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SUPEREXPONENTIALLY CONVERGENT ALGORITHM FOR AN ABSTRACT EIGENVALUE PROBLEM WITH APPLICATIONS TO ODES

СУПЕРЕКСПОНЕНЦІАЛЬНО ЗБІЖНИЙ АЛГОРИТМ ДЛЯ АБСТРАКТНОЇ ЗАДАЧІ НА ВЛАСНІ ЗНАЧЕННЯ ІЗ ЗАСТОСУВАННЯМ ДО ЗВИЧАЙНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ

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A new algorithm for eigenvalue problems for the linear operators of the type $\mathcal{A} = A + B$ with a special application to high order ordinary differential equations is proposed and justified. The algorithm is based on the approximation of $\mathcal{A}$ by an operator $\bar{\mathcal{A}} = A + \bar{B}$ where the eigenvalue problem for $\bar{\mathcal{A}}$ is supposed to be simpler then that for $\mathcal{A}$. The algorithm for this eigenvalue problem is based on the homotopy idea and for a given eigenpair number computes recursively a sequence of the approximate eigenpairs which converges to the exact eigenpair with an superexponential convergence rate. The eigenpairs can be computed in parallel for all prescribed indexes. The case of multiple eigenvalues of the operator $\bar{\mathcal{A}}$ is emphasized. Examples of the eigenvalue problems for the high order ordinary differential operators are presented to support the theory.

Запропоновано та обґрунтовано новий алгоритм для задач на власні значення для лінійних операторів типу $\mathcal{A} = A + B$ із спеціальним застосуванням до звичайних диференціальних рівнянь високого порядку. Алгоритм полягає в апроксимації оператора $\mathcal{A}$ таким оператором $\bar{\mathcal{A}} = A + \bar{B}$, що задача на власні значення для $\bar{\mathcal{A}}$ стає простішою, ніж для $\mathcal{A}$. Особливу увагу приділено випадку, коли оператор $\bar{\mathcal{A}}$ має кратні власні значення. Запропонований підхід базується на ідеї гомотопії. Послідовність наближень до власних пар обчислюється в ході рекурентного процесу та збігається до точного розв'язку із суперекспоненціальною швидкістю. Власні пари можна обчислювати паралельно для всіх заданих індексів. Наведені числові приклади задач на власні значення для звичайних диференціальних операторів високого порядку підтверджують одержані теоретичні результати.

1. Introduction. The eigenvalue problem (EVP), i.e., the problem of finding of eigenpairs (eigenvalues (frequencies) and eigenfunctions (vibration shapes)), play an important role in various applications concerned with vibrations and wave processes [1, 21]. Such popular methods as the finite-difference method (FDM), finite element method (FEM) and other variational methods, spectral methods allow to efficiently compute some lower eigenvalues only. At the same time there are applied problems requiring the computation of a great number (hundreds of thousands) of eigenvalues and eigenfunctions including eigenpairs with great indexes (see, e.g., [21, p. 273]).

In order to find numerically the higher eigenvalues we propose a new approach described below which we will (following [4, 5]) refer to as the FD-method. This approach is based on the perturbation and homotopy ideas (see, e.g., [2] and references therein). The perturbation in the case of ODE operators can be similar to that of the Pruess method [19 – 21] for the second-order ODEs or of methods from [1], where the coefficients of the differential operator are replaced by piecewise constant ones.

Our approach can be applied also to EVPs for nonlinear operators [7] and possesses the following main advantages:

1. It produces eigenpairs with an arbitrary prescribed indices and the accuracy increases with the index growth unlike matrix methods such as the finite difference (FDM), finite elements (FEM) or variational methods (VM).

2. It possesses the superexponential convergence rate contrary to the FDM, FEM or VM methods which converge polynomially.

2. The homotopy based method for EVPs. Let us briefly explain the ideas of perturbation and homotopy for the eigenvalue problem

$$(A + B)u_n - \lambda_n u_n = \theta, \tag{1}$$

in a Banach space X with the null-element θ under the assumption that the spectrum of the operator $A + B$ is discrete and we are looking for the eigenpair $\{\lambda_n, u_n\}$ with a given fixed index n. Let X^* be the dual Banach space of linear functionals on X and let $(\cdot, \cdot)$ be the duality relation.

Let $\overline{B}$ be an approximating operator for B in the sense that the eigenvalue problem

$$(A + \overline{B})u_n^{(0)} - \lambda_n^{(0)} u_n^{(0)} = \theta \tag{2}$$

is “simpler” than problem (1).

Formally, a homotopy between two problems P_1 and P_2 with solutions u_1 and u_2 from some topological space X is defined to be a parametric problem $P_H(t)$ with a solution $u(t)$ continuously depending on the parameter $t \in [0, 1]$ and such that $u(0) = u_1$ and $u(1) = u_2$.

Following the homotopy idea for a given eigenpair number n we imbed our problem into the parametric family of problems

$$(A + W(t))u_n(t) - \lambda_n(t)u_n(t) = \theta, \quad t \in [0, 1], \tag{3}$$

with $W(t) = \overline{B} + t(B - \overline{B})$ containing the both problems (1) and (2), so that we obviously have

$$u_n(0) = u_n^{(0)}, \quad \lambda_n(0) = \lambda_n^{(0)}, \quad u_n(1) = u_n, \quad \lambda_n(1) = \lambda_n.$$

This suggests the idea to look for the solution of (3) in the form

$$\lambda_n(t) = \sum_{j=0}^{\infty} \lambda_n^{(j)} t^j, \quad u_n(t) = \sum_{j=0}^{\infty} u_n^{(j)} t^j, \tag{4}$$

where formally

$$\lambda_n^{(j)} = \frac{1}{j!} \frac{d^j \lambda_n(t)}{dt^j}\bigg|_{t=0}, \quad u_n^{(j)} = \frac{1}{j!} \frac{d^j u_n(t)}{dt^j}\bigg|_{t=0}.$$

Setting $t = 1$ in (4) we obtain

$$\lambda_n = \sum_{j=0}^{\infty} \lambda_n^{(j)}, \quad u_n = \sum_{j=0}^{\infty} u_n^{(j)}$$

provided that series (4) converge for all $t \in [0,1]$. The truncated sums

$$\overset{N}{\lambda}_n = \sum_{j=0}^{N} \lambda_n^{(j)}, \quad \overset{N}{u} = \sum_{j=0}^{N} u_n^{(j)}$$

are approximations (of rank N) to the exact eigenvalue and eigenfunction of problem (1) and together with the formulas below for $\lambda_n^{(j)}$, $u_n^{(j)}$ represent the algorithms for their computation.

In order to find the coefficients we substitute (4) into (3) and by matching the coefficients in front of the same powers of t we arrive at the following recurrence sequence of equations:

$$(A + \overline{B}) u_n^{(j+1)} - \lambda_n^{(0)} u_n^{(j+1)} = F_n^{(j+1)}, \quad j = -1, 0, 1, \ldots, \tag{5}$$

with $F_n^{(0)} = 0$ and

$$F_n^{(j+1)} = F_n^{(j+1)}(\lambda_n^{(0)}, \ldots, \lambda_n^{(j+1)}; u_n^{(0)}, \ldots, u_n^{(j)}) = -\varphi(B) u_n^{(j)} + \sum_{p=0}^{j} \lambda_n^{(j+1-p)} u_n^{(p)} =$$

$$= \lambda_n^{(j+1)} u_n^{(0)} - \varphi(B) u_n^{(j)} + \sum_{p=1}^{j} \lambda_n^{(j+1-p)} u_n^{(p)}, \quad \varphi(B) = B - \overline{B}. \tag{6}$$

For the pair $\lambda_n^{(0)}$, $u_n^{(0)}$ we get the so called base problem

$$(A + \overline{B}) u_n^{(0)} - \lambda_n^{(0)} u_n^{(0)} = \theta, \tag{7}$$

which is assumed to be “simpler” than the original one and produces an initial data for problems (5), (6).

Let the base problem possesses real eigenvalues

$$0 \le \lambda_1^{(0)} \le \lambda_2^{(0)} \le \ldots \le \lambda_n^{(0)} \le \ldots . \tag{8}$$

We suppose that in (8) each eigenvalue $\lambda_n^{(0)}$ is represented k_n times in according to its multiplicity k_n. Let $e_{n,p}$, $p = \overline{1,k_n}$, $n = 1, 2, \ldots$, build a basis in X, where $e_{n,p}$, $p = \overline{1,k_n}$, are the eigenvectors corresponding to $\lambda_n^{(0)}$. We denote by $\vec{e} = (e_{n,1}, e_{n,2}, ..., e_{n,k_n})^T$ and by $f_{n,p}, p = \overline{1,k_n}$, $n = 1, 2, \ldots$, the corresponding biorthogonal system of functionals in X^* to $e_{n,p}, p = \overline{1,k_n}, n = 1, 2, ...$, i.e., $(e_{m,i}, f_{n,j}) = \delta_{n,j}\delta_{m,i}$, $n, m = 1, 2, \ldots, i = 1, 2, \ldots, k_m$, $j = 1, 2, \ldots, k_n$ (due to the Riesz representation theorem in the case of Hilbert space we can consider the scalar product as the duality relation) which build a basis of X^* [15]. Here and below $\delta_{n,j}$ is the Kronecker delta. The recurrence equations (5) of our method are of the kind

$$\tilde{A}u - \lambda_n^{(0)}u = g,$$

where $\lambda_n^{(0)}$ is an eigenvalue of the operator $\tilde{A} = A + \overline{B}$, i.e., the operator $\tilde{A} + \lambda_n^{(0)}E$ with the identity operator E is singular. We look for the particular solution of the form

$$\hat{u} = \sum_{p=1}^{\infty}\sum_{i=1}^{k_p} c_{p,i}e_{p,i}.$$

By substituting this ansatz into the equation and using the biorthogonality of the systems $\{e_{p,i}\}$ and $\{f_{p,i}\}$, we obtain $c_{n,i} = 0$, $c_{p,i} = (g, f_{p,i})/(\lambda_p^{(0)} - \lambda_n^{(0)})$, i.e.,

$$\hat{u} = \Gamma_n^+ g = \sum_{p=1, p\neq n}^{\infty} \frac{1}{\lambda_p^{(0)} - \lambda_n^{(0)}} \sum_{i=1}^{k_p} (g, f_{p,i})e_{p,i}, \tag{9}$$

where Γ_n^+ denotes a pseudoinverse Moore – Penrose operator to $\tilde{A} + \lambda_n^{(0)}E$.

The general solution of problem (7) has the form

$$u_n^{(0)} = \sum_{p=1}^{k_n} C_{n,p}^{(0)} e_{n,p},$$

where the constants $C_{n,p}^{(0)}$, $p = \overline{1,k_n}$, will be defined below.

Under the solvability conditions

$$(F_n^{(j+1)}, f_{n,p}) = 0, \quad p = \overline{1,k_n}, \tag{10}$$

the solution of (5) can be represented by

$$u_n^{(j+1)} = \sum_{p=1}^{k_n} C_{n,p}^{(j+1)} e_{n,p} + \hat{u}_n^{(j+1)}, \tag{11}$$

where the first summand represents the general solution of the homogeneous equation and

$$\hat{u}_n^{(j+1)} = \Gamma_n^+ \left(\sum_{p=1}^{j} \lambda_n^{(j+1-p)} u_n^{(p)} - \varphi(B) u_n^{(j)} \right) \tag{12}$$

is a particular solution of the inhomogeneous equation. Note that the index p runs from 1 and not from 0 due to the property of Γ_n^+ formulated by the lemma below.

Lemma 1. *We have the following:*

$$\Gamma_n^+ e_{n,p} = 0, \quad p = \overline{1, k_n}.$$

The proof follows from representation (9).

Condition (10) leads to the system of equations

$$\sum_{s=1}^{k_n} C_s^{(j)} \left((\varphi(B) - \lambda_n^{(1)} E) e_{n,s}, f_{n,m} \right) = -\left(\varphi(B) \hat{u}_n^{(j)}, f_{n,m} \right) + \sum_{p=0}^{j-1} \lambda_n^{(j+1-p)} C_m^{(p)}, \quad m = \overline{1, k_n}. \quad (13)$$

Here $\lambda_n^{(1)}$ denotes one of the eigenvalues $\lambda_n^{(1)} = \lambda_{n,i}^{(1)}$, $i = \overline{1, k_n}$, of the matrix $[(\varphi(B)e_{n,s}, f_{n,m})]_{s,m=\overline{1,k_n}}$ corresponding to the ordering

$$\lambda_{n,1}^{(1)} \le \lambda_{n,2}^{(1)} \le \ldots \le \lambda_{n,k_n}^{(1)},$$

where each eigenvalue is repeated accordance to its multiplicity. Let us introduce a vector $\vec{C}^{(j)}$ with the components $C_s^{(j)}$, $s = 1, \ldots, k_n$, and the matrix

$$D^{[\nu]} = [d_{s,m}^{[\nu]}]_{s,m=\overline{1,k_n}}, \quad d_{s,m}^{[\nu]} = \left((\varphi(B) - \lambda_{n,\nu}^{(1)} E) e_{n,s}, f_{n,m} \right), \quad 1 \le \nu \le k_n,$$

and rewrite equations (13) in a matrix form,

$$D^{[\nu]} \vec{C}^{(j)} = \sum_{p=0}^{j-1} \lambda_n^{(j+1-p)} \vec{C}^{(p)} - \left\langle \varphi(B) \hat{u}^{(j)}, \vec{f} \right\rangle, \quad j = 0, 1, \ldots, \quad (14)$$

with $\vec{f} = [f_1, f_2, \ldots, f_{k_n}]^T$, $\langle v, \vec{f} \rangle = [(v, f_1), (v, f_2), \ldots, (v, f_{k_n})]^T$. From (14) with $j = 0$ allowing for the condition $\hat{u}^{(0)} = 0$, we obtain

$$D^{[\nu]} \vec{C}^{(0)} = \vec{0}. \quad (15)$$

Here the vectors

$$\vec{C}_i^{(0)}, \quad i = \overline{1, \mu_\nu}, \quad 1 \le \mu_\nu < k_n, \quad (\vec{C}_i^{(0)}, \vec{C}_t^{(0)})_R = \delta_{i,t}$$

are solutions of system (15), i.e., $\lambda_\nu^{(1)}$ is an eigenvalue of the matrix $[(\varphi(B)e_{n,s}, f_{n,m})]_{s,m=\overline{1,k_n}}$ of multiplicity μ_ν, where $(\cdot,\cdot)_R$ is the scalar product in R^{k_n}.

We require the coefficients $C_{n,p}^{(j+1)}$, $p = \overline{1, k}$, to satisfy

$$(u_n^{(j+1)}, f_n^{(0)}) = \left(u_n^{(j+1)}, \sum_{i=1}^{k_n} C_{n,i}^{(0)} f_{n,i} \right) = 0,$$

where $f_n^{(0)} = \sum_{i=1}^{k_n} C_{n,i}^{(0)} f_{n,i}$ is the conjugate element to $u_n^{(0)}$. This yields

$$\left(\vec{C}_i^{(j+1)}, \vec{C}_i^{(0)}\right)_R = \sum_{p=1}^{k} C_{n,p}^{(j+1)} C_{n,p}^{(0)} = 0, \quad j = 0, 1, \ldots .$$

Multiplying (13) by $C_m^{(0)}$ and summing up over m from 1 to k_n, we arrive at the following relation:

$$\lambda_n^{(j+1)} = \left(\varphi(B)\hat{u}_n^{(j)}, f_n^{(0)}\right), \tag{16}$$

from where

$$\left|\lambda_n^{(j+1)}\right| \le \left\|\varphi(B)\hat{u}_n^{(j)}\right\| \left\|f_n^{(0)}\right\|_*, \tag{17}$$

where $\|\cdot\|_*$ is the norm in X^*.

Having regard to (16) one can see that the right-hand side of (14) is orthogonal to the vectors $\vec{C}_i^{(0)}$, $i = \overline{1, \mu_\nu}$, i.e., the necessary and sufficient solvability conditions are fulfilled.

Let us consider the following solution of system (14) (this solution is not unique):

$$\vec{C}_i^{(j)} = \left(D^{[\nu]}\right)^+ \left(\sum_{p=0}^{j-1} \lambda_i^{(j+1-p)} \vec{C}_i^{(p)} - \left\langle \varphi(B)\hat{u}_i^{(j)}, \vec{f} \right\rangle\right), \tag{18}$$

where $(D^{[\nu]})^+$ is the Moore – Penrose pseudoinverse matrix of the matrix $D^{[\nu]}$. Let us note that in a detailed description of the matrix, the triple n, ν, i should be kept in mind instead of the index i or n.

It is easy to show that

$$\left(D^{[\nu]}\right)^+ \vec{C}_i^{(0)} = \vec{0}, \tag{19}$$

which means that for the solution of system (14) in the form (18) the following orthogonality conditions are fulfilled:

$$(\vec{C}_i^{(j+1)}, \vec{C}_i^{(0)})_R = 0, \quad j = 0, 1, \ldots .$$

Further we give the error estimates of our method. Having regard to (16), (19) we obtain from (18) that

$$\left\|\vec{C}_i^{(j)}\right\|_R \le w \cdot \sum_{p=0}^{j-1} \left\|\hat{u}_i^{(j-p)}\right\| \left\|\vec{C}_i^{(p)}\right\|_R, \tag{20}$$

where

$$w = \|\varphi(B)\| \left\|\left(D^{[\nu]}\right)^+\right\|_R \left(\left\|f_n^{(0)}\right\|_* + \left|\left\langle \vec{f} \right\rangle\right|\right), \quad \left|\left\langle \vec{f} \right\rangle\right| = \left(\sum_{s=1}^{k} \|f_s\|_*^2\right)^{1/2}, \quad \|\vec{a}\|_R = (\vec{a}, \vec{a})_R .$$

From (11), (12) we deduce the estimates

$$\left\|\tilde{u}_n^{(j+1)}\right\| \le M_n \left\{\sum_{s=0}^{j} \|\tilde{u}_n^{(j-s)}\| \|\tilde{u}_n^{(s)}\| + \left\|\vec{C}_n^{(j)}\right\|_R\right\},$$
$$\left\|\varphi(B)\tilde{u}_n^{(j+1)}\right\| \le N_n \left\{\sum_{s=0}^{j} \|\tilde{u}_n^{(j-s)}\| \|\tilde{u}_n^{(s)}\| + \left\|\vec{C}_n^{(j)}\right\|_R\right\}, \tag{21}$$

where

$$M_n = \max\left\{\|\varphi(B)\| \left\|\Gamma_n^+\right\| \left\|f_n^{(0)}\right\|_*, \left\|\Gamma_n^+\varphi(B)\right\| k_n\right\},$$
$$N_n = \max\left\{\|\varphi(B)\| \left\|\varphi(B)\Gamma_n^+\right\| \left\|f_n^{(0)}\right\|_*, \left\|\varphi(B)\Gamma_n^+\varphi(B)\right\| k_n\right\},$$

and

$$\tilde{u}_i^{(s)} = [1 - \operatorname{sgn}(s)]u_i^{(0)} + \operatorname{sgn}(s)\hat{u}_i^{(s)}$$

$\left(\text{here we have kept in mind that } \left\|u_i^{(j+1)}\right\| \le \left\|\hat{u}_i^{(j+1)}\right\| + \left\|\vec{C}_i^{(j+1)}\right\|_R\right).$

With the notations

$$u_{j+1} = \left\|\tilde{u}_n^{(j+1)}\right\|, \quad v_{j+1} = \left\|\varphi(B)\tilde{u}_n^{(j+1)}\right\|, \quad c_{j+1} = \left\|\vec{C}_n^{(j+1)}\right\|_R, \quad j = 0, 1, \ldots,$$

the recurrence system of inequalities (20), (21) can be rewritten in the form

$$c_{j+1} \le w \sum_{p=0}^{j} u_{j-p} c_p,$$
$$u_{j+1} \le M_n \left\{\sum_{s=0}^{j} u_{j-s} u_s + c_j\right\},$$
$$v_{j+1} \le N_n \left\{\sum_{s=0}^{j} u_{j-s} u_s + c_j\right\}, \quad j = 0, 1, \ldots.$$

Replacing the inequality signs by the equalities ones we obtain the following majorant system:

$$C_{j+1} = w\sum_{p=0}^{j} U_{j-p}C_p,$$

$$U_{j+1} = M_n\left\{\sum_{s=0}^{j} U_{j-s}U_s + C_j\right\},$$

$$V_{j+1} = N_n\left\{\sum_{s=0}^{j} U_{j-s}U_s + C_j\right\}, \quad j = 0,1,\ldots,$$

$$C_0 = 1, \quad U_0 = \left\|u_n^{(0)}\right\|$$

that is $c_j \le C_j$, $u_j \le U_j$, $v_j \le V_j$, $j = 0,1,\ldots$. Having regard to

$$N_nU_{j+1} = M_nV_{j+1}$$

the first and the second equations imply

$$C_{j+1} = w\sum_{p=0}^{j} U_{j-p}C_p,$$

$$U_{j+1} = M_n\left\{\sum_{s=0}^{j} U_{j-s}U_s + C_j\right\}, \quad j = 0,1,\ldots,$$

$$C_0 = 1, \quad U_0 = \left\|u_n^{(0)}\right\|.$$

Introducing the new variables and the new majorant variables by

$$M_n^{-j}U_j = \tilde{U}_j, \quad M_n^{-j}C_j = \tilde{C}_j$$

we can switch to the majorant system

$$\tilde{C}_{j+1} = w\sum_{p=0}^{j} \tilde{U}_{j-p}\tilde{C}_p,$$

$$\tilde{U}_{j+1} = \sum_{s=0}^{j} \tilde{U}_{j-s}\tilde{U}_s + \tilde{C}_j, \quad j = 0,1,\ldots, \tag{22}$$

$$\tilde{C}_0 = 1, \quad \tilde{U}_0 = \left\|u_n^{(0)}\right\|.$$

To solve (22) we introduce the following generating functions:

$$f(z) = \sum_{j=0}^{\infty} z^j\tilde{U}_j, \quad g(z) = \sum_{j=0}^{\infty} z^j\tilde{C}_j.$$

Then starting with (22) we obtain the system of equations

$$f(z) - U_0 = z[f^2(z) + g(z)], \quad g(z) - 1 = w[f(z) - U_0]g(z).$$

We define from the second equation $g(z) = 1/\{1 - w\,[f(z) - U_0]\}$ and substitute it into the first one. Then we have

$$-zwf^3(z) + (w + z + zwU_0)f^2(z) - (zwU_0 + 1)f(z) + (wU_0^2 + U_0 + z) = 0. \tag{23}$$

Let us interchange in (23) the variables and consider z as a function of f. Then

$$z(f) = \frac{w(f - U_0)\left(f - \frac{wU_0+1}{w}\right)}{wf^2\left(f - \frac{wU_0+1}{w}\right) - 1}. \tag{24}$$

Considering function (24) we see that

$$z(U_0) = z\left(\frac{wU_0+1}{w}\right) = 0, \quad z(f) > 0 \quad \forall f \in \left(U_0, \frac{wU_0+1}{w}\right),$$

$$z'(U_0) = \frac{1}{U_0^2 + 1} > 0, \quad z'\left(\frac{wU_0+1}{w}\right) = -1 < 0.$$

This implies the existence of some $z_{\max} = z(f_{\max}), f_{\max} \in \left(U_0, \dfrac{w\,U_0+1}{w}\right)$, which represents the convergence radius of the series for $f(z)$, i.e., there exist some constants L, ε independent of j, n such that

$$(z_{\max})^j \tilde{U}_j \leq \frac{L}{j^{1+\varepsilon}}, \quad j = 1, 2, \ldots. \tag{25}$$

Having regard to (25) we obtain for $z \geq 0$ that

$$\sum_{j=0}^{\infty} z^j \left\|\tilde{u}_i^{(j)}\right\| \leq \sum_{j=0}^{\infty} \left(\frac{z}{z_{\max}} M_n\right)^j \tilde{U}_j (z_{\max})^j \leq 1 + \sum_{j=1}^{\infty} \left(\frac{z}{z_{\max}} M_n\right)^j \frac{L}{j^{1+\varepsilon}}. \tag{26}$$

Under the assumption

$$q_n = \frac{M_n}{z_{\max}} < 1 \tag{27}$$

the inequality (26) is fulfilled for all $z \in [0, 1]$, therefore we get

$$\left\|\tilde{u}_i^{(j)}\right\| \leq \frac{L\,[q_n]^j}{j^{1+\varepsilon}}, \quad j = 1, 2, \ldots. \tag{28}$$

From (20) and (28) we obtain the following recurrence inequality system:

$$\left\|\vec{C}_n^{(j)}\right\|_R \leq wL \sum_{p=0}^{j-1} [q_n]^{j-p} \left\|\vec{C}_n^{(p)}\right\|_R, \quad j = 1, 2, \ldots,$$

with a solution which is majorized by the solution of

$$C_j = wL \sum_{p=0}^{j-1} [q_n]^{j-p} C_p, \quad j = 1, 2, \ldots, \quad C_0 = 1, \quad \left\| \vec{C}_n^{(j)} \right\|_R \le C_j.$$

Applying the method of generating functions we obtain

$$g(z) = \frac{1}{1 - 2wL(\tilde{f}(z) - 1)}, \quad \tilde{f}(z) = \sum_{j=0}^{\infty} z^j \, [q_n]^j \, .$$

The Taylor series for $\tilde{g}(z) = \sum_{j=0}^{\infty} z^j \tilde{C}_j$, $C_j \le \tilde{C}_j$ implies

$$\tilde{C}_j = \frac{wL}{1 + wL} \left[q_n(1 + wL) \right]^j . \tag{29}$$

This series is convergent for all $z \in [0, 1]$ provided that

$$q_n(1 + wL) < 1. \tag{30}$$

These considerations analogous to [16] imply the following assertion.

Theorem 1. *Under the assumptions (27), (30), the FD-method for problem (1) is superexponentially convergent with the estimates*

$$\left\| u_{n,i} - \overset{m}{u}_{n,i} \right\| = \left\| u_{n,i} - \sum_{j=0}^{m} u_{n,i}^{(j)} \right\| \le L \left[wk_n + 1 \right] \frac{\left[q_n(1 + wL) \right]^{m+1}}{1 - q_n(1 + wL)},$$

$$\left| \lambda_{n,i} - \overset{m}{\lambda}_{n,i} \right| = \left| \lambda_{n,i} - \sum_{j=0}^{m} \lambda_{n,i}^{(j)} \right| \le L \left\| \varphi(B) \right\| \left\| f_n^{(0)} \right\|_* \frac{[q_n]^m}{(m+1)^{1+\varepsilon}(1 - q_n)}.$$

The proof is based on (28), (17) and (29).

Remark 1. One can control (27) by choosing n and $\overline{B}$ in such a way that

$$M_n = \max \left\{ \left\| \varphi(B) \right\| \left\| \Gamma_n^+ \right\| \left\| f_n^{(0)} \right\|_*, \left\| \Gamma_n^+ \varphi(B) \right\| k_n \right\}$$

is small enough.

Then due to (9) we have

$$\left\| \Gamma_n^+ \right\| \le \max \left\{ \frac{1}{\lambda_n^{(0)} - \lambda_{n-1}^{(0)}}, \frac{1}{\lambda_{n+1}^{(0)} - \lambda_n^{(0)}} \right\}. \tag{31}$$

Let the operator $\varphi(B) = B - \overline{B}$ be subordinated to the operator $(A + B)^\alpha$, $\alpha > 0$, i.e.,

$$\left\| (A + \bar{B})^{-\alpha} \varphi(B) v \right\| \le c \|v\|$$

with some positive constant c. Then, on the other hand, in the case of a Hilbert space, of simple eigenvalues and of a self-adjoint $A+\bar{B}$, we have for an arbitrary v that

$$\Gamma_n^+\varphi(B)v = -\sum_{p=1,p\neq n}^{\infty} \frac{((A+\bar{B})^{-\alpha}\varphi(B)v,(A+\bar{B})^{\alpha}u_p^{(0)})}{\lambda_n^{(0)}-\lambda_p^{(0)}}\,u_p^{(0)} =$$
$$= -\sum_{p=1,p\neq n}^{\infty} \frac{[\lambda_p^{(0)}]^{\alpha}((A+\bar{B})^{-\alpha}\varphi(B)v,u_p^{(0)})}{\lambda_n^{(0)}-\lambda_p^{(0)}}\,u_p^{(0)},$$

from where

$$\|\Gamma_n^+\varphi(B)v\|^2 = \sum_{p=1,p\neq n}^{\infty} \frac{[\lambda_p^{(0)}]^{2\alpha}}{(\lambda_n^{(0)}-\lambda_p^{(0)})^2}((A+\bar{B})^{-\alpha}\varphi(B)v,u_p^{(0)})^2 \leq$$
$$\leq \max\left\{\frac{[\lambda_{n+1}^{(0)}]^{2\alpha}}{(\lambda_{n+1}^{(0)}-\lambda_n^{(0)})^2},\frac{[\lambda_n^{(0)}]^{2\alpha}}{(\lambda_n^{(0)}-\lambda_{n-1}^{(0)})^2}\right\}\|(A+\bar{B})^{-\alpha}\varphi(B)v\|^2 \leq$$
$$\leq c^2\max\left\{\frac{[\lambda_{n+1}^{(0)}]^{2\alpha}}{(\lambda_{n+1}^{(0)}-\lambda_n^{(0)})^2},\frac{[\lambda_n^{(0)}]^{2\alpha}}{(\lambda_n^{(0)}-\lambda_{n-1}^{(0)})^2}\right\}\|v\|^2.$$

One can see that $\|\Gamma_n^+\varphi(B)\| \to 0$ as $n\to\infty$ provided that the quotients in the curly brackets tend to zero as $n\to\infty$. The behavior of this norm as a function of n depends on the asymptotic of the eigenvalues. For example, the ODE operators of order m with regular boundary conditions possess the asymptotic $\lambda_n^{(0)} = \mathcal{O}(n^m)$ (see, e.g., [17, 18]) so that both values $\frac{[\lambda_{n+1}^{(0)}]^{2\alpha}}{(\lambda_{n+1}^{(0)}-\lambda_n^{(0)})^2}$ and $\frac{[\lambda_n^{(0)}]^{2\alpha}}{(\lambda_n^{(0)}-\lambda_{n-1}^{(0)})^2}$ are of the order $\mathcal{O}(n^{2(\alpha-1)m+2})$. This implies $\|\Gamma_n^+\|$, $\|\Gamma_n^+\varphi(B)\| \to 0$ as $n\to\infty$ provided that

$$0<\alpha<1-\frac{1}{m}.$$

If for some fixed n convergence condition (30) is not valid then its validity can be reestablished by using the better approximation of operator B, or equivalently by reducing the value $\|\varphi(B)\|$.

Remark 2. One can show that FD-method is superexponentially convergent for eigenfunctions as well for approximation of eigenvalues.

The case of Banach space and eigenvalues of arbitrary (finite) multiplicity can be considered analogously as above.

Example 1. The following calculations made with Maple demonstrate the principle difference of the behavior of the iterations of the FD-method for a second order differential equation with and without the first derivative.

We consider the EVP

$$\frac{d^2u(x)}{dx^2}+r(x)\frac{du(x)}{dx}+(\lambda-x)u(x)=0,\quad x\in(0,1),\quad u(0)=0,\quad u(1)=0,$$

where we define $Au = \dfrac{d^2u}{dx^2}$, $Bu = r(x)\,\dfrac{du(x)}{dx} - xu(x)\ \forall u \in \overset{\circ}{H}{}^1(0,1) \cap H^2$. We choose $\overline{B} = 0$. Our algorithm in the case $r(x) \equiv 0$ provides the iterations

$$\lambda_n^{(0)} = (n\pi)^2, \quad \lambda_n^{(1)} = \frac{1}{2}, \quad \lambda_n^{(2)} = \frac{1}{48(n\pi)^2} - \frac{15}{48(n\pi)^4}, \quad \lambda_n^{(3)} = 0,$$

$$\lambda_n^{(4)} = \frac{1}{2304(n\pi)^6} - \frac{35}{384(n\pi)^8} + \frac{55}{64(n\pi)^{10}},$$

i.e., we have $\lim_{n\to\infty} \lambda_n^{(j)} = 0$.

For the case where $r(x) = 0,5(\operatorname{sgn}(x) + 1)$ we obtain

$$\lambda_n^{(0)} = (n\pi)^2, \quad \lambda_n^{(1)} = \frac{1}{2}\,(2 - \cos(n\pi)),$$

$$\lambda_n^{(2)} = \frac{1}{48}\left(1 - \frac{5}{(n\pi)^2} - \frac{15}{(n\pi)^4}\right), \quad \lambda_n^{(3)} = \frac{1}{1536}\left(1 + \frac{7}{(n\pi)^2} - \frac{78}{(n\pi)^4}\right),$$

$$\lambda_n^{(4)} = \frac{-1}{2580480}\left(49 - \frac{7081}{(n\pi)^2} + \frac{5027}{(n\pi)^4} - \frac{26320}{(n\pi)^6} - \frac{195720}{(n\pi)^8} - \frac{2217600}{(n\pi)^{10}}\right).$$

The principal difference to the previous example is that now $\lim_{n\to\infty} \lambda_n^{(j)} = \text{const} \neq 0$.

These results are consistent with our theory. In fact, we have for the norm of the operator $\Gamma_n^+ : L_2(0,1) \to L_2(0,1)$:

$$\left\|\Gamma_n^+\right\|_{L_2(0,1)\to L_2(0,1)} = \sup_{v\in L_2(0,1)} \|v\|^{-1} \left\{\int_0^1 \left[\sum_{p=1,p\neq n}^{\infty} \frac{2v_n \sin(p\pi x)}{\pi^2(n^2 - p^2)}\right]^2 dx\right\}^{\frac{1}{2}} \leq \frac{1}{\pi^2(2n-1)},$$

where $v_n = \displaystyle\int_0^1 v(\xi)\sin(p\pi\xi)d\xi$ is the Fourier coefficient of v. Note that general estimate (31) is of the same order with respect to the parameter n. Actually, $\lambda_n^{(0)} = n^2\pi^2$, so that $\lambda_{n+1}^{(0)} - \lambda_n^{(0)} = \mathcal{O}(n)$, $\lambda_n^{(0)} - \lambda_{n-1}^{(0)} = \mathcal{O}(n)$, i.e., $\|\Gamma_n^+\|_{L_2(0,1)\to L_2(0,1)} = \mathcal{O}(n^{-1})$.

To obtain an estimate for $\varphi(B)\Gamma_n^+$, let us estimate each summand separately. For $k(x)\dfrac{d}{dx}\,\Gamma_n^+$ we obtain

$$\left\|k(\cdot)\frac{d}{dx}\,\Gamma_n^+\right\|_{L_2(0,1)\to L_2(0,1)} \leq \frac{\|k\|_\infty}{\pi^2} \sup_{v\in L_2(0,1)} \|v\|^{-1} \left\{\int_0^1 \left[\sum_{p=1,p\neq n}^{\infty} \frac{2p^2\pi^2 v_n \sin(p\pi x)}{(n^2 - p^2)^2}\right]^2 dx\right\}^{\frac{1}{2}} \leq$$

$$\leq \frac{\|k\|_\infty}{\pi}\,\frac{n+1}{(n+1)^2 - n^2} \leq \frac{2\|k\|_\infty}{3\pi}$$

and further

$$\left\|\varphi(B)\Gamma_n^+\right\|_{L_2(0,1)\to L_2(0,1)} \le \left\|k(x)\frac{d}{dx}\,\Gamma_n^+\right\|_{L_2(0,1)\to L_2(0,1)} + \left\|x\Gamma_n^+\right\|_{L_2(0,1)\to L_2(0,1)} \le$$

$$\le \frac{2\left\|k\right\|_\infty}{3\pi} + \frac{1}{\pi^2(2n-1)} \le \frac{2\left\|k\right\|_\infty}{3\pi} + \frac{1}{\pi^2}.$$

Analogously after integration by parts we have

$$\Gamma_n^+\left(k(\xi)\frac{dv}{d\xi}\right) = \sum_{p=1,p\ne n}^{\infty} \frac{-p\pi\int_0^1 k(\xi)v(\xi)\sqrt{2}\cos(p\pi\xi)d\xi}{\pi^2(n^2-p^2)}\sqrt{2}\sin(p\pi x)-$$

$$-\sum_{p=1,p\ne n}^{\infty} \frac{\int_0^1 [k'(\xi)+\xi]v(\xi)\sqrt{2}\sin(p\pi\xi)d\xi}{\pi^2(n^2-p^2)}\sqrt{2}\sin(p\pi x)$$

and estimating analogously as above we get

$$\left\|\Gamma_n^+\varphi(B)\right\|_{L_2(0,1)\to L_2(0,1)} \le c\cdot\max\{\|k\|_\infty, \|k'+\xi\|_\infty\}$$

with some constant c. These estimates show that M_n remains bounded in the case where the coefficient in the front of the first derivative is not equal to zero and $M_n = \mathcal{O}(n^{-1}) \to 0$ as $n \to \infty$ otherwise.

3. Application to the fourth order differential operators. In this section we adapt our exponentially convergent algorithm from above for the numerical solution of the following class of eigenvalue problems (see also [8]):

$$y^{(4)}(\xi) + g_3(\xi)y^{(3)}(\xi) + g_2(\xi)y''(\xi) + g_1(\xi)y'(\xi) + g_0(\xi)y(\xi) - g(\xi)\lambda y(\xi) = 0,$$

$$y^{(p)}(0) = y^{(q)}(0) = y^{(r)}(1) = y^{(s)}(1) = 0, \tag{32}$$

$$0 \le p < q \le 3, \quad 0 \le r < s \le 3,$$

which play a special role in the applications.

The type of boundary conditions is defined by the four natural numbers $(p, q; r, s)$, $p, q, r, s \in \in \{0, 1, 2, 3\}$. All these boundary conditions are regular and affect among others such important property as the multiplicity of eigenvalues.

One of the oldest and probably mostly famous applications of this mathematical model is the description of free and forced vibrations of a Bernoulli – Euler beam [12, 22] (there are also good reasons to call this theory as "The Da Vinci – Euler – Bernoulli Beam Theory" [3]). Euler – Bernoulli beam theory emerged in the middle of the 18th century as a simplification from the linear isotropic theory of elasticity. Due to its simplicity and at the same time to its adequate accuracy (demonstrated by many practical applications, amongst others during assembly of the Eiffel Tower and the Ferris Wheel in the late 19th century) the beam theory became an important tool in the sciences, especially structural and mechanical engineering.

Equation (32) as well as the fourth order equation in the self-adjoint form

$$\frac{d^2}{d\xi^2}\left(a(\xi)\frac{d^2}{d\xi^2}y(\xi)\right)-\frac{d}{d\xi}\left(b(\xi)\frac{d}{d\xi}y(\xi)\right)+(c(\xi)-\lambda d(\xi))y(\xi)=0 \tag{33}$$

can be reduced to the form

$$u^{(4)}(x)+k_2(x)u''(x)+k_1(x)u'(x)+k_0(x)u(x)-\lambda u(x)=0, \tag{34}$$

i.e., we can make the coefficient in the front of the third derivative equal to zero and the coefficient at the front of $\lambda u(x)$ equal to one. Note that in the case of the nth order differential equation

$$u^{(n)}(x)+k_{n-1}(x)u^{(n-1)}(x)+\ldots+k_1(x)u'(x)+\rho^n\cdot 1\cdot u(x)=0$$

with the coefficient in the front of λ equal to the constant 1 one can make the coefficient in the front of the $(n-1)$th derivative equal to zero by the variables transform [18]

$$u=e^{-\frac{1}{n}\int k_{n-1}(x)dx}\tilde{u}.$$

Equation (32) can be converted to form (34) by the variable transform $\xi=\varphi(x)$, $y(\xi)=$ $=\psi(x)u(x)$ with the appropriate functions $\varphi(x)$, $\psi(x)$ (compare with the Liouville transform [17] for the second order differential equation). The functions $\varphi(x),\psi(x)$ can be found in [8].

Mechanics often use the following approximation $A+\overline{B}$ for the differential operator $A+B$, $A=\dfrac{d^2}{d\xi^2}\left(a(\xi)\,\dfrac{d^2}{d\xi^2}\,y(\xi)\right)$ or $A=\dfrac{d^4}{d\xi^4}$ of the form (32), (33) with the corresponding boundary conditions: the interval $(0,1)$ is covered by a grid $\omega=\{t_i\,:\,i=1,\ldots,N-1, 0=t_0<t_1<\ldots$ $\ldots<t_{N-1}<t_N=1\}$ with a maximal step $h=\max_{i=1,\ldots,N}(t_i-t_{i-1})$, the variable coefficients on each subinterval are replaced by constants (for example, by some fixed values of the corresponding variable coefficients) and the solution of such problem is accepted as an approximate solution of (32) or (33). The basic idea here is the approximation of the differential equation (i.e., its coefficients). The corresponding methods for the second-order Sturm–Liouville problems are known as the Pruess methods [21] because S. Pruess in 1973 provided a rigorous convergence and error analysis [19]. But in fact variants of this method were in use since the beginning of the past century and for the piecewise coefficient approximation the method was theoretically justified for linear second order ordinary differential equations (as the so-called method of "stumps"(metodo dei tronconi)) by N. N. Bogoliouboff and N. M. Kryloff in 1928 [6, 10].

The higher order eigenvalue problems were treated, e.g., in [1, 9, 11, 23]. Theoretical and numerical methods of solving eigenvalue problems for higher order equations (especially of those whose coefficients have considerable variation) have not been developed to a desirable extent or are altogether absent. Some constructive approaches to the solution of self-conjugate fourth-order eigenvalue problems with different types of boundary conditions have been suggested in [1].

Let us estimate the accuracy of the method of "tronconi"for the following test eigenvalue problem:

$$\frac{d^2}{dt^2}\left(a(t)\,\frac{d^2}{dt^2}\,v(t)\right)-\lambda v(t)=0,\quad v(0)=v(1)=v''(0)=v''(1)=0, \tag{35}$$

with $0 < \kappa \leq a(t) \leq K < \infty$.

Following the described method we consider instead of (35) the problem

$$\frac{d^2}{dt^2}\left(\overline{a}(t)\frac{d^2}{dt^2}v^{(0)}(t)\right) - \lambda^{(0)}v^{(0)}(t) = 0, \;\; v^{(0)}(0) = v^{(0)}(1) = \frac{d^2}{dt^2}v^{(0)}(0) = \frac{d^2}{dt^2}v^{(0)}(1) = 0 \tag{36}$$

with $\bar{a}(t) = \min_{t\in[t_{i-1},t_i]} a(t)$, $t \in [t_{i-1}, t_i]$, $i = 1, \ldots, N$. At the discontinuity points of the coefficient $\overline{a}(t)$ we require that the following consistency conditions hold:

$$[v(t)]_{t=t_i} = \left[\frac{d}{dt}v(t)\right]_{t=t_i} = \left[\bar{a}(t)\frac{d^2}{dt^2}v(t)\right]_{t=t_i} = \left[\frac{d}{dt}\left(\bar{a}(t)\frac{d^2}{dt^2}v(t)\right)\right]_{t=t_i} = 0$$

where $[w(t)]_{t=t_i} = w(t_i+0) - w(t_i-0)$ is the jump of $w(t)$ at the point $t = t_i$. It can be shown (see, e.g., [17]) that the spectra of the both problems (35), (36) are discrete and the eigenvalues can be ordered in ascending order. We are interesting in the error due to replacement of (35) by (36). With this aim let us consider the following auxiliary the differential equation with the parameter $s \in [0, 1]$:

$$\begin{gathered}\frac{\partial^2}{\partial t^2}\left(\bar{a}(t,s)\frac{\partial^2}{\partial t^2}v(t,s)\right) - \lambda(s)v(t,s) = 0,\\ v(0,s) = v(1,s) = \frac{\partial^2}{\partial t^2}v(0,s) = \frac{\partial^2}{\partial t^2}v(1,s) = 0,\end{gathered} \tag{37}$$

where $\bar{a}(t,s) = \bar{a}(t) + s(a(t) - \bar{a}(t))$ and the solution is normalized by $\int_0^1 v^2(t,s)dt = 1 \; \forall s \in$ $\in [0,1]$. We have obviously that $v(t,1) = v(t)$, $\lambda(1) = \lambda$, $v(t,0) = v^{(0)}(t)$, $\lambda(0) = \lambda^{(0)}$. Since $v(t,s)$ depends on the parameter s analytically we can differentiate (37) with respect to s and obtain

$$\frac{\partial^2}{\partial t^2}\left(\bar{a}(t,s)\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial s}v(t,s)\right) - \lambda(s)\frac{\partial}{\partial s}v(t,s) =$$

$$= -\frac{\partial^2}{\partial t^2}\left((a(t) - \bar{a}(t))\frac{\partial^2}{\partial t^2}v(t,s)\right) + v(t,s)\frac{d}{ds}\lambda(s),$$

$$\frac{\partial}{\partial s}v(0,s) = \frac{\partial}{\partial s}v(1,s) = \frac{\partial^2}{\partial t^2}\frac{\partial}{\partial s}v(0,s) = \frac{\partial^2}{\partial t^2}\frac{\partial}{\partial s}v(1,s) = 0, \quad s \in [0,1].$$

The solvability condition allowing for the normalization condition and the integration by parts lead to the formula

$$\frac{d\lambda(s)}{ds} = \int_0^1 (a(t) - \bar{a}(t))\left(\frac{\partial^2}{\partial t^2}v(t,s)\right)^2 dt.$$

On the other hand, the equalities (37) imply

$$\lambda(s) = \int_0^1 a(t,s)\left(\frac{\partial^2}{\partial t^2}v(t,s)\right)^2 ds.$$

These two relations yield

$$0 < \frac{d\lambda(s)}{ds} \le \max_{t\in[0,1]} |a(t) - \bar{a}(t)| \int\limits_0^1 \left(\frac{\partial^2}{\partial t^2} v(t,s)\right)^2 dt \le \max_{t\in[0,1]} |a(t) - \bar{a}(t)| \frac{\lambda(s)}{\kappa}$$

which, together with

$$\lambda(s) = \lambda^{(0)} + \int\limits_0^s \frac{d}{d\eta} \lambda(\eta) d\eta,$$

leads to the following estimate:

$$0 \le \lambda_n - \lambda_n^{(0)} \le \max_{t\in[0,1]} (a(t) - \bar{a}(t)) \frac{\lambda_n}{\kappa}. \tag{38}$$

Assuming that $a(t) \in C^{(2)}[0,1]$ due to the asymptotic $\lambda_n = \mathcal{O}(n^4)$ we obtain from (38) that

$$0 \le \lambda_n - \lambda_n^{(0)} = \mathcal{O}(hn^4), \quad h = \max_{i=\overline{1,N}} (t_i - t_{i-1})$$

(compare with the estimate $|\lambda_n - \lambda_n^{(0)}| \le C \max\{1, n^2\} h$ for the second-order Sturm – Liouville problems [21, p. 119]). This estimate shows that the method under consideration is appropriate for not very large n, in fact, for some lowest eigenvalues only.

3.1. FD-method in the case of multiple eigenvalues of the base problem.

Example 2. Let us consider the eigenvalue problem

$$(A + B)u - \lambda u = 0,$$

where the operators A, B are defined by

$$D(A) = \left\{ v \in C^4(0,1) : v^{(p)}(0) = v^{(p)}(1), p = 0,1,2,3 \right\}, \quad D(A) \subseteq D(B),$$

$$Au = u^{(4)}(x) \ \ \forall u \in D(A),$$

$$B = B_1 + B_2 + B_3 + B_0, \quad B_j u = k_{4-j}(x) u^{(j)}(x) \ \ \forall u \in D(A), \quad j = 0,1,2,3.$$

Let us choose $\overline{B} = 0$, then the base problem of the FD-method is

$$(A + \overline{B})u - \lambda u = u^{(4)}(x) - \lambda u = 0, \quad x \in (0,1),$$

$$u^{(p)}(0) = u^{(p)}(1), \quad p = 0,1,2,3.$$

Each eigenvalue

$$\lambda_n = (2n\pi)^4$$

of this spectral problem is double and corresponds to the following two eigenfunctions:

$$u_{n,1}(x) = \sqrt{2}\sin(2n\pi x), \quad u_{n,2}(x) = \sqrt{2}\cos(2n\pi x).$$

The operator $A+\overline{B}-\lambda_n E = A-\lambda_n E$ is singular, i.e., its pseudoinverse Γ^+ acts on the functions g satisfying the solvability conditions

$$\int_0^1 \sin(2n\pi x) g(x)dx = 0, \quad \int_0^1 \cos(2n\pi x) g(x)dx = 0.$$

It is easy to check that $\Gamma^+ g(\cdot)$ for a fixed x can be represented as the linear continuous functional

$$l(\varphi(\cdot)) = \int_0^1 \varphi(t) d \int_0^t g(s)ds,$$

with the function of bounded variation $\int_0^t g(s)ds$, on the function

$$\varphi(t) = \frac{1}{32n^3\pi^3}\left(\sin(2n\pi(x-t)) + \sin(2n\pi|x-t|) + \frac{\cosh(n\pi(2|x-t|-1))}{\sinh(n\pi)}\right).$$

This functional is defined on continuous functions $\varphi(t) \in C[0,1]$ (see, e.g., [14]) and its norm is given by

$$\|l\| = V_0^1 \left[\int_0^t g(s)ds\right],$$

where

$$V_a^b[\varphi] = \sup \sum_{k=1}^{r} |\varphi(t_k) - \varphi(t_{k-1})|$$

is the total variation of the function $\varphi(t)$ on the interval $[a,b]$.

Let us denote by $V[0,1]$ the set of functions of bounded variation continuous from left and vanishing on the left end of the interval $[0,1]$. Then we have

$$\left\|\Gamma^+\right\| = \sup_{\tilde{g}(t)\in V[0,1]} \frac{\|\Gamma^+ g\|_{C[0,1]}}{V_0^1[\tilde{g}]} \le \frac{2+\coth(n\pi)}{32n^3\pi^3},$$

$$\left\|\varphi(B_1)\Gamma^+\right\| = \sup_{\tilde{g}(t)\in V[0,1]} \frac{\|\varphi(B_1)\Gamma^+ g\|_{C[0,1]}}{V_0^1[\tilde{g}]} = \sup_{\tilde{g}(t)\in V[0,1]} \frac{\left\|k_1(\cdot)\frac{d}{dx}\Gamma^+ g\right\|_{C[0,1]}}{V_0^1[\tilde{g}]} \le$$

$$\le \frac{\|k_1\|_{C[0,1]}}{16n^2\pi^2} \sup_{\tilde{g}(t)\in V[0,1]} \max_{x\in[0,1]} \frac{\left[3V_0^x[\tilde{g}] + V_x^1[\tilde{g}]\right]}{V_0^1[\tilde{g}]} \le \frac{3\|k_1\|_{C[0,1]}}{16n^2\pi^2},$$

$$\left\|\varphi(B_2)\Gamma^+\right\| = \sup_{\tilde{g}(t)\in V[0,1]} \frac{\|\varphi(B_2)\Gamma^+ g\|_{C[0,1]}}{V_0^1[\tilde{g}]} =$$

$$= \sup_{\tilde{g}(t)\in V[0,1]} \frac{\left\|k_2(\cdot)\frac{d^2}{dx^2}\Gamma^+ g\right\|_{C[0,1]}}{V_0^1[\tilde{g}]} \le \frac{2+\coth(n\pi)}{8n\pi}\|k_2\|_{C[0,1]},$$

$$\left\|\varphi(B_3)\Gamma^+\right\| = \sup_{\tilde{g}(t)\in V[0,1]} \frac{\|\varphi(B_3)\Gamma^+ g\|_{C[0,1]}}{V_0^1[\tilde{g}]} = \sup_{\tilde{g}(t)\in V[0,1]} \frac{\left\|k_3(\cdot)\frac{d^3}{dx^3}\,\Gamma^+ g\right\|_{C[0,1]}}{V_0^1[\tilde{g}]} \le \frac{3}{4}\|k_3\|_{C[0,1]},$$

where $\tilde{g}(t) = \int_0^t g(s)ds$.

To obtain estimates from below we choose $\tilde{g}(t) = \mathrm{H}\left(t - \frac{1}{4}\right) - \mathrm{H}\left(t - \frac{3}{4}\right)$ with the Heaviside function H $\left(\text{i.e., } g(x) = \delta\left(x - \frac{1}{4}\right) - \delta\left(x - \frac{3}{4}\right)\right)$ in the case of an even n we get

$$\left\|\Gamma^+\right\| = \sup_{\tilde{g}(t)\in V[0,1]} \frac{\|\Gamma^+ g\|_{C[0,1]}}{V_0^1\left[\tilde{g}\right]} \ge$$

$$\ge \max_{x\in[0,1]} \frac{1}{32(n\pi)^3\cosh(\frac{n\pi}{2})} \left|\begin{cases} -\sinh(2n\pi x), & x\in[0,1/4], \\ \sinh\left(2n\pi\left(x-\frac{1}{2}\right)\right) - 2\sin\left(2n\pi\left(x-\frac{1}{4}\right)\right)\times \\ \times\cosh\left(\frac{n\pi}{2}\right), & x\in[1/4,3/4], \\ -\sinh(2n\pi\,(x-1)), & x\in[3/4,1] \end{cases}\right| \ge$$

$$\ge \frac{\left|\sinh\left(\frac{n\pi}{2}\right) - 2\cosh\left(\frac{n\pi}{2}\right)\right|}{32(n\pi)^3\cosh\left(\frac{n\pi}{2}\right)} \ge \frac{2-\tanh\left(\frac{\pi}{2}\right)}{32(n\pi)^3},$$

since $V_0^1\left[\tilde{g}\right] = 2$. Analogously we obtain

$$\left\|\varphi(B_1)\Gamma^+\right\| = \sup_{\tilde{g}(t)\in V[0,1]} \frac{\|\varphi(B_1)\Gamma^+ g\|_{C[0,1]}}{V_0^1\left[\tilde{g}\right]} = \sup_{\tilde{g}(t)\in V[0,1]} \frac{\left\|k_1(\cdot)\frac{d}{dx}\,\Gamma^+ g\right\|_{C[0,1]}}{V_0^1\left[\tilde{g}\right]} \ge$$

$$\ge \max_{x\in[0,1]} \frac{|k_1(x)|}{16(n\pi)^2\cosh\left(\frac{n\pi}{2}\right)} \left|\begin{cases} -\cosh(2n\pi x), & x\in[0,1/4], \\ \cosh\left(2n\pi\left(x-\frac{1}{2}\right)\right) - 2\cos\left(2n\pi\left(x-\frac{1}{4}\right)\right)\times \\ \times\cosh\left(\frac{n\pi}{2}\right), & x\in[1/4,3/4], \\ -\cosh(2n\pi\,(x-1)), & x\in[3/4,1] \end{cases}\right| \ge$$

$$\ge \frac{1}{16(n\pi)^2}\,\|k_1\|_{C[0,1]},$$

and

$$\left\|\varphi(B_2)\Gamma^+\right\| = \sup_{\tilde{g}(t)\in V[0,1]} \frac{\left\|k_2(\cdot)\frac{d^2}{dx^2}\,\Gamma^+ g\right\|_{C[0,1]}}{V_0^1\,[\tilde{g}]} \geq$$

$$\geq \max_{x\in[0,1]} \frac{|k_2(x)|}{8n\pi\cosh\left(\frac{n\pi}{2}\right)} \left| \begin{cases} -\sinh(2n\pi x), & x\in[0,1/4]\,, \\ \sinh\left(2n\pi\left(x-\dfrac{1}{2}\right)\right)+2\sin\left(2n\pi\left(x-\dfrac{1}{4}\right)\right)\times \\ \times\cosh\left(\dfrac{n\pi}{2}\right), & x\in[1/4,3/4], \\ -\sinh(2n\pi\,(x-1))\,, & x\in[3/4,1] \end{cases} \right| \geq$$

$$\geq \frac{2-\tanh\left(\frac{\pi}{2}\right)}{8n\pi}\,\|k_2\|_{C[0,1]}\,,$$

$$\left\|\varphi(B_3)\Gamma^+\right\| = \sup_{\tilde{g}(t)\in V[0,1]} \frac{\left\|k_3(\cdot)\frac{d^3}{dx^3}\,\Gamma^+ g\right\|_{C[0,1]}}{V_0^1\,[\tilde{g}]} \geq$$

$$\geq \max_{x\in[0,1]} \frac{|p_3(x)|}{4\cosh\left(\frac{n\pi}{2}\right)} \left| \begin{cases} -\sinh(2n\pi x), & x\in[0,1/4]\,, \\ \sinh\left(2n\pi\left(x-\dfrac{1}{2}\right)\right)+2\cos\left(2n\pi\left(x-\dfrac{1}{4}\right)\right)\times \\ \times\cosh\left(\dfrac{n\pi}{2}\right), & x\in[1/4,3/4], \\ -\sinh(2n\pi\,(x-1))\,, & x\in[3/4,1] \end{cases} \right| \geq$$

$$\geq \frac{\|k_3\|_{C[0,1]}}{4}\frac{\left|\sinh\left(\frac{n\pi}{2}\right)-2\cosh\left(\frac{n\pi}{2}\right)\right|}{\cosh\left(\frac{n\pi}{2}\right)} \geq \frac{1}{4}\,\|k_3\|_{C[0,1]}\,.$$

Thus we have estimates of the same order with respect to n from the both sides. The case of odd n can be considered analogously. Note that we have evaluate $\Gamma^+ g$ by solving the BVP

$$\frac{d^4v(x)}{dx^4}-(2n\pi)^4v(x)=\delta\left(x-\frac{1}{4}\right)-\delta\left(x-\frac{3}{4}\right), \quad x\in(0,1),$$

$$\frac{d^iv(0)}{dx^i}=\frac{d^iv(1)}{dx^i}, \quad i=\overline{0,3},$$

which is equivalent to

$$\frac{d^4v(x)}{dx^4}-(2n\pi)^4v(x)=0, \quad x\in(0,1), \quad x\neq\frac{1}{4},\frac{3}{4},$$

$$\frac{d^iv(0)}{dx^i}=\frac{d^iv(1)}{dx^i}, \quad i=\overline{0,2},$$

$$\frac{d^3v(\frac{1}{4}+0)}{dx^3}-\frac{d^3v(\frac{1}{4}-0)}{dx^3}=1, \quad \frac{d^3v(\frac{3}{4}+0)}{dx^3}-\frac{d^3v(\frac{3}{4}-0)}{dx^3}=-1.$$

It is easy to find that

$$v(x) = \frac{1}{32(n\pi)^3 \cosh\left(\frac{n\pi}{2}\right)} \begin{cases} -\sinh(2n\pi x), & x \in [0, 1/4], \\ \sinh\left(2n\pi\left(x-\frac{1}{2}\right)\right) - 2\sin\left(2n\pi\left(x-\frac{1}{4}\right)\right) \times \\ \times \cosh\left(\frac{n\pi}{2}\right), & x \in [1/4, 3/4], \\ -\sinh(2n\pi(x-1)), & x \in [3/4, 1]. \end{cases}$$

The estimates for $\|\Gamma^+\varphi(B)\|$ can be obtained analogously and they are of the same order with respect to n as $\|\varphi(B)\Gamma^+\|$.

3.2. FD-method for boundary conditions of type $(p, q; r, s)$**.** In this section we consider a variant of the algorithm when $Au = u^{(4)}(x)$ subject to the corresponding boundary conditions and $\overline{B}$ is an approximation of the part with lower derivations where the coefficients in the front of these derivatives in (34) are approximated by some piecewise constant coefficients on some chosen grid. All assumptions of Theorem 1 holds true for the FD-method described above in Section 2, i.e., we obtain all eigenpairs λ_n, $u_n(x)$ with an exponential accuracy for all $n \geq n_0$ beginning with some n_0. The next example with boundary conditions of type $(0, 2; 0, 2)$ confirms this result.

Example 3. Let us consider problem (34) with $k_2(x) = x$, $k_1(x) \equiv 0$, $k_0(x) \equiv 0$. The smallest eigenvalue of this problem computed with the standard tool of the computer algebra system Maple is

$$\lambda_1^{ex} = 102{,}3353144965013. \tag{39}$$

Let us compare this result with the one obtained by our algorithm.

For the problem under consideration we have the base problem

$$\frac{d^4}{dx^4} u_n^{(0)}(x) - \lambda_n^{(0)} u_n^{(0)}(x) = 0, \quad u_n^{(0)}(0) = \frac{d^2}{dx^2} u_n^{(0)}(0) = u_n^{(0)}(1) = \frac{d^2}{dx^2} u_n^{(0)}(1) = 0$$

with the solution $u_n^{(0)}(x) = \sqrt{2}\sin(n\pi x)$, $\lambda_n^{(0)} = (n\pi)^4$, $n = 1, 2, \ldots$. The functions $u_n^{(j+1)}(x)$, $j = 0, 1, \ldots$, are defined as solutions of the following sequence of problems:

$$\frac{d^4}{dx^4} u_n^{(j+1)}(x) - \lambda_n^{(0)} u_n^{(j+1)}(x) = -k_2(x) \frac{d^2}{dx^2} u_n^{(j)}(x) - k_1(x) \frac{d}{dx} u_n^{(j)}(x) - k_0(x) u_n^{(j)}(x) + \\ + \sum_{s=0}^{j} \lambda^{(j+1-s)} u_n^{(s)}(x), \quad x \in (0, 1), \tag{40}$$

$$u_n^{(j+1)}(0) = \frac{d^2}{dx^2} u_n^{(j+1)}(0) = u_n^{(j+1)}(1) = \frac{d^2}{dx^2} u_n^{(j+1)}(1) = 0, \quad j = 0, 1, \ldots, m,$$

and are subject to the following normalizing condition:

$$\left(u_n^{(j)}, u_n^{(0)}\right) = \left(u_n^{(j)}, u_n^{(0)}\right)_{L_2(0,1)} = \int_0^1 u_n^{(j)}(x) u_n^{(0)}(x) dx = 0. \tag{41}$$

The solvability condition for problem (40) together with (41) yields

$$\lambda_n^{(j+1)} = \int\limits_0^1 \left(k_2(x)\frac{d^2}{dx^2}\,u_n^{(j)}(x) + k_1(x)\,\frac{d}{dx}\,u_n^{(j)}(x) + k_0(x)u_n^{(j)}(x)\right) u_n^{(0)}(x)dx. \tag{42}$$

The solution of problem (40) is given by

$$u_n^{(j+1)}(x) = \sum_{\alpha=1}^{4} u_{n,\alpha}^{(j+1)}(x) = -\frac{1}{\pi^4}\sum_{l=1,l\neq n}^{\infty}\frac{u_l^{(0)}(x)}{n^4-l^4}\int\limits_0^1 \Bigg(- k_2(\xi)\frac{d^2}{d\xi^2}\,u_n^{(j)}(\xi) -$$

$$- k_1(\xi)\frac{d}{d\xi}\,u_n^{(j)}(\xi) - k_0(\xi)u_n^{(j)}(\xi) + \sum_{s=0}^{j}\lambda_n^{(j+1-s)}u_n^{(s)}(\xi)\Bigg)u_n^{(0)}(\xi)d\xi =$$

$$= -\frac{2}{\pi^4}\sum_{l=1,l\neq n}^{\infty}\frac{\sin(l\pi x)}{n^4-l^4}\int\limits_0^1 \Bigg(\Bigg[(l\pi)^2 k_2(\xi)\sin(l\pi\xi) + l\pi\left(-2k_2'(\xi) + \right.$$

$$\left. + \; k_1(\xi)\right)\cos(l\pi\xi) + \left(-k_2''(\xi) + k_1'(\xi) - k_0(\xi)\right)\sin(l\pi\xi)\Bigg]u_n^{(j)}(\xi) +$$

$$+ \sum_{s=0}^{j}\lambda_n^{(j+1-s)}u_n^{(s)}(\xi)\sin(l\pi\xi)\Bigg)d\xi. \tag{43}$$

Let us estimate each summand on the right-hand side separately:

$$\left\|u_{n,1}^{(j+1)}\right\| = \frac{1}{\pi^4}\left\{\sum_{l=1,l\neq n}^{\infty}\frac{(l\pi)^4}{(n^4-l^4)^2}\left[\int\limits_0^1 k_2(\xi)u_n^{(j)}(\xi)\sin(l\pi\xi)d\xi\right]^2\right\}^{\frac{1}{2}} \leq$$

$$\leq \frac{1}{\pi^2}\frac{(n+1)\|k_2\|_\infty}{2n^2-2n+1}\left\|u_n^{(j)}\right\|,$$

$$\left\|u_{n,2}^{(j+1)}\right\| = \frac{2}{\pi^3}\left\{\sum_{l=1,l\neq n}^{\infty}\frac{(l\pi)^2}{(n^4-l^4)^2}\left[\int\limits_0^1 (-2k_2'(\xi) + k_1(\xi))u_n^{(j)}(\xi)\cos(l\pi\xi)d\xi\right]^2\right\}^{\frac{1}{2}} \leq$$

$$\leq \frac{1}{\pi^3}\,\frac{\|-2k_2' + k_1\|_\infty}{2n^2-2n+1}\left\|u_n^{(j)}\right\|,$$

$$\left\|u_{n,3}^{(j+1)}\right\| \leq \frac{1}{\pi^4}\frac{\|-k_2'' + k_1' - k_0\|_\infty}{n(2n^2-2n+1)}\left\|u_n^{(j)}\right\|,$$

$$\left\|u_{n,4}^{(j+1)}\right\| \leq \frac{1}{\pi^4}\,\frac{1}{n(2n^2-2n+1)}\left\|\sum_{s=0}^{j}\lambda_n^{(j+1-s)}u_n^{(s)}\right\|.$$

From (42) and (43) we obtain

$$\left|\lambda_n^{(j+1-s)}\right| \le \sqrt{2}\,\mu[(n\pi)^2 + n\pi + 1]\,\left\|u_n^{(j-s)}\right\|$$

and

$$\begin{aligned}\left\|u_n^{(j+1)}\right\| &\le \sum_{t=1}^{4}\left\|u_{n,t}^{(j+1)}\right\| \le \frac{1}{\pi^2}\,\frac{1}{2n^2-2n+1}\left[\left[(n+1)\,\|q_2\|_\infty + \frac{1}{\pi}\left\|-2q_2' + q_1\right\|_\infty + \right.\right.\\
&\quad \left. + \frac{1}{\pi^2}\frac{\|-q_2'' + q_1' - q_0\|_\infty}{n}\right]\left\|u_n^{(j)}\right\| + \\
&\quad \left. + \sqrt{2}\mu\left[n + \frac{1}{\pi} + \frac{1}{n\pi^2}\right]\sum_{s=0}^{j}\left\|u_n^{(j-s)}\right\|\left\|u_n^{(s)}\right\|\right] \le \\
&\le \frac{1}{\pi^2}\frac{\mu}{2n^2-2n+1}\left[\left[n+1+\frac{1}{\pi}+\frac{1}{n\pi^2}\right]\left\|u_n^{(j)}\right\| + \right.\\
&\quad \left. + \sqrt{2}\left[n + \frac{1}{\pi} + \frac{1}{n\pi^2}\right]\sum_{s=0}^{j}\left\|u_n^{(j-s)}\right\|\left\|u_n^{(s)}\right\|\right] \le \\
&\le M_n\sum_{s=0}^{j}\left\|u_n^{(j-s)}\right\|\left\|u_n^{(s)}\right\|,\end{aligned}$$

where

$$M_n = \frac{1}{\pi^2}\frac{\mu\sqrt{2}}{2n^2-2n+1}\left[n+1+\frac{1}{\pi}+\frac{1}{n\pi^2}\right],$$

$$\mu = \max\left(\|k_2\|_\infty, \left\|-2k_2' + k_1\right\|_\infty, \left\|-k_2'' + k_1' - k_0\right\|_\infty\right).$$

The solution of the last inequality is

$$\left\|u_n^{(j)}\right\| \le (4M_n)^j\,2\frac{(2j-1)!!}{(2j+2)!!} \le \frac{(4M_n)^j}{(j+1)\sqrt{\pi j}}.$$

It means that under the condition $r_n = 4M_n < 1$ the following modification of the estimates from our main theorem holds true:

$$\begin{aligned}\left|\lambda_n - \overset{m}{\lambda}_n\right| &= \left|\lambda_n - \sum_{j=0}^{m}\lambda_n^{(j)}\right| \le \mu\sqrt{2}\left[(n\pi)^2 + n\pi + 1\right]\frac{(r_n)^m}{1-r_n}\,\frac{(2m-1)!!}{(2m+2)!!} \le \\
&\le \frac{\mu}{\sqrt{2}}\left[(n\pi)^2 + n\pi + 1\right]\frac{(r_n)^m}{1-r_n}\,\frac{1}{(m+1)\sqrt{\pi m}},\end{aligned}$$

$$\left\|u_n - \overset{m}{u}_n\right\| = \left\|u_n - \sum_{j=0}^{m} u_n^{(j)}\right\| \le 2\frac{(r_n)^{m+1}}{1-r_n}\,\frac{(2m+1)!!}{(2m+4)!!} \le \frac{(r_n)^{m+1}}{(m+2)\sqrt{\pi(m+1)}}.$$

The Maple-calculations provide

$$\lambda_1^{(0)} = \pi^4, \quad \left|\lambda_1^{ex} - \lambda_1^{(0)}\right| = 4,926223462,$$

$$\lambda_1^{(1)} = \frac{\pi^2}{2}, \quad \left|\lambda_1^{ex} - \overset{1}{\lambda}_1\right| = \left|\lambda_1^{ex} - \lambda_1^{(0)} - \lambda_1^{(1)}\right| = 0,008578738,$$

$$\lambda_1^{(2)} = -\frac{1}{96}\left(1 + \frac{15}{\pi^2} - \frac{48}{\pi^3} - \frac{96}{(e^{\pi}-1)\pi^3}\right),$$

$$\left|\lambda_1^{ex} - \overset{2}{\lambda}_1\right| = \left|\lambda_1^{ex} - \lambda_1^{(0)} - \lambda_1^{(1)} - \lambda_1^{(2)}\right| = 0,000086933,$$

which is in a good agreement with (39).

Analogously one can consider equation (34) subject to the boundary condition of type $(0,1;0,1)$ and the following example is the experimental confirmation of the main result.

Example 4. Let us consider equation (33) with $a(\xi) = 1+\xi$, $b(\xi) = c(\xi) \equiv 0$, $d(\xi) \equiv 1$. After the change of variables

$$\xi = \left(1 + \frac{3}{4}\,x\right)^{\frac{4}{3}} - 1, \quad v(\xi) = \left(1 + \frac{3}{4}\,x\right)^{-\frac{1}{6}} u(x) \tag{44}$$

equation (33) becomes

$$u^{(4)}(x) + \frac{13}{18\left(x+\frac{4}{3}\right)^2}\,u''(x) + \frac{13}{9\left(x+\frac{4}{3}\right)^3}\,u'(x) + \left(-\lambda + \frac{17}{16\left(x+\frac{4}{3}\right)^4}\,u(x)\right) = 0. \tag{45}$$

The boundary conditions of the type $(0,1;0,1)$ after substitution (44) switch to the boundary conditions of the same type for equation (45) on the interval $(0,a)$ with $a = \dfrac{4}{3}\,(2^{3/4}-1)$:

$$u(0) = u'(0) = u(a) = u'(a) = 0. \tag{46}$$

The smallest exact eigenvalue of problem (45), (46) is $\lambda_1^{ex} = 729,5132640790354497$. Our method of rank 1 provides the following results:

$$\lambda_1^{(0)} = 729,0804175123859275, \quad \left|\lambda_1^{ex} - \lambda_1^{(0)}\right| = 0,432846566,$$

$$\lambda_1^{(1)} = 0,4329291815470396, \quad \left|\lambda_1^{ex} - \overset{1}{\lambda}_1\right| = \left|\lambda_1^{ex} - \lambda_1^{(0)} - \lambda_1^{(1)}\right| = 0,000082614.$$

Now, let us consider the approach where the coefficients of the differential equation are changed by the piecewise constant functions (metodo dei tronconi), for example, let us consider instead of

$$\frac{d^2}{d\xi^2}\left((1+\xi)\frac{d^2}{d\xi^2}\,y(\xi)\right)-\tilde{\lambda}y(\xi)=0,\quad y(0)=y'(0)=y(1)=y'(1)=0$$

the problem

$$\frac{d^4}{d\xi^4}\,\tilde{y}(\xi)-\frac{2}{3}\,\tilde{\lambda}\tilde{y}(\xi)=0,\quad \tilde{y}(0)=\tilde{y}'(0)=\tilde{y}(1)=\tilde{y}'(1)=0.$$

Then by our method we obtain the following approximation for the smallest eigenvalue

$$\tilde{\lambda_1}=750,8458526104090604,\quad \lambda_1^{ex}-\tilde{\lambda_1}=21,33258853$$

which is much coarser as the approximation $\lambda_1^{(0)}$ obtained by the FD-method of the rank zero.

Example 5. Let us consider differential equation (34) with $k_2(x)=k_1(x)\equiv 0$, $k_0(x)= = k_0(1-x)=\left(x-\frac{1}{2}\right)^2$ subject to the boundary conditions of type $(2,3;2,3)$, i.e.,

$$u^{(4)}(x)+\left[\left(x-\frac{1}{2}\right)^2-\lambda\right]u(x)=0,\quad x\in(0,1),$$
$$u^{(k)}(0)=u^{(k)}(1)=0,\quad k=2,3. \tag{47}$$

The smallest eigenvalues of problem (47) computed with the computer algebra tool Maple is

$$\lambda_{0,1}^{ex}=0,0833223112249938\ldots,\quad \lambda_{0,2}^{ex}=0,14999891773580\ldots.$$

The base problem

$$\frac{d^4u_n^{(0)}(x)}{dx^4}-\lambda_n^{(0)}u_n^{(0)}(x)=0,\quad x\in(0,1),\quad \frac{d^ju_n^{(0)}(0)}{dx^j}=\frac{d^ju_n^{(0)}(1)}{dx^j}=0,\quad j=2,3,$$

possesses the double eigenvalue $\lambda^{(0)}=0$ corresponding to the orthonormal eigenfunctions

$$u_{0,1}^{(0)}(x)=1\quad\text{and}\quad u_{0,2}^{(0)}(x)=2\sqrt{3}\left(\frac{1}{2}-x\right)$$

(see, e.g., [13]). All other eigenvalues are simple. The results obtained by the FD-method of rank $m=\overline{0,3}$ are

$$\overset{0}{\lambda}_0=0,\quad \overset{1}{\lambda}_{0,1}=\frac{1}{12}=0,08(3),\quad \overset{1}{\lambda}_{0,2}=\frac{3}{20}=0,15,$$

$$\overset{2}{\lambda}_{0,1}=\frac{7559}{90720}=0,0833223104056437\ldots,\quad \overset{2}{\lambda}_{0,2}=\frac{138599}{924000}=0,14999891774891\ldots,$$

$$\overset{3}{\lambda}_{0,1}=\frac{163437676007}{1961511552000}=0,0833223112249098\ldots,$$

$$\overset{3}{\lambda}_{0,2}=\frac{34306024477}{228708480000}=0,14999891773580\ldots.$$

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