UDC 517.9

ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A THIRD ORDER NONLINEAR DIFFERENTIAL EQUATION

АСИМПТОТИКА РОЗВ'ЯЗКІВ НЕЛІНІЙНОГО ДИФЕРЕНЦІАЛЬНОГО РІВНЯННЯ ТРЕТЬОГО ПОРЯДКУ

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The asymptotic properties of solutions of some third order differential equation are examined. Sufficient conditions for the square integrability and oscillation of solutions are established.

Вивчаються асимптотичні властивості розв'язків деякого рівняння третього порядку. Встановлено достатні умови квадратичної інтегровності та осциляційності розв'язків.

1. Introduction. The aim of this paper is to study the qualitative behavior of solutions of a class of nonlinear differential equations of the third order. In particular, we give sufficient conditions for the existence of square integrable solutions of the nonlinear equation

$$(x''(t) + p(t)x(t))' + p(t)x'(t) + q(t)f(x(t)) = 0, \tag{1.1}$$

where we assume that $p, q\colon [t_0, \infty) \to \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$ are continuous, $q(t) > 0$, and $xf(x) > 0$ for $x \neq 0$. Some necessary and sufficient relationships between the square integrability of the first and second derivatives of solutions are also presented.

The problem of obtaining sufficient conditions to ensure that all solutions of certain classes of third order nonlinear differential equations are oscillatory or nonoscillatory is an important one in the study of qualitative theory of ordinary differential equations. This problem has been the subject of intensive study in the last three decades, and we refer the reader to the monographs of Greguš [6] and Kiguradze and Chanturia [10], as well as the papers [2–5, 7–9, 11–16] and the references contained therein. By a *solution* of (1.1), we mean a function $x \in C^3([T_x, \infty))$, $T_x \geq t_0$, that satisfies (1.1) on $[T_x, \infty)$. We only consider those solutions $x(t)$ of (1.1) that are continuable and nontrivial, i.e., that satisfy $\sup\{|x(t)|\colon t \geq T_x\} > 0$ for all $T_x \geq t_0$. We assume that (1.1) possesses such a solution. A nontrivial solution of (1.1) is said to be *oscillatory* if it has a sequence of zeros tending to infinity, and it is called *nonoscillatory* otherwise. An equation is said to be oscillatory if all its solutions are oscillatory.

2. Preliminary results. First, we give some lemmas that we will use in the proofs of our main results.

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Lemma 2.1. *Let* $y(t)$ *be a continuous and twice differentiable function on the interval* $[t_0, +\infty)$ *such that* $y(t) > 0$ *for* $t \geq t_0$. *If* $\lim_{t\to\infty} \dfrac{y'(t)}{y(t)} = -\infty$, *then* $\lim_{t\to\infty} y(t) = 0$. *If, in addition,* $y''(t) > 0$, *then* $\lim_{t\to\infty} y'(t) = 0$.

Proof. Since $y(t) > 0$, it follows that $y'(t) < 0$ for large t and hence $\lim_{t\to\infty} y(t)$ exists and $y(t)$ is eventually decreasing. Suppose that $\lim_{t\to\infty} y(t) = \lambda > 0$. Let $\delta < 0$ be a number satisfying $\dfrac{y'(t)}{y(t)} < \delta$ on $[t_1, +\infty)$ for some $t_1 \geq t_0$. Then $y'(t) < \delta y(t) < \delta\lambda < 0$. This implies $y(t) < 0$ for large t, which is a contradiction. If we have $y''(t) > 0$, then $\lim_{t\to\infty} y'(t) = c < 0$ would also give a contradiction.

Lemma 2.2. *Let* $x(t)$ *be a solution of (1.1). Then*

$$F[x(t)] = x(t)\left[x''(t) + p(t)x(t)\right] - \frac{1}{2}\, x'^2(t) \tag{2.1}$$

is nonincreasing on $[T, +\infty)$ *for some* $T \geq t_0$.

Proof. Differentiating and using the fact that $x(t)$ satisfies (1.1), we see that

$$F'[x(t)] = -q(t)x(t)f(x(t)) \leq 0. \tag{2.2}$$

Thus, $F[x(t)]$ is nonincreasing for large t.

Next we define two classes of solutions of equation (1.1) as follows.

Definition 2.1. *We say that a solution* $x(t)$ *of (1.1) belongs to Class I if* $F[x(t)] \geq 0$ *on* $[T_x, +\infty)$ *for some* $T_x \geq t_0$.

Definition 2.2. *We say that a solution* $x(t)$ *of (1.1) belongs to Class II if* $F[x(T)] < 0$ *for some* $T > T_x$.

3. Main results I. In this section, we study asymptotic properties of solutions of equation (1.1) that belong to the Class I. Note that while we have assumed that $q(t) > 0$ for $t \geq t_0$, the function $p(t)$ need not be of constant sign. We also assume that the following conditions hold:

(H_1) there are constants α and β such that

$$-\infty < \alpha \leq \inf p(t) \leq \sup p(t) \leq \beta < \infty \quad \text{and} \quad \omega = \max\{|\alpha|, |\beta|\};$$

(H_2) $\displaystyle\int_{t_0}^{\infty} p(s)ds = +\infty$;

(H_3) there exist constants $N \geq M > 0$ such that

$$0 < M \leq \frac{f(x)}{x} \leq N \quad \text{for all} \quad x \neq 0;$$

(H_4) there exists $\lambda > 0$ such that $\lim_{t\to\infty} \dfrac{1}{t}\displaystyle\int_{t_0}^{t} q(s)ds = \lambda < \infty$;

(H_5) there exists $\mu > 0$ such that $|p'(t)| \leq \mu < \infty$ for all $t \geq t_0$.

We are now ready to prove our first result.

Theorem 3.1. *Let* $x(t)$ *be a solution of equation (1.1) belonging to Class I and assume that* $q'(t) \geq 0$ *and conditions (*H_1*) and (*H_3*)–(*H_5*) hold. Then:*

(i) $\displaystyle\int^{\infty} x^2(s)ds < \infty$;

(ii) $\int^{\infty} x'^2(s)ds < \infty$;

(iii) $\int^{\infty} x''^2(s)ds < \infty$.

Proof. (i) Let $x(t)$ be a solution in Class I and choose $t_1 \geq t_0$ such that $F(x(t)) \geq 0$ for $t \geq t_1$. Multiplying (1.1) by $x(t)$, we have

$$x(t)\left[x''(t)+p(t)x(t)\right]' + p(t)x(t)x'(t) + q(t)x(t)f(x(t)) = 0,$$

and then integrating by parts from t_1 to t, we obtain

$$F[x(t)] - F[x(t_1)] + \int_{t_1}^{t} q(s)x(s)f(x(s))ds = 0. \tag{3.1}$$

Now $x(t)f(x(t)) \geq Mx^2(t)$ by (H_3), and since $x(t)$ belongs to Class I, from (3.1) we see that there is a positive constant c such that

$$\int_{t_1}^{t} x^2(s)ds \leq c < \infty$$

for all $t \geq t_1$. Hence, (i) holds.

(ii) Suppose $\int_{t_1}^{\infty} x'^2(t) = \infty$. Since x belongs to Class I,

$$x(t)\left[x''(t)+p(t)x(t)\right] \geq \frac{1}{2}x'^2(t)$$

for $t \geq t_1$. Integrating from t_1 to t, we obtain

$$\frac{3}{2}\int_{t_1}^{t} x'^2(s)ds \leq k_0 + x(t)x'(t) + \int_{t_1}^{t} p(s)x^2(s)ds,$$

where $k_0 = x(a)x'(a)$. Thus, from (H_1),

$$\frac{3}{2}\int_{t_1}^{t} x'^2(s)ds \leq k_1 + x(t)x'(t)$$

where $k_1 = k_0 + \omega c$. Now $\lim_{t\to\infty}\int_{t_1}^{t} x'^2(s)ds = \infty$ implies that for any $A > \frac{2k_1}{3}$ there exist $t_2 \geq t_1$ such that

$$0 < \frac{3}{2}A - k_1 \leq x(t)x'(t) \quad \text{for} \quad t \geq t_2.$$

But

$$\int_{t_2}^{t} x(s)x'(s)ds = \frac{1}{2}\,x^2(t) - \frac{1}{2}\,x^2(t_2) > \left(\frac{3}{2}\,A - k_1\right)(t - t_2) \quad \text{for} \quad t \geq t_2.$$

This implies $\lim_{t\to\infty}\int_{t_2}^{t} x^2(s)ds = \infty$ contradicting part (i). Hence, there exists $d > 0$ such that

$$\int_{t_1}^{t} x'^2(s)ds \leq d < \infty$$

for all $t \geq t_1$, and so (ii) holds.

(iii) Now assume that $\int_{t_1}^{t} x''^2(s)ds = \infty$. Multiplying (1.1) by $x'(t)$ and integrating from t_1 to t, we obtain

$$\int_{t_1}^{t} x'(s)\left(x''(s) + p(s)x(s)\right)' ds = -\int_{t_1}^{t} p(s)x'^2(s)ds - \int_{t_1}^{t} q(s)x'(s)f(x(s))ds.$$

Integrating the term on the left-hand side by parts, we have

$$\int_{t_1}^{t} x'(s)\left(x''(s) + p(s)x(s)\right)' ds = x'(t)\left(x''(t) + p(t)x(t)\right) -$$

$$-\int_{t_1}^{t} x''^2(s)ds - \int_{t_1}^{t} p(s)x(s)x''(s)ds - C_0,$$

where $C_0 = x'(t_1)\left(x''(t_1) + p(t_1)x(t_1)\right)$. Thus,

$$\int_{t_1}^{t} x''^2(s)ds = -\int_{t_1}^{t} p(s)x(s)x''(s)ds + \int_{t_1}^{t} p(s)x'^2(s)ds + x'(t)x''(t) +$$

$$+ p(t)x'(t)x(t) + \int_{t_1}^{t} q(s)x'(s)f(x(s))ds - C_0. \tag{3.2}$$

Integrating the first term on the right by parts, applying conditions (H_1), (H_3), and (H_5) as well as parts (i) and (ii) in the proof, we obtain

$$\int_{t_1}^{t} x''^2(s)ds = C_1 p(t_1) + 2\omega d + \int_{t_1}^{t} p'(s)x(s)x'(s)ds + x'(t)x''(t) +$$

$$+ \int_{t_1}^{t} q(s)x'(s)f(x(s))ds - C_0 \le C_1 p(t_1) + 2\omega d - C_0 +$$

$$+ \frac{1}{2}\int_{t_1}^{t} \left|p'(s)\right| \left(x^2(s) + x'^2(s)\right) ds + x'(t)x''(t) + \int_{t_1}^{t} q(s)x'(s)f(x(s))ds \le$$

$$\le C_1 p(t_1) + 2\omega d - C_0 + \frac{1}{2}\mu(c+d) + x'(t)x''(t) +$$

$$+ \frac{1}{2}\int_{t_1}^{t} q(s)\left[x'^2(s) + f^2(x(s))\right] ds \le C_1 p(t_1) + 2\omega d - C_0 +$$

$$+ \frac{1}{2}\mu(c+d) + x'(t)x''(t) + \frac{1}{2} q(t)\left[d + N^2 c\right] \le$$

$$\le K_0 + x'(t)x''(t) + K_1 q(t), \tag{3.3}$$

since $q(t)$ is increasing, where $C_1 = x(t_1)x'(t_1)$, $K_0 = C_1 p(t_1) + 2\omega d - C_0 + \frac{1}{2}\mu(c+d)$, and $K_1 = \frac{1}{2}[d + N^2 c]$.

Since $\int_{t_1}^{\infty} x''^2(s)ds = \infty$, we can find $C > K_0 + k_1\lambda > 0$ and $t_2 \ge t_1$ such that

$$0 < C - K_0 \le x'(t)x''(t) + K_1 q(t) \tag{3.4}$$

for $t \ge t_2$. Now, integrating (3.4) from t_2 to t, we obtain

$$(C - K_0)(t - t_2) - K_1 \int_{t_2}^{t} q(s)ds \le \frac{1}{2} x'^2(t),$$

or

$$t\left[(C - K_0) - \frac{t_2}{t}(C - K_0) - \frac{K_1}{t}\int_{t_2}^{t} q(s)ds)\right] \le \frac{1}{2} x'^2(t).$$

In view of our choice of C and condition (H_4), we see that $x'^2(t) \to \infty$ as $t \to \infty$. This contradicts part (ii) of the theorem and completes the proof.

In our next theorem, we replace condition (H_4) by asking instead that the function q be bounded above.

Theorem 3.2. *Let $x(t)$ be a solution of equation (1.1) belonging to Class I and assume that (H$_1$), (H$_3$), and (H$_5$) hold, and there exists $q_1 > 0$ such that $q(t) \le q_1$ for all $t \ge t_0$. Then the conclusion of Theorem 3.1 holds.*

Proof. The proof of cases (i) and (ii) are the same as in the proof of the previous theorem. In case (iii), from (3.3), we have

$$\int_{t_1}^{t} x''^2(s)ds \le K_0 + x'(t)x''(t) + K_1 q_1.$$

Similar to what we did in the proof of Theorem 3.1, there exist $C_1 > 0$ and $t_3 \ge t_0$ such that

$$0 < C_1 \le x'(t)x''(t).$$

An integration again contradicts part (ii) and completes the proof.

4. Main result II. In this section of the paper we consider solutions of equation (1.1) that belong to the Class II. In our first result, we give sufficient conditions for a solution to be either oscillatory or to converge to zero.

Theorem 4.1. *Assume that condition (H$_2$) holds and let $x(t)$ be a solution of equation (1.1) belonging to the Class II. Then either $x(t)$ is oscillatory or $x(t) \to 0$ as $t \to \infty$.*

Proof. Let x be a nonoscillatory solution of (1.1), say $x(t) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$. (The proof if $x(t) < 0$ for $t \ge t_1$ is similar.) To prove the theorem, we need to show that $x(t) \to 0$ as $t \to \infty$. In view of Lemma 2.2, $F[x(t)] < 0$ for $t \ge t_2$ for some $t_2 \ge t_1$. Define

$$R[x(t)] = \frac{x'(t)}{x(t)} + \int_{t_2}^{t} p(s)ds;$$

then

$$R'[x(t)] = \frac{F[x(t)]}{x^2(t)} - \frac{1}{2}\left[\frac{x'(t)}{x(t)}\right]^2 < 0,$$

and so $R[x(t)]$ is decreasing on $[t_2, \infty)$. This, together with condition (H$_2$), implies

$$\lim_{t\to\infty} \frac{x'(t)}{x(t)} = -\infty.$$

By Lemma 2.1, $x(t) \to 0$ as $t \to \infty$, and this proves the theorem.

The above theorem easily yields the following corollary.

Corollary 4.1. *Assume that (H$_2$) holds. Then, any solution $x(t)$ of (1.1) that has one zero is either oscillatory or converges to 0 as $t \to \infty$.*

Proof. Suppose $x(t)$ is a solution of (1.1) that has a zero at some $t_1 \ge t_0$. Since $F[x(t_1)] = -\frac{1}{2} x'^2(t_1) \le 0$, $x(t)$ belongs to Class II, and the conclusion follows from the theorem.

Next, we consider equation (1.1) with:

(H$_6$) there exists a constant σ such that $p(t) \ge \sigma > 0$;

(H$_7$) $q'(t) \le 0$.

Theorem 4.2. *Assume that conditions (H_2) and (H_5) – (H_7) hold and let $x(t)$ be a solution of equation (1.1) belonging to Class II. Then:*

(iv) $\int^{\infty} x'^2(s)ds = \infty$;

(v) $\int^{\infty} x''^2(s)ds = \infty$.

Proof. Let $x(t)$ be a solution of equation (1.1) belonging to the Class II. Then

$$F\left[x(t)\right] \le F\left[x(t_1)\right] < 0$$

for $t \ge t_1$ for some $t_1 \ge t_0$. Define

$$J[x(t)] = x(t)x'(t) - \frac{3}{2}\int\limits_{t_1}^{t} x'^2(s)ds. \tag{4.1}$$

Then, by (4.1),

$$J'[x(t)] = F\left[x(t)\right] - p(t)x^2(t) < F\left[x(t)\right] \le F\left[x(t_1)\right] < 0 \tag{4.2}$$

for $t \ge t_1$. Integrating (4.2) from t_1 to t gives

$$J[x(t)] < F[x(t_1)](t - t_1) - J[x(t_1)].$$

Hence,

$$J[x(t)] = x(t)x'(t) - \frac{3}{2}\int\limits_{t_1}^{t} x'^2(s)ds \to -\infty \quad \text{as} \quad t \to \infty.$$

If $\int_0^{\infty} x'^2(s)ds < \infty$, then $x(t)x'(t) \to -\infty$ as $t \to \infty$, so there exist $B > 0$ and $t_2 \ge t_1$ such that $x(t)x'(t) \le -B < 0$ for $t \ge t_2$. An integration shows that

$$x^2(t)/2 \le -B(t - t_2) + x^2(t_2)$$

for $t \ge t_2$ which is impossible. Thus, (iv) holds.

To prove (v), note that

$$F[x(t)] = x(t)\left[x''(t) + p(t)x(t)\right] - \frac{1}{2}x'^2(t) \le 0, \tag{4.3}$$

so from (3.2), (4.3), and (H_6), we obtain

$$-\int_T^t q(s)x'(s)f(x(s))ds + V_1(t) + C_0 + \int_T^t x''^2(s)ds =$$

$$= -\left[\int_T^t p(s)\left(x(s)x''(s) - \frac{1}{2}x'^2(s)\right)ds\right] + \frac{1}{2}\int_T^t p(s)x'^2(s)ds \geq$$

$$\geq \frac{1}{2}\int_T^t p(s)x'^2(s)ds \geq \frac{\sigma}{2}\int_T^t x'^2(s)ds,$$

where $V_1(t) = -x'(t)(x''(t) + p(t)x(t))$. Furthermore,

$$-\int_T^t q(s)x'(s)f(x(s))ds = -q(t)H(x(t)) + \int_T^t q'(t)H(x(s))ds + C_1,$$

where $H(x) = \int_0^x f(u)du \geq 0$ and $C_1 = q(T)H(x(T))$. Since $q'(t) \leq 0$, it follows that

$$C_1 + C_0 + V_1(t) + \int_T^t x''^2(s)ds \geq \frac{\sigma}{2}\int_T^t x'^2(s)ds. \tag{4.4}$$

By Theorem 4.1, either $x(t)$ oscillates or $x(t) \to 0$ as $t \to \infty$. Assume that $\int_{t_0}^t x''^2(s)ds < < \infty$. If $x(t)$ is oscillatory, then $x'(t)$ is oscillatory, so choose an increasing sequence $\{t_n\}$ of zeros of $x'(t)$. In view of (4.4), we must have $V_1(t) \to \infty$ as $t \to \infty$. But $V_1(t_n) = 0$ for $t = t_n$, $n = 1, 2, \ldots$, which contradicts $V_1(t) \to \infty$ as $t \to \infty$.

If $x(t) \to 0$ as $t \to \infty$, then there exists $\epsilon > 0$ such that $|x(t)| \leq \epsilon$ for large t, say for $t \geq T_1$ for some $T_1 \geq T$. Since $V_1(t) \to \infty$ as $t \to \infty$, there exist $D > \mu\epsilon^2$ and $T_2 \geq T_1$ such that

$$D < -x'(t)x''(t) - p(t)x(t)x'(t)$$

for $t \geq T_2$. Integrating, we obtain

$$D(t - T_2) < -\int_{T_2}^t x'(s)x''(s)ds - \int_{T_2}^t p(s)x(s)x'(s)ds \leq$$

$$\leq -\frac{1}{2}x'^2(t) - \frac{1}{2}p(t)x^2(t) + k + \int_{T_2}^t \left|p'(s)\right| x^2(s)ds,$$

where $k = \dfrac{1}{2}\,x'^2(T_2) + \dfrac{1}{2}\,p(T_2)x^2(T_2)$. From (H$_5$),

$$(D - \mu\epsilon^2)t < k + DT_2,$$

for $t \geq T_2$, which is impossible. This contradiction shows that part (v) holds and completes the proof of the theorem.

The following result is an immediate consequence of Theorems 3.1 and 4.1.

Theorem 4.3. *Assume that $q'(t) \geq 0$ and conditions (H_1) – (H_5) hold. If $x(t)$ is a nonoscillatory solution of equation (1.1), then* $\liminf_{t\to\infty} \|x(t)\| = 0$.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). If $x(t)$ belongs to Class II, the conclusion follows from Theorem 4.1. If $x(t)$ is in Class I, and $\liminf_{t\to\infty} |x(t)| > 0$, then there exists $\lambda > 0$ such that

$$|x(t)| \geq \lambda > 0.$$

This implies that $\lim_{t\to\infty} \displaystyle\int_{t_1}^{t} x^2(s)ds = \infty$, which contradicts part (i) of Theorem 3.1. This proves the theorem.

Before presenting our final result in this paper, notice that it follows from Theorems 3.1 and 4.2 that, if x is a solution in Class I, then

$$\int^{\infty} x'^2(s)ds < \infty,$$

and if

$$\int^{\infty} x'^2(s)ds < \infty,$$

then x is not in Class II, and so it must be in Class I. Therefore, we have the following necessary and sufficient relationships holding.

Theorem 4.4. *Assume that (H_1) – (H_3) and (H_5) – (H_7) hold, and let $x(t)$ be a solution of (1.1). Then:*

$$x(t) \quad \textit{belongs to Class I} \quad \Longleftrightarrow \left\{ \int_{t_1}^{\infty} x'^2(s)ds < \infty \right\} \Longleftrightarrow \left\{ \int_{t_1}^{\infty} x''^2(s)ds < \infty \right\},$$

$$x(t) \quad \textit{belongs to Class II} \quad \Longleftrightarrow \left\{ \int_{t_1}^{\infty} x'^2(s)ds = \infty \right\} \Longleftrightarrow \left\{ \int_{t_1}^{\infty} x''^2(s)ds = \infty \right\}.$$

Proof. Once we note that condition (H$_7$) implies $q(t)$ is bounded from above, the conclusions follow from Theorems 3.2 and 4.2.

Concluding remarks. In view of the results above, it would be reasonable to ask what, if anything, can be said about the square integrability of solutions belonging to Class II. In this case we do know that $F[x(t)]$ is negative and decreasing for $t \geq T$, T large enough, so $F[x(t)] \to$

$\to F_0 \geq -\infty$. If we assume for the moment that $0 < q_0 \leq q(t) \leq q_1$, then from ($H_3$) and (2.2), we have

$$q_0 M \int\limits_T^\infty x^2(s)ds \leq -F[x(t)] + F[x(T)] \leq q_1 N \int\limits_T^\infty x^2(s)ds. \tag{4.5}$$

If $F_0 = -\infty$, then

$$\int\limits_T^\infty x^2(s)ds = \infty$$

as one might suspect for a solution in Class II. However, if $F_0 > -\infty$, then from the left-hand side of (4.5) we see that

$$\int\limits_T^\infty x^2(s)ds < \infty.$$

In order to exclude this situation, a condition implying $q(t) \to 0$ at $t \to \infty$ might be needed. This is an open question at this time. Another interesting problem would be to establish sufficient conditions for solutions of equation (1.1) to belong to Class I or II.

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Received 18.09.15,
after revision — 07.10.16