in each direction $\mathbf{b}$. Is it true or false?*

In this paper, we prove that mentioned proposition is true. Note that for the function $\cos\left(\sqrt{z_1z_2}\right)$ there exists a partial differential equation and a positive continuous function $L\colon\mathbb{C}^2\to\to\mathbb{R}_+$ with properties [14]:

(i) $f\left(\sqrt{z_1z_2}\right)$ is its solution;

(ii) every entire solution of the PDE has bounded L-index in the direction $\mathbf{b}$.

2. Auxiliary proposition. We need some notation. If for a given $z^0\in\mathbb{C}^n$ one has $g_{z^0}(t)\neq 0$ for all $t\in\mathbb{C}$, then $G_r^{\mathbf{b}}\left(F,z^0\right):=\varnothing$; if for a given $z^0\in\mathbb{C}^n$ we get $g_{z^0}(t)\equiv 0$, then

$$G_r^{\mathbf{b}}\left(F,z^0\right):=\left\{z^0+t\mathbf{b}\colon t\in\mathbb{C}\right\}.$$

And if for a given $z^0\in\mathbb{C}^n$ we have $g_{z^0}(t)\not\equiv 0$ and a_k^0 are zeros of $g_{z^0}(t)$, then

$$G_r^{\mathbf{b}}\left(F,z^0\right):=\bigcup_k\left\{z^0+t\mathbf{b}\colon\ \left|t-a_k^0\right|\le r\right\},\quad r>0.$$

Let

$$G_r^{\mathbf{b}}(F)=\bigcup_{z^0\in\mathbb{C}^n}G_r^{\mathbf{b}}(F,z^0).$$

By $n\left(r,z^0,t_0,1/F\right)=\sum_{|a_k^0-t_0|\le r}1$ we denote the counting function of the zero sequence $\left(a_k^0\right)$.

The following criterion is convenient for a proof of index boundedness in direction.

Theorem 2 [1, 2]**.** *Let $F(z)$ be an entire function in $\mathbb{C}^n$. A function $F(z)$ is of bounded index in the direction $\mathbf{b}$ if and only if:*

(i) *for every $r>0$ there exists $P=P(r)>0$ that for each $z\in\mathbb{C}^n\backslash G_r^{\mathbf{b}}(F)$*

$$\left|\frac{1}{F(z)}\frac{\partial F(z)}{\partial\mathbf{b}}\right|\le P;$$

(ii) *for every $r>0$ there exists $\widetilde{n}(r)\in\mathbb{Z}_+$ that for every $z^0\in\mathbb{C}^n$ satisfying $F(z^0+t\mathbf{b})\not\equiv 0$, and for all $t_0\in\mathbb{C}$*

$$n\left(r,z^0,t_0,\frac{1}{F}\right)\le\widetilde{n}(r). \tag{2}$$

Recently, we obtained weaker sufficient conditions [15, 16] than in Theorem 2 replacing the universal quantifier by the existence quantifier. Besides, we need following property of bounded index in direction.

Theorem 3 [1, 2]**.** *Let $m\in\mathbb{C}\setminus\{0\}$. An entire function $F(z),\ z\in\mathbb{C}^n$, is of bounded index in the direction $\mathbf{b}$ if and only if $F(z)$ is of bounded index in the direction $m\mathbf{b}$.*

3. Main theorem.

Theorem 4. *If $f(t),\ t\in\mathbb{C}$, is an even entire transcendental function of bounded index, which has an infinite number of zeros, then the function $f\left(\sqrt{z_1z_2}\right)$ is of unbounded index in each direction $\mathbf{b}$.*

Proof. Let $(c_l)_{l=1}^{\infty}$ be an infinite sequence of zeros of $f(t)$, $|c_1| < |c_2| < \ldots < |c_k| < \ldots$. In view of Theorem 1 and Remark 1 we suppose that there are located a finite number of zeros c_k on every ray with origin. It remains to prove that function $f\left(\sqrt{z_1 z_2}\right)$ is of unbounded index in the direction $\mathbf{b}$. We show that condition (2) of Theorem 2 does not hold.

Obviously, for every $\varepsilon > 0$ there exists a sector with a vertex at the origin and a central angle 2ε where an infinite number of zeros c_k is located inside. Let $arg\, z = \theta$ be a bisector of the sector. In view of Remark 1 we suppose $\theta = 0$ without loss of generality. Below in the proof we consider only zeros c_l containing in the specified sector. In fact, it shall be proven that the condition (2) does not hold for zeros of function $f\left(\sqrt{\left(z_1^0 + b_1 t\right)\left(z_2^0 + b_2 t\right)}\right)$, generating by c_l from that sector.

Case 1. Let $b_1 \neq 0$, $b_2 = 0$ and $a_k \in \mathbb{R}_+$, $a_k \to \infty$. Later we will impose more conditions on the sequence $(a_k)_{k=1}^{\infty}$. We put $z^0 = \left(z_1^0, z_2^0\right)$, where

$$z_1^0 = 1, \qquad z_2^0 = a_k^2, \qquad t_0 = 0. \tag{3}$$

The zeros of the function $f\left(\sqrt{\left(z_1^0 + b_1 t\right)\left(z_2^0 + b_2 t\right)}\right)$ are found from the equation

$$\left(z_1^0 + b_1 t\right) z_2^0 = z_2^0 b_1 t + z_1^0 z_2^0 = c_l^2, \quad l \in \mathbb{Z}.$$

Consider its root

$$t_l = \frac{-(b_2 z_1^0 + b_1 z_2^0) + \sqrt{(b_2 z_1^0 - b_1 z_2^0)^2 + 4 c_l^2 b_1 b_2}}{2 b_1 b_2}.$$

The condition of a zero t_l belongs to r-neighborhood of the point t_0 has the form $|t_l - t_0| < r$. Let $\varphi_l = arg\, c_l$. It implies $\left|\dfrac{|c_l|^2 e^{2i\varphi_l} - a_k^2}{|b_1| \cdot a_k^2}\right| < r$ or

$$r|b_1|\, a_k^2 > \left||c_l|^2 e^{2i\varphi_l} - a_k^2\right| = \left||c_l|^2 \cos 2\varphi_l - a_k^2 + i|c_l|^2 \sin 2\varphi_l\right| =$$

$$= \sqrt{(|c_l|^2 \cos 2\varphi_l - a_k^2)^2 + |c_l|^4 \sin^2 2\varphi_l} = \sqrt{|c_l|^4 - 2|c_l|^2 \cos 2\varphi_l a_k^2 + a_k^4}.$$

Hence, we deduce a biquadratic inequality

$$|c_l|^4 - 2|c_l|^2 \cos 2\varphi_l \cdot a_k^2 + a_k^4 - r^2 |b_1|^2 \cdot a_k^4 < 0. \tag{4}$$

Solving (4), we obtain an estimate for $|c_l|$:

$$a_k^2 \left(\cos 2\varphi_l - \sqrt{r^2 |b_1|^2 - \sin^2 2\varphi_l}\right) < c_l^2 < a_k^2 \left(\cos 2\varphi_l + \sqrt{r^2 |b_1|^2 - \sin^2 2\varphi_l}\right). \tag{5}$$

We choose $r \geq \dfrac{1}{|b_1|}$. It follows $r^2 |b_1|^2 \geq 1$ or $r^2 |b_1|^2 - \sin^2 2\varphi_l \geq \cos^2 2\varphi_l$. Therefore, the left-hand side in (5) is nonpositive. Besides, we pick $\varepsilon \in \left(0, \dfrac{\pi}{4}\right)$, i.e. $\varphi_l \in (-\varepsilon; \varepsilon) \subset \left(-\dfrac{\pi}{4}; \dfrac{\pi}{4}\right)$. The right-hand side in (5) implies

$$a_k^2 > \frac{|c_l|^2}{\cos 2\varphi_l + \sqrt{r^2 |b_1|^2 - \sin^2 2\varphi_l}}.$$

For $k+1$ zeros of $f(t)$ we choose

$$a_k^2 > \max_{1\le l\le k+1} \frac{|c_l|^2}{\cos 2\varphi_l + \sqrt{r^2|b_1|^2 - \sin^2 2\varphi_l}}.$$

Thus, $\exists r>0$ $\left(\text{we have } r \in \left[\frac{1}{|b_1|};\infty\right)\right)$ $\forall k\in\mathbb{N}$ $\exists z^0\in\mathbb{C}^2$ $\exists t^0\in\mathbb{C}$ (see (3)) that

$$n\left(r, z^0, t^0, \frac{1}{f}\right) \ge k+1 > k.$$

Hence, the function $f\left(\sqrt{z_1 z_2}\right)$ is of unbounded index in the direction $\mathbf{b}$.

Case 2. Let $b_1\ne 0,\ b_2\ne 0$ and $a_k\in\mathbb{R}_+,\ a_k\to\infty$. Later we will impose more conditions on the sequence $(a_k)_{k=1}^{\infty}$. By Theorem 3 an entire function $F(z),\ z\in\mathbb{C}^2$ is of bounded index in direction (b_1,b_2) if and only if $F(z)$ is of bounded index in direction $\left(\frac{b_1}{\sqrt{|b_1 b_2|}}, \frac{b_2}{\sqrt{|b_2 b_2|}}\right)$. Hence, without loss of generality we suppose that $|b_1 b_2| = 1$.

Put $\varphi = \arg(b_1 b_2),\ z^0 = \left(z_1^0, z_2^0\right)$, where z_1^0 is arbitrary complex number,

$$z_2^0 = \frac{b_2 z_1^0 + (1-a_k^2)e^{i\varphi/2}}{b_1}, \qquad t_0 = \frac{a_k^2 e^{i\varphi/2} - b_2 z_1^0}{b_1 b_2}. \tag{6}$$

The zeros of $f\left(\sqrt{\left(z_1^0 + b_1 t\right)\left(z_2^0 + b_2 t\right)}\right)$ can be found from the equation

$$\left(z_1^0 + b_1 t\right)\left(z_2^0 + b_2 t\right) = b_1 b_2 t^2 + \left(z_1^0 b_2 + z_2^0 b_1\right) t + z_1^0 z_2^0 = c_l^2, \quad l\in\mathbb{N}.$$

Consider its roots

$$t_l = \frac{-\left(b_2 z_1^0 + b_1 z_2^0\right) + \sqrt{\left(b_2 z_1^0 - b_1 z_2^0\right)^2 + 4c_l^2 b_1 b_2}}{2b_1 b_2}.$$

Let $\varphi_l = \arg c_l$. The condition of a zero t_l belongs to r-neighborhood of the point t_0 has the form $|t_l - t_0| < r$ or

$$r > \left| a_k^2 e^{i\varphi/2} - b_2 z_1^0 - \frac{-\left(2b_2 z_1^0 + \left(1-a_k^2\right)e^{i\varphi/2}\right) + \sqrt{\left(a_k^2-1\right)^2 e^{i\varphi} + 4e^{i\varphi}|c_l|^2}}{2} \right| \Longleftrightarrow$$

$$\Longleftrightarrow 2r > \left| a_k^2 + 1 + \sqrt{\left(a_k^2 - 1\right)^2 + 4|c_l|^2(\cos 2\varphi_l + i\sin 2\varphi_l)} \right| =$$

$$= \left| a_k^2 + 1 + \sqrt{\left(a_k^4 - 2a_k^2 + 1 + 4|c_l|^2\cos 2\varphi_l\right) + 4i|c_l^2|\sin 2\varphi_l} \right|. \tag{7}$$

To calculate a square root of complex number we suppose $x\in\mathbb{R},\ y\in\mathbb{R}$ and use

$$\sqrt{x+iy} = \pm\left(\sqrt{\frac{\sqrt{x^2+y^2}+x}{2}} + i\sqrt{\frac{\sqrt{x^2+y^2}-x}{2}}\right). \tag{8}$$

Applying (8) to (7), we obtain

$$2r > \Bigg| a_k^2 + 1 \pm \frac{1}{\sqrt{2}} \Bigg(\sqrt{\Big(\big(a_k^2-1\big)^2 + 4|c_l|^2\cos 2\varphi_l\Big)^2 + 16|c_l|^4 \sin^2 2\varphi_l} + $$

$$+ 4|c_l|^2 \cos 2\varphi_l + \big(a_k^2-1\big)^2 \Bigg)^{1/2} \pm$$

$$\pm \frac{i}{\sqrt{2}} \Bigg(\sqrt{\Big(\big(a_k^2-1\big)^2 + 4|c_l|^2\cos 2\varphi_l\Big)^2 + 16|c_l|^4 \sin^2 2\varphi_l} - $$

$$- 4|c_l|^2 \cos 2\varphi_l - \big(a_k^2-1\big)^2 \Bigg)^{1/2} \Bigg|. \tag{9}$$

We choose a minus before square roots in (9). But for real x and y the inequality $|x+iy| \le |x|+|y|$ holds. We suppose that real and image parts of expression in modulus (9) don't exceed $r,$ i.e.,

$$\sqrt{2}r > \Bigg| \sqrt{2}\left(a_k^2+1\right) - \Big(4|c_l|^2 \cos 2\varphi_l + \big(a_k^2-1\big)^2 + $$

$$+ \sqrt{\Big(\big(a_k^2-1\big)^2 + 4|c_l|^2\cos 2\varphi_l\Big)^2 + 16|c_l|^4 \sin^2 2\varphi_l} \Big)^{1/2} \Bigg| \tag{10}$$

and

$$\frac{1}{\sqrt{2}} \left| 4|c_l|^2 \cos 2\varphi_l + \big(a_k^2-1\big)^2 - \sqrt{\Big(\big(a_k^2-1\big)^2 + 4|c_l|^2\cos 2\varphi_l\Big)^2 + 16|c_l|^4 \sin^2 2\varphi_l} \right|^{1/2} < r. \tag{11}$$

At first we prove that for some r and some ε and for all a_k the inequality (11) holds. From (11) it follows that

$$\big(4|c_l|^2 \cos 2\varphi_l + a_k^4 - 2a_k^2 + 1 - 2r^2\big)^2 < \big(a_k^4 - 2a_k^2 + 1 + 4|c_l|^2 \cos 2\varphi_l\big)^2 +$$

$$+ 16|c_l|^2 \sin^2 2\varphi_l < \big(4|c_l|^2 \cos 2\varphi_l + a_k^4 - 2a_k^2 + 1 + 2r^2\big)^2$$

or

$$\big(4|c_l|^2 \cos 2\varphi_l + a_k^4 - 2a_k^2 + 1\big)^2 + 4r^4 - 4r^2 \left(4|c_l|^2 \cos 2\varphi_l + a_k^4 - 2a_k^2 + 1\right) <$$

$$< \big(a_k^4 - 2a_k^2 + 1 + 4|c_l|^2 \cos 2\varphi_l\big)^2 + 16|c_l|^2 \sin^2 2\varphi_l <$$

$$< \big(4|c_l|^2 \cos 2\varphi_l + a_k^4 - 2a_k^2 + 1\big)^2 + 4r^4 + 4r^2 \left(4|c_l|^2 \cos 2\varphi_l + a_k^4 - 2a_k^2 + 1\right).$$

Simplifying and reducing by 4 we deduce

$$r^4 - r^2 \left(4|c_l|^2 \cos 2\varphi_l + a_k^4 - 2a_k^2 + 1\right) < 4|c_l|^2 \sin^2 2\varphi_l <$$

$$< r^4 + r^2\left(4|c_l|^2\cos 2\varphi_l + a_k^4 - 2a_k^2 + 1\right). \tag{12}$$

Now we choose $\varepsilon \le \dfrac{\pi}{4}$, $r < 3$. Since $a_k \to \infty$ we suppose $\left(a_k^2 - 1\right)^2 > 9$, i.e., $|a_k|^2 > 4$. From here it follows that

$$r^4 - r^2\left(4|c_l|^2\cos 2\varphi_l + a_k^4 - 2a_k^2 + 1\right) = r^2\Big(r^2 - 4|c_l|^2\cos 2\varphi_l - \left(a_k^2 - 1\right)^2\Big) < 0$$

for $|\varphi_l| < \dfrac{\pi}{4}$. Therefore, the left-hand side in (12) is negative.

The right-hand side in (12) is equivalent to

$$4|c_l|^2\left(\sin^2 2\varphi_l - r^2\cos 2\varphi_l\right) < r^4 + r^2\left(a_k^2 - 1\right)^2. \tag{13}$$

For validity of (13) we choose $r \in (2;3)$.

Then $\sin^2 2\varphi_l - r^2\cos 2\varphi_l < \sin^2 2\varphi_l - 4\cos 2\varphi_l$. Now we require $\sin^2 2\varphi_l - 4\cos 2\varphi_l < 0$. It means that

$$\cos^2 2\varphi_l + 4\cos 2\varphi_l - 1 > 0.$$

Its solution is

$$\cos 2\varphi_l < -2 - \sqrt{5} \qquad \text{or} \qquad \cos 2\varphi_l > -2 + \sqrt{5}.$$

Hence, for $\varepsilon < \dfrac{1}{2}\arccos\left(-2 + \sqrt{5}\right) < \dfrac{\pi}{4}$ and $r \in (2;3)$ the inequality (11) is valid for all $|a_k|^2 > 4$.

Now we shall choose r and ε and construct sequence (a_k) such that inequality (10) is true.

That inequality is equivalent to the following estimate:

$$\begin{aligned}
&\sqrt{2}(a_k^2 + 1 - r) < \\
&\quad < \left(4|c_l|^2\cos 2\varphi_l + \left(a_k^2 - 1\right)^2 + \sqrt{\left(\left(a_k^2 - 1\right)^2 + 4|c_l|^2\cos 2\varphi_l\right)^2 + 16|c_l|^4\sin^2 2\varphi_l}\right)^{1/2} < \\
&\quad < \sqrt{2}(a_k^2 + 1 + r) \Longleftrightarrow 2\left(a_k^2 + 1 - r\right)^2 < \\
&\quad < 4|c_l|^2\cos 2\varphi_l + \left(a_k^2 - 1\right)^2 + \sqrt{\left(\left(a_k^2 - 1\right)^2 + 4|c_l|^2\cos 2\varphi_l\right)^2 + 16|c_l|^4\sin^2 2\varphi_l} < \\
&\quad < 2\left(a_k^2 + 1 + r\right)^2.
\end{aligned} \tag{14}$$

Putting $r = 2$ in (14), we strengthen that inequality:

$$\begin{aligned}
&2\left(a_k^2 - 1\right)^2 < \\
&\quad < 4|c_l|^2\cos 2\varphi_l + \left(a_k^2 - 1\right)^2 + \sqrt{\left(\left(a_k^2 - 1\right)^2 + 4|c_l|^2\cos 2\varphi_l\right)^2 + 16|c_l|^4\sin^2 2\varphi_l} < \\
&\quad < 2\left(a_k^2 + 3\right)^2 \Longleftrightarrow \\
&\Longleftrightarrow \left(a_k^2 - 1\right)^2 < 4|c_l|^2\cos 2\varphi_l + \sqrt{\left(\left(a_k^2 - 1\right)^2 + 4|c_l|^2\cos 2\varphi_l\right)^2 + 16|c_l|^4\sin^2 2\varphi_l} <
\end{aligned}$$

$$< a_k^4 + 14a_k^2 + 17. \tag{15}$$

But

$$4|c_l|^2\cos 2\varphi_l + \sqrt{\Big(\big(a_k^2-1\big)^2 + 4|c_l|^2\cos 2\varphi_l\Big)^2 + 16|c_l|^4\sin^2 2\varphi_l} >$$

$$> \big(a_k^2-1\big)^2 + 4|c_l|^2\cos 2\varphi_l > \big(a_k^2-1\big)^2.$$

Thus, the left inequality in (15) is obvious. The middle expression in (15) does not exceed $\big(a_k^2-1\big)^2 + 8|c_l|^2\cos 2\varphi_l + 4|c_l|^2|\sin 2\varphi_l|$. We obtain inequality

$$a_k^4 - 2a_k^2 + 1 + 4|c_l|^2(2\cos 2\varphi_l + |\sin 2\varphi_l|) < a_k^4 + 14a_k^2 + 17 \iff$$

$$\iff 4|c_l|^2(2\cos 2\varphi_l + |\sin 2\varphi_l|) - 16 < 16a_k^2.$$

Hence, we must choose a_k such that

$$a_k^2 > \frac{1}{4}|c_l|^2(2\cos 2\varphi_l + |\sin 2\varphi_l|) - 1.$$

For $k+1$ zeros of $f(t)$ we pick a_k satisfying this condition

$$a_k^2 > \max_{1\le l\le k+1} \frac{|c_l|^2}{4}(2\cos 2\varphi_l + |\sin 2\varphi_l|) - 1.$$

Thus, $\exists r > 0$ (we have $r \in (2;3)$) $\forall k \in \mathbb{N}$ $\exists z^0 \in \mathbb{C}^2$ $\exists t^0 \in \mathbb{C}$ (see (6)) that

$$n\left(r, z^0, t^0, \frac{1}{f}\right) \ge k+1 > k.$$

We conclude that function $f\big(\sqrt{z_1 z_2}\big)$ is of unbounded index in direction $\mathbf{b}$.

Combining Theorem 1 and Theorem 4 in one statement, we get the following

Theorem 5. *Let $f(t)$, $t \in \mathbb{C}$, be an even entire transcendental function of bounded index. Then:*

(i) *for each direction $\mathbf{b} = (b_1, b_2) \in \mathbb{C}^2 \setminus \{0\}$ and for every fixed $z_1^0, z_2^0 \in \mathbb{C}$ the function $g(t) = f\left(\sqrt{\big(z_1^0 + b_1 t\big)\big(z_2^0 + b_2 t\big)}\right)$ is an entire function of bounded index $(t \in \mathbb{C})$;*

(ii) *the function $f\big(\sqrt{z_1 z_2}\big)$ is of unbounded index in each direction $\mathbf{b}$.*

References

1. *Bandura A. I, Skaskiv O. B.* Entire functions of bounded l-index in direction // Mat. Stud. – 2007. – **27**, № 1. – P. 30 – 52.
2. *Bandura A. I., Skaskiv O. B.* Entire functions of several variables of bounded index // Lviv: Publisher I. E. Chyzhykov, 2016. – 128 p.
3. *Kuzyk A. D, Sheremeta M. N.* Entire functions of bounded l-distribution of values // Math. Notes. – 1986. – **39**, № 1. – P. 3 – 8.
4. *Lepson B.* Differential equations of infinite order, hyperdirichlet series and entire functions of bounded index // Proc. Sympos. Pure Math. – 1968. – **11**. – P. 298 – 307.
5. *Shah S. M.* Entire function of bounded index // Lect. Notes Math. – Berlin; Heidelberg: Springer, 1977. – **599**. – P. 117 – 145.

6. *Hayman W. K.* Differential inequalities and local valency // Pacific J. Math. – 1973. – **44**, № 1. – P. 117 – 137.

7. *Shah S. M.* Entire functions of bounded index // Proc. Amer. Math. Soc. – 1968. – **19**, № 5. – P. 1017 – 1022.

8. *Bandura A. I., Skaskiv O. B.* Sufficient sets for boundedness L-index in direction for entire functions // Mat. Stud. – 2008. – **30**, № 2. – P. 177 – 182.

9. *Nuray F., Patterson R. F.* Entire bivariate functions of exponential type // Bull. Math. Sci. – 2015. – **5**, № 2. – P. 171 – 177.

10. *Nuray F., Patterson R. F.* Multivalence of bivariate functions of bounded index // Matematiche (Catania). – 2015. – **70**, № 2. – P. 225 – 233.

11. *Bandura A. I., Skaskiv O. B.* Open problems for entire functions of bounded index in direction // Mat. Stud. – 2015. – **43**, № 1. – P. 103 – 109.

12. *Bandura A. I., Skaskiv O. B.* Entire functions of bounded and unbounded index in direction // Mat. Stud. – 2007. – **27**, № 2. – P. 211 – 215.

13. *Bandura A. I.* A class of entire functions of unbounded index in each direction // Mat. Stud. – 2015. – **44**, № 1. – P. 107 – 112.

14. *Bandura A., Skaskiv O.* Boundedness of L-index in direction of entire solutions of second order PDE // Acta Comment. Univ. Tartu. Math. (in print).

15. *Bandura A. I., Skaskiv O. B.* Directional logarithmic derivative and the distribution of zeros of an entire function of bounded L-index along the direction // Ukr. Math. J. – 2017. – **69**, № 3. – P. 500 – 508.

16. *Bandura A. I.* Some improvements of criteria of L-index boundedness in direction // Mat. Stud. – 2017. – **47**, № 1. – P. 27 – 32.

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