

**BOUNDED SOLUTIONS TO BOUNDARY-VALUE
HYPERBOLIC PROBLEMS**

ОБМЕЖЕНІ РОЗВ'ЯЗКИ ГІПЕРБОЛІЧНИХ ГРАНИЧНИХ ЗАДАЧ

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We investigate linear boundary-value problems for first-order one-dimensional hyperbolic systems in a strip. We establish conditions for existence and uniqueness of bounded continuous solutions. For that we suppose that the nondiagonal part of the zero-order coefficients vanish at infinity. Moreover, we establish a dissipativity condition in terms of the boundary data and the diagonal part of the zero-order coefficients.

Вивчаються лінійні граничні задачі для одновимірних гіперболічних систем першого порядку у смугі. Знайдено умови існування та єдиності обмежених неперервних розв'язків за умови занулення на нескінченності недіагональної частини коефіцієнтів нульового порядку. Також встановлено умови дисипативності в термінах граничних даних та діагональної частини коефіцієнтів нульового порядку.

1. Introduction. 1.1. Problem setting and our result. We investigate the general linear first-order hyperbolic system in a single space variable

$$\partial_t u_j + a_j(x, t) \partial_x u_j + \sum_{k=1}^n b_{jk}(x, t) u_k = f_j(x, t), \quad (x, t) \in (0, 1) \times \mathbb{R}, \quad j \leq n, \quad (1.1)$$

subject to the boundary conditions

$$\begin{aligned} u_j(0, t) &= (Ru)_j(t), \quad 1 \leq j \leq m, \quad t \in \mathbb{R}, \\ u_j(1, t) &= (Ru)_j(t), \quad m < j \leq n, \quad t \in \mathbb{R}, \end{aligned} \quad (1.2)$$

where $0 \leq m \leq n$ are positive integers and $R = (R_1, \dots, R_n)$ is a linear bounded operator from BC_n to $BC_n(\mathbb{R})$. Here and below by BC_n we will denote the vector space of all bounded and continuous maps $u: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}^n$, with the norm

$$\|u\|_\infty = \max_{j \leq n} \max_{x \in [0, 1]} \sup_{t \in \mathbb{R}} |u_j|.$$

Similarly, by BC_n^1 we will denote a Banach space of all $u \in BC_n$ such that $\partial_x u, \partial_t u \in BC_n$, with the norm

$$\|u\|_1 = \|u\|_\infty + \|\partial_x u\|_\infty + \|\partial_t u\|_\infty.$$

Also, we use the notation $BC_n(\mathbb{R})$ for the space of all bounded and continuous maps $v: \mathbb{R} \rightarrow \mathbb{R}^n$ and the notation $BC_n^1(\mathbb{R})$ for the space of all $v \in BC_n(\mathbb{R})$ with $v' \in BC_n(\mathbb{R})$. For simplicity, we will skip the subscript n if $n = 1$.

We make the following assumptions on the coefficients of (1.1):

$$a_j, b_{jk} \in BC^1 \quad \text{for all } j \leq n \quad \text{and } k \leq n, \quad (1.3)$$

$$\inf_{j,x,t} a_j > 0 \quad \text{for all } j \leq m \quad \text{and} \quad \sup_{j,x,t} a_j < 0 \quad \text{for all } m < j \leq n. \quad (1.4)$$

Suppose also that

$$\begin{aligned} & \text{the restriction of the operator } R \text{ to } BC_n^1 \\ & \text{is a linear bounded operator from } BC_n^1 \text{ to } BC_n^1(\mathbb{R}) \end{aligned} \quad (1.5)$$

and

$$\text{for all } 1 \leq j \neq k \leq n \text{ there exists } \tilde{b}_{jk} \in BC^1 \text{ such that } b_{jk} = \tilde{b}_{jk}(a_k - a_j). \quad (1.6)$$

Let us introduce the characteristics of the hyperbolic system (1.1). Given $j \leq n$, $x \in [0, 1]$, and $t \in \mathbb{R}$, the j th characteristic is defined to be the solution $\xi \in [0, 1] \mapsto \omega_j(\xi, x, t) \in \mathbb{R}$ of the initial value problem

$$\partial_\xi \omega_j(\xi, x, t) = \frac{1}{a_j(\xi, \omega_j(\xi, x, t))}, \quad \omega_j(x, x, t) = t. \quad (1.7)$$

To shorten the notation, we will simply write $\omega_j(\xi) = \omega_j(\xi, x, t)$. Set

$$c_j(\xi, x, t) = \exp \int_x^\xi \left(\frac{b_{jj}}{a_j} \right) (\eta, \omega_j(\eta)) d\eta, \quad d_j(\xi, x, t) = \frac{c_j(\xi, x, t)}{a_j(\xi, \omega_j(\xi))}. \quad (1.8)$$

Integration along the characteristic curves transforms the system (1.1), (1.2) to an integral form,

$$u_j(x, t) = c_j(0, x, t)(Ru)_j(w_j(0)) - \int_0^x d_j(\xi, x, t) \sum_{k \neq j} b_{jk}(\xi, \omega_j(\xi)) u_k(\xi, \omega_j(\xi)) d\xi + \\ + \int_0^x d_j(\xi, x, t) f_j(\xi, \omega_j(\xi)) d\xi, \quad 1 \leq j \leq m, \quad (1.9)$$

$$u_j(x, t) = c_j(1, x, t)(Ru)_j(w_j(1)) - \int_1^x d_j(\xi, x, t) \sum_{k \neq j} b_{jk}(\xi, \omega_j(\xi)) u_k(\xi, \omega_j(\xi)) d\xi + \\ + \int_1^x d_j(\xi, x, t) f_j(\xi, \omega_j(\xi)) d\xi, \quad m < j \leq n. \quad (1.10)$$

By a straightforward calculation, one can easily show that a C^1 -map $u: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}^n$ is a solution to the PDE problem (1.1), (1.2) if and only if it satisfies the system (1.9), (1.10). This motivates the following definition.

Definition 1.1. A function $u \in BC_n$ is called a bounded continuous solution to the problem (1.1), (1.2) if it satisfies (1.9) and (1.10).

Introduce an operator $C: BC_n \rightarrow BC_n$ by

$$(Cv)_j(x, t) = \begin{cases} c_j(0, x, t)(Rv)_j(\omega_j(0)) & \text{for } 1 \leq j \leq m, \\ c_j(1, x, t)(Rv)_j(\omega_j(1)) & \text{for } m < j \leq n. \end{cases} \quad (1.11)$$

Theorem 1.1. Suppose that the conditions (1.3)–(1.6) are fulfilled. Moreover, assume that there exists $\ell \in \mathbb{N}$ such that

$$\|C^\ell\|_{\mathcal{L}(BC_n)} < 1, \quad (1.12)$$

and

$$\text{for all } \varepsilon > 0 \text{ there exists a compact interval } I \subset \mathbb{R} \text{ such that} \\ |b_{jk}(x, t)| < \varepsilon \text{ for all } 1 \leq j \neq k \leq n, x \in [0, 1] \text{ and } t \in \mathbb{R} \setminus I. \quad (1.13)$$

Then the problem (1.1), (1.2) has a unique continuous bounded solution u .

Example 1.1. The following example shows that if the conditions (1.6) and (1.13) are not satisfied, then the statement of Theorem 1.1 is not true, in general. Specifically, we consider the problem

$$\partial_t u_1 + \frac{2}{\pi} \partial_x u_1 - u_2 = 0, \\ \partial_t u_2 + \frac{2}{\pi} \partial_x u_2 + u_1 = 0, \quad (1.14)$$

$$u_1(0, t) = 0, \quad u_2(1, t) = 0. \quad (1.15)$$

Obviously, the problem (1.14), (1.15) is a particular case of the problem (1.1), (1.2) and satisfies all assumptions of Theorem 1.1 with the exception of (1.6) and (1.13). It is straightforward to check that

$$u_1 = \sin \frac{\pi}{2} x \sin l \left(t - \frac{\pi}{2} x \right), \quad u_2 = \cos \frac{\pi}{2} x \sin l \left(t - \frac{\pi}{2} x \right), \quad l \in \mathbb{N},$$

make an infinite set of linearly independent bounded continuous (2π -periodic in t) solutions to the problem (1.14), (1.15). This means that the kernel of the operator of the problem (1.14), (1.15) is infinite dimensional. Thus, the uniqueness conclusion of Theorem 1.1 is not true.

In Subsection 1.2 we give a brief motivation of our investigations. In Section 2 we prove the Fredholm alternative for (1.1), (1.2) (Theorem 2.1), while in Section 3 — a uniqueness result for (1.1), (1.2) (Theorem 3.1). Theorem 1.1 will then straightforwardly follow from Theorems 2.1 and 3.1.

1.2. Motivation and state of the art. Systems of the type (1.1), (1.2) are used to model problems of laser dynamics [12, 14, 16], chemical kinetics [17], chemotaxis [15] and population dynamics [3]. Another area of applications of such models is boundary control problems [1, 13]. In many mathematical models the system (1.1) is controlled by the so-called reflection boundary conditions what is a particular case of (1.2).

In [2] Coppel proved the Dichotomy theorem for the linear ODE, namely $x' = A(t)x$. It claims that the inhomogeneous equation $x' = A(t)x + f(t)$ has a unique bounded continuous solution on \mathbb{R} for every bounded and continuous function f if and only if the homogeneous equation $x' = A(t)x$ has an exponential dichotomy on \mathbb{R} . In [10] the authors provide a criterion of the existence of exponential dichotomy on \mathbb{R} for a strongly continuous exponentially bounded evolution family on a Banach space in terms of existence and uniqueness of a bounded continuous mild solutions. In this respect, our result is an important step towards the existence of the exponential dichotomy for boundary-value hyperbolic problems.

The well-posedness of a particular case of (1.1), (1.2) was investigated in [5]. Specifically, the authors studied the system (1.1) with the boundary conditions

$$\begin{aligned} u_j(0, t) &= \mu_j(t), \quad j \leq m, \\ u_j(1, t) &= \mu_j(t), \quad m < j \leq n. \end{aligned} \quad (1.16)$$

and investigated existence and uniqueness of continuous but *not necessarily bounded* solutions. The main assumption imposed in [5] is a smallness of all b_{jk} in a neighborhood of $-\infty$. It comes from the Banach fixed point argument used in the proof of the main result. In comparison, in the present paper, we allow for b_{jj} to be elements of BC only, and for b_{jk} with $j \neq k$ we impose the assumption (1.13). Moreover, after the change of variables $u_j \rightarrow v_j = u_j - \mu_j(t)$ in (1.1) and (1.16) we get $C = 0$. This means that the dissipativity conditions (1.12) is satisfied here automatically (with $\ell = 1$).

In [7–9] time-periodic solutions to the system (1.1) with reflection boundary conditions are investigated. It is suggested a rather general approach to proving the Fredholm alternative in spaces of time-periodic functions (in the autonomous case [8]) and in the space of continuous and time-periodic functions (in the nonautonomous case [9]). In the present paper, we extend

this approach from the spaces of periodic functions to the spaces of bounded functions and prove the Fredholm alternative for quite general boundary conditions. Note that, when the problem (1.1), (1.2) is considered in the space of continuous and periodic in time functions, then according to the standard terminology (1.12) means non-resonant behavior of (1.1), (1.2).

2. Fredholm alternative. We use the notation

$$x_j = \begin{cases} 0 & \text{if } 1 \leq j \leq m, \\ 1 & \text{if } m < j \leq n. \end{cases}$$

Define linear bounded operators $D, F: BC_n \rightarrow BC_n$ by

$$(Du)_j(x, t) = - \int_{x_j}^x d_j(\xi, x, t) \sum_{k \neq j} b_{jk}(\xi, \omega_j(\xi)) u_k(\xi, \omega_j(\xi)) d\xi, \quad j \leq n, \quad (2.1)$$

and

$$(Ff)_j(x, t) = \int_{x_j}^x d_j(\xi, x, t) f_j(\xi, \omega_j(\xi)) d\xi, \quad j \leq n. \quad (2.2)$$

On the account of (1.11), (2.1), and (2.2), the system (1.9), (1.10) can be written as the operator equation

$$u = Cu + Du + Ff. \quad (2.3)$$

Theorem 2.1. *Suppose that all conditions of Theorem 1.1 are fulfilled. Let \mathcal{K} denote the vector space of all bounded continuous solutions to (1.1), (1.2) with $f \equiv 0$. Then we have the following:*

(i) *$\dim \mathcal{K} < \infty$ and the vector space of all $f \in BC_n$ such that there exists a bounded continuous solution to (1.1), (1.2) is a closed subspace of codimension $\dim \mathcal{K}$ in BC_n .*

(ii) *If $\dim \mathcal{K} = 0$, then for any $f \in BC_n$ there exists a unique bounded continuous solution u to (1.1), (1.2).*

One of the technical tools we employ for the proof is a generalized Arzela–Ascoli compactness criteria for unbounded domains, see [11]. To formulate it, we need a corresponding notion of equicontinuity.

Definition 2.1. *A family $\Phi \subset BC_n$ is called equicontinuous on $[0, 1] \times \mathbb{R}$ if*

Φ is equicontinuous on any compact set in $[0, 1] \times \mathbb{R}$, and

for any $\varepsilon > 0$ there exists $T > 0$ such that

$$\left| u(x', t') - u(x'', t'') \right| < \varepsilon \quad (2.4)$$

for all $x', x'' \in [0, 1]$, all $t', t'' \in \mathbb{R} \setminus [-T, T]$, and all $u \in \Phi$.

Theorem 2.2 (a generalized Arzela–Ascoli theorem). *A family $\Phi \subset BC_n$ is precompact in BC_n if and only if Φ is bounded in BC_n and equicontinuous on $[0, 1] \times \mathbb{R}$.*

Proof of Theorem 2.1. First observe that the operator $I - C: BC_n \rightarrow BC_n$ is bijective, what straightforwardly follows from the Banach fixed-point theorem and the condition (1.12). Then the operator $I - C - D$ is Fredholm of index zero if and only if

$$I - (I - C)^{-1}D: BC_n \rightarrow BC_n \text{ is Fredholm of index zero.} \quad (2.5)$$

Nikolsky's criterion [4] (Theorem XIII.5.2) says that an operator $I + K$ on a Banach space is Fredholm of index zero whenever K^2 is compact. Hence, we are done with (2.5) if we show that the operator $[(I - C)^{-1}D]^2 = (I - C)^{-1}D(I - C)^{-1}D$ is compact. As the composition of a compact and a bounded operator is a compact operator, it is enough to show that

$$D(I - C)^{-1}D: BC_n \rightarrow BC_n \text{ is compact.}$$

Since $D(I - C)^{-1}D = D^2 + DC(I - C)^{-1}D$ and $(I - C)^{-1}D$ is bounded, it is sufficient to prove that

$$D^2, DC: BC_n \rightarrow BC_n \text{ are compact.} \quad (2.6)$$

To show (2.6), we use Theorem 2.2. Given $T > 0$, set $Q(T) = \{(x, t) \in [0, 1] \times \mathbb{R} : -T \leq t \leq T\}$. Fix an arbitrary bounded set $X \subset BC_n$. For (2.6) it is sufficient to prove the following two statements:

$$D^2X \text{ and } DCX \text{ are equicontinuous on } Q(T) \text{ for an arbitrary fixed } T > 0 \quad (2.7)$$

and

$$\text{given } \varepsilon > 0, \text{ there exists } T > 0 \text{ such that (2.4) is fulfilled for all} \quad (2.8)$$

$$x', x'' \in [0, 1], t', t'' \in \mathbb{R} \setminus [-T, T], u \in D^2X \text{ and } u \in DCX.$$

We start with the proving (2.7). Denote by $C_n(Q(T))$ (respectively $C_n^1(Q(T))$) the Banach space of continuous (respectively continuously differentiable) vector functions on $Q(T)$. As $C_n^1(Q(T))$ is compactly embedded into $C_n(Q(T))$ (due to the Arzela–Ascoli theorem), it is sufficient to show that

$$\left\| D^2u|_{Q(T)} \right\|_{C_n^1(Q(T))} + \left\| DCu|_{Q(T)} \right\|_{C_n^1(Q(T))} = O(\|u\|_\infty) \text{ for all } u \in X. \quad (2.9)$$

It should be noted that for all sufficiently large T the functions D^2u and DCu restricted to $Q(T)$ depend only on u restricted to $Q(2T)$.

We will use the following formulas:

$$\partial_x \omega_j(\xi) = -\frac{1}{a_j(x, t)} \exp \int_\xi^x \left(\frac{\partial_2 a_j}{a_j^2} \right) (\eta, \omega_j(\eta)) d\eta, \quad (2.10)$$

$$\partial_t \omega_j(\xi) = \exp \int_\xi^x \left(\frac{\partial_2 a_j}{a_j^2} \right) (\eta, \omega_j(\eta)) d\eta, \quad (2.11)$$

that hold for all $j \leq n$, all $\xi, x \in [0, 1]$, and all $t \in \mathbb{R}$. Here and below by ∂_i we denote the partial derivative with respect to the i th argument. Then for all sufficiently large $T > 0$ the partial derivatives $\partial_x D^2 u$, $\partial_t D^2 u$, $\partial_x DCu$, and $\partial_t DCu$ on $Q(T)$ exist and are continuous for all $u \in C^1(Q(2T))$. Since $C^1(Q(2T))$ is dense in $C(Q(2T))$, the desired property (2.9) will follow from the bound

$$\begin{aligned} \left\| D^2 u|_{Q(T)} \right\|_{C_n^1(Q(T))} + \left\| DCu|_{Q(T)} \right\|_{C_n^1(Q(T))} &= O(\|u\|_{C_n(Q(2T))}) \\ \text{for all } u \in C_n^1(Q(2T)). \end{aligned} \quad (2.12)$$

This bound is proved similarly to [9] (Lemma 4.2):

We start with the estimate

$$\left\| D^2 u|_{Q(T)} \right\|_{C_n^1(Q(T))} = O(\|u\|_{C_n(Q(2T))}) \quad \text{for all } u \in C_n^1(Q(2T)).$$

Given $j \leq n$ and $u \in C_n^1(Q(2T))$, let us consider the following representation for $(D^2 u)_j(x, t)$ obtained after the application of the Fubini's theorem:

$$(D^2 u)_j(x, t) = \sum_{k \neq j} \sum_{l \neq k} \int_{x_j}^x \int_{\eta}^x d_{jkl}(\xi, \eta, x, t) b_{jk}(\xi, \omega_j(\xi)) u_l(\eta, \omega_k(\eta, \xi, \omega_j(\xi))) d\xi d\eta, \quad (2.13)$$

where

$$d_{jkl}(\xi, \eta, x, t) = d_j(\xi, x, t) d_k(\eta, \xi, \omega_j(\xi)) b_{kl}(\eta, \omega_k(\eta, \xi, \omega_j(\xi))). \quad (2.14)$$

It is easy to see that from (2.13) it follows that

$$\left\| D^2 u|_{Q(T)} \right\|_{C_n(Q(T))} = O(\|u\|_{C_n(Q(2T))}).$$

Since

$$(\partial_t + a_j(x, t) \partial_x) \varphi(\omega_j(\xi, x, t)) = 0$$

for all $j \leq n$, $\varphi \in C^1(\mathbb{R})$, $x, \xi \in [0, 1]$, and $t \in \mathbb{R}$, one can easily check that

$$\left\| \left[(\partial_t + a_j(x, t) \partial_x) (D^2 u)_j|_{Q(T)} \right] \right\|_{C_n(Q(T))} = O(\|u\|_{C_n(Q(2T))})$$

$$\text{for all } j \leq n \quad \text{and} \quad u \in C_n^1(Q(2T)).$$

Hence the estimate

$$\left\| \partial_x D^2 u|_{Q(T)} \right\|_{C_n(Q(T))} = O(\|u\|_{C_n(Q(2T))})$$

will follow from the following one:

$$\left\| \partial_t D^2 u|_{Q(T)} \right\|_{C_n(Q(T))} = O(\|u\|_{C_n(Q(2T))}). \quad (2.15)$$

We are therefore reduced to prove (2.15). To this end, we start with the following consequence of (2.13):

$$\begin{aligned} \partial_t[(D^2u)_j(x, t)] &= \sum_{k \neq j} \sum_{l \neq k} \int_{x_j}^x \int_{\eta}^x \frac{d}{dt} [d_{jkl}(\xi, \eta, x, t) b_{jk}(\xi, \omega_j(\xi))] u_l(\eta, \omega_k(\eta, \xi, \omega_j(\xi))) d\xi d\eta + \\ &+ \sum_{k \neq j} \sum_{l \neq k} \int_{x_j}^x \int_{\eta}^x d_{jkl}(\xi, \eta, x, t) b_{jk}(\xi, \omega_j(\xi)) \times \\ &\times \partial_t \omega_k(\eta, \xi, \omega_j(\xi)) \partial_t \omega_j(\xi) \partial_2 u_l(\eta, \omega_k(\eta, \xi, \omega_j(\xi))) d\xi d\eta. \end{aligned}$$

Let us transform the second summand. Using (1.7), (2.10), and (2.11), we get

$$\begin{aligned} \frac{d}{d\xi} u_l(\eta, \omega_k(\eta, \xi, \omega_j(\xi))) &= [\partial_x \omega_k(\eta, \xi, \omega_j(\xi)) + \partial_t \omega_k(\eta, \xi, \omega_j(\xi)) \partial_\xi \omega_j(\xi)] \times \\ &\times \partial_2 u_l(\eta, \omega_k(\eta, \xi, \omega_j(\xi))) = \left(\frac{1}{a_j(\xi, \omega_j(\xi))} - \frac{1}{a_k(\xi, \omega_j(\xi))} \right) \times \\ &\times \partial_t \omega_k(\eta, \xi, \omega_j(\xi)) \partial_2 u_l(\eta, \omega_k(\eta, \xi, \omega_j(\xi))). \end{aligned} \quad (2.16)$$

Therefore,

$$\begin{aligned} b_{jk}(\xi, \omega_j(\xi)) \partial_t \omega_k(\eta, \xi, \omega_j(\xi)) \partial_2 u_l(\eta, \omega_k(\eta, \xi, \omega_j(\xi))) &= \\ = a_j(\xi, \omega_j(\xi)) a_k(\xi, \omega_j(\xi)) \tilde{b}_{jk}(\xi, \omega_j(\xi)) \frac{d}{d\xi} u_l(\eta, \omega_k(\eta, \xi, \omega_j(\xi))), \end{aligned} \quad (2.17)$$

where the functions $\tilde{b}_{jk} \in BC$ are fixed to satisfy (1.6). Note that \tilde{b}_{jk} are not uniquely defined by (1.6) for (x, t) with $a_j(x, t) = a_k(x, t)$. Nevertheless, as it follows from (2.16), the right-hand side (and, hence, the left-hand side of (2.17)) do not depend on the choice of \tilde{b}_{jk} , since $\frac{d}{d\xi} u_l(\eta, \omega_k(\eta, \xi, \omega_j(\xi))) = 0$ if $a_j(x, t) = a_k(x, t)$.

Write

$$\tilde{d}_{jkl}(\xi, \eta, x, t) = d_{jkl}(\xi, \eta, x, t) \partial_t \omega_j(\xi) a_k(\xi, \omega_j(\xi)) a_j(\xi, \omega_j(\xi)) \tilde{b}_{jk}(\xi, \omega_j(\xi)),$$

where d_{jkl} are introduced by (2.14) and (1.8). Using (1.7) and (2.10), we see that the function $\tilde{d}_{jkl}(\xi, \eta, x, t)$ is C^1 -smooth in ξ due to the regularity assumptions (1.3) and (1.6). Similarly, using (2.11), we see that the functions $d_{jkl}(\xi, \eta, x, t)$ and $b_{jk}(\xi, \omega_j(\xi))$ are C^1 -smooth in t .

By (2.17) we have

$$\begin{aligned}
\partial_t \left[(D^2 u)_j(x, t) \right] &= \sum_{k \neq j} \sum_{l \neq k} \int_{x_j}^x \int_{\eta}^x \frac{d}{dt} [d_{jkl}(\xi, \eta, x, t) b_{jk}(\xi, \omega_j(\xi))] u_l(\eta, \omega_k(\eta, \xi, \omega_j(\xi))) d\xi d\eta + \\
&+ \sum_{k \neq j} \sum_{l \neq k} \int_{x_j}^x \int_{\eta}^x \tilde{d}_{jkl}(\xi, \eta, x, t) \frac{d}{d\xi} u_l(\eta, \omega_k(\eta, \xi, \omega_j(\xi))) d\xi d\eta = \\
&= \sum_{k \neq j} \sum_{l \neq k} \int_{x_j}^x \int_{\eta}^x \frac{d}{dt} [d_{jkl}(\xi, \eta, x, t) b_{jk}(\xi, \omega_j(\xi))] u_l(\eta, \omega_k(\eta, \xi, \omega_j(\xi))) d\xi d\eta - \\
&- \sum_{k \neq j} \sum_{l \neq k} \int_{x_j}^x \int_{\eta}^x \partial_{\xi} \tilde{d}_{jkl}(\xi, \eta, x, t) u_l(\eta, \omega_k(\eta, \xi, \omega_j(\xi))) d\xi d\eta + \\
&+ \sum_{k \neq j} \sum_{l \neq k} \int_{x_j}^x \left[\tilde{d}_{jkl}(\xi, \eta, x, t) u_l(\eta, \omega_k(\eta, \xi, \omega_j(\xi))) \right]_{\xi=\eta}^{\xi=x} d\eta. \tag{2.18}
\end{aligned}$$

The desired estimate (2.15) now easily follows from the assumptions (1.3), (1.4), (1.6) and the equations (2.13) and (2.18).

To finish with (2.9), it remains to show that

$$\begin{aligned}
\|DCu|_{Q(T)}\|_{C_n(Q(T))} + \|\partial_t DCu|_{Q(T)}\|_{C_n(Q(T))} &= O(\|u\|_{C_n(Q(2T))}) \\
\text{for all } u \in C_n^1(Q(2T)), & \tag{2.19}
\end{aligned}$$

as the estimate for $\partial_x DCu$ follows similarly to the case of $\partial_x D^2 u$. In order to prove (2.19), we consider an arbitrary integral contributing into DCu , namely

$$\int_x^{x_j} e_{jk}(\xi, x, t) b_{jk}(\xi, \omega_j(\xi)) (Ru)_k(\omega_k(x_k, \xi, \omega_j(\xi))) d\xi, \tag{2.20}$$

where

$$e_{jk}(\xi, x, t) = d_j(\xi, x, t) c_k(x_k, \xi, \omega_j(\xi))$$

and $j \leq n$ and $k \leq n$ are arbitrary fixed. From (2.20) it follows that

$$\|DCu|_{Q(T)}\|_{C_n(Q(T))} = O(\|u\|_{C_n(Q(2T))}).$$

Differentiating (2.20) in t , we get

$$\begin{aligned} & \int_x^{x_j} \frac{d}{dt} [e_{jk}(\xi, x, t) b_{jk}(\xi, \omega_j(\xi))] (Ru)_k(\omega_k(x_k, \xi, \omega_j(\xi))) d\xi + \\ & + \int_x^{x_j} e_{jk}(\xi, x, t) b_{jk}(\xi, \omega_j(\xi)) \partial_t \omega_k(x_k, \xi, \omega_j(\xi)) \partial_t \omega_j(\xi) \partial_2 (Ru)_k(\omega_k(x_k, \xi, \omega_j(\xi))) d\xi. \end{aligned} \quad (2.21)$$

Our task is to estimate the second integral; for the first one the desired estimate is obvious. Similarly to the above, we use (1.7), (2.10), and (2.11) to obtain

$$\begin{aligned} \frac{d}{d\xi} (Ru)_k(\omega_k(x_k, \xi, \omega_j(\xi))) &= [\partial_x \omega_k(x_k, \xi, \omega_j(\xi)) + \partial_t \omega_k(x_k, \xi, \omega_j(\xi)) \partial_\xi \omega_j(\xi)] \times \\ &\times \partial_2 (Ru)_k(\omega_k(x_k, \xi, \omega_j(\xi))) = \left(\frac{1}{a_j(\xi, \omega_j(\xi))} - \frac{1}{a_k(\xi, \omega_j(\xi))} \right) \times \\ &\times \partial_t \omega_k(x_k, \xi, \omega_j(\xi)) \partial_2 (Ru)_k(\omega_k(x_k, \xi, \omega_j(\xi))). \end{aligned}$$

Taking into account (1.6), the last expression reads

$$\begin{aligned} & b_{jk}(\xi, \omega_j(\xi)) \partial_t \omega_k(x_k, \xi, \omega_j(\xi)) \partial_2 (Ru)_k(\omega_k(x_k, \xi, \omega_j(\xi))) = \\ & = a_j(\xi, \omega_j(\xi)) a_k(\xi, \omega_j(\xi)) \tilde{b}_{jk}(\xi, \omega_j(\xi)) \frac{d}{d\xi} (Ru)_k(\omega_k(x_k, \xi, \omega_j(\xi))). \end{aligned} \quad (2.22)$$

Set

$$\tilde{e}_{jk}(\xi, x, t) = e_{jk}(\xi, x, t) \partial_t \omega_j(\xi) a_k(\xi, \omega_j(\xi)) a_j(\xi, \omega_j(\xi)) \tilde{b}_{jk}(\xi, \omega_j(\xi)).$$

Using (2.10) and (2.22), let us transform the second summand in (2.21) as

$$\begin{aligned} & \int_x^{x_j} e_{jk}(\xi, x, t) b_{jk}(\xi, \omega_j(\xi)) \partial_t \omega_k(x_k, \xi, \omega_j(\xi)) \partial_t \omega_j(\xi) \partial_2 (Ru)_k(\omega_k(x_k, \xi, \omega_j(\xi))) d\xi = \\ & = \int_x^{x_j} \tilde{e}_{jk}(\xi, x, t) \frac{d}{d\xi} (Ru)_k(\omega_k(x_k, \xi, \omega_j(\xi))) d\xi = \\ & = [\tilde{e}_{jk}(\xi, x, t) (Ru)_k(\omega_k(x_k, \xi, \omega_j(\xi)))]_{\xi=x}^{\xi=x_j} - \\ & - \int_x^{x_j} \partial_\xi \tilde{e}_{jk}(\xi, x, t) (Ru)_k(\omega_k(x_k, \xi, \omega_j(\xi))) d\xi. \end{aligned} \quad (2.23)$$

The bound (2.19) now easily follows from (2.21) and (2.23). This finishes the proof of the bound (2.12) and, hence the statement (2.7).

It remains to prove (2.8). Fix an arbitrary $\varepsilon > 0$. We have to prove the estimates

$$\left| (D^2u)(x', t') - (D^2u)(x'', t'') \right| < \varepsilon \quad (2.24)$$

and

$$\left| (DCu)(x', t') - (DCu)(x'', t'') \right| < \varepsilon \quad (2.25)$$

for all $u \in X$ and all $x', x'' \in [0, 1]$, $t', t'' \in \mathbb{R} \setminus [-T, T]$ and some $T > 0$.

Let us prove (2.24). By (2.13), given $j \leq n$ and $u \in X$, we have

$$\begin{aligned} \left| (D^2u)_j(x', t') - (D^2u)_j(x'', t'') \right| &\leq \left| (D^2u)_j(x', t') \right| + \left| (D^2u)_j(x'', t'') \right| = \\ &= 2 \max_{j \leq n} \max_{x \in [0, 1]} \max_{t \in \mathbb{R} \setminus [-T, T]} \left| \sum_{k \neq j} \sum_{l \neq k} \int_{x_j}^x \int_{\eta}^x d_{jkl}(\xi, \eta, x, t) \times \right. \\ &\quad \left. \times b_{jk}(\xi, \omega_j(\xi)) u_l(\eta, \omega_k(\eta, \xi, \omega_j(\xi))) d\xi d\eta \right| \leq \\ &\leq M \|u\|_{\infty} \max_{k \neq j, l \neq k} \max_{x, \xi, \eta \in [0, 1]} \max_{t \in \mathbb{R} \setminus [-T, T]} |b_{jk}(\xi, \omega_j(\xi)) b_{kl}(\eta, \omega_k(\eta, \xi, \omega_j(\xi)))|, \end{aligned}$$

the constant M being dependent on n , a_j and b_{jj} but not on $u \in X$ and b_{jk} with $j \neq k$. Since $\|u\|_{\infty}$ is bounded on X , the desired estimate (2.24) now straightforwardly follows from the assumption (1.13) and the fact that $\omega_j(\xi, x, t) \rightarrow \infty$ as $t \rightarrow \pm\infty$.

The estimate (2.25) can be obtained by the same argument, what finishes the proof of (2.8).

The theorem is proved.

3. Uniqueness of a bounded continuous solution.

Theorem 3.1. *Suppose that the conditions (1.3), (1.4), and (1.12) are fulfilled. Then there is $\varepsilon > 0$ and $T > 0$ such that a bounded continuous solution to (1.1), (1.2) (if any) is unique whenever*

$$|b_{jk}(x, t)| < \varepsilon \text{ for all } 1 \leq j \neq k \leq n, \quad x \in [0, 1], \text{ and } t \in (-\infty, T]. \quad (3.1)$$

Proof. Given $T \in \mathbb{R}$, let $\Pi^T = [0, 1] \times (-\infty, T]$ and $\Pi_T = [0, 1] \times [T, \infty)$. Given $T > 0$, consider the problem (1.1), (1.2) in Π^{-T} . The system of integral equations can again be written in the operator form $u = \tilde{C}u + \tilde{D}u + \tilde{F}f$ with operators $\tilde{C}, \tilde{D}, \tilde{F}: BC_n(\Pi^{-T}) \rightarrow BC_n(\Pi^{-T})$ given by the rules (1.11), (2.1) and (2.2), respectively. As the operator $I - \tilde{C}: BC_n(\Pi^{-T}) \rightarrow BC_n(\Pi^{-T})$ is bijective (by the condition (1.12)), the operator equation reads

$$u = (I - \tilde{C})^{-1} \tilde{D}u + (I - \tilde{C})^{-1} \tilde{F}f.$$

Because of assumption (3.1), the value of $T > 0$ can be chosen so large that the norm of the operator \tilde{D} is sufficiently small. Consequently, for such T we have

$$\|(I - \tilde{C})^{-1} \tilde{D}\|_{\mathcal{L}(BC_n(\Pi^{-T}))} < 1.$$

By the Banach fixed-point theorem, there exists a unique function $u \in BC_n(\Pi^{-T})$ satisfying (1.9), (1.10) in Π^{-T} .

Now consider the problem (1.1), (1.2) in Π_{-T} with the initial condition

$$u_j|_{t=-T} = u_j(x, -T), \quad j \leq n. \quad (3.2)$$

Existence and uniqueness of a continuous solution $u \in C_n(\bar{\Pi}_{-T})$ to the initial boundary-value problem (1.1), (1.2), (3.2) follows from [6]. Summarizing, the problem (1.1), (1.2) in the strip $[0, 1] \times \mathbb{R}$ has a unique continuous solution bounded at $-\infty$. This immediately entails that a bounded continuous solution to the problem (1.1), (1.2) (if any) is unique. The proof is therewith complete.

To finish with Theorem 1.1, it remains to note that, by Theorem 3.1, $\dim \mathcal{K} = 0$. Then Theorem 1.1 immediately follows from Theorem 2.1 (ii).

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