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ON THE CAUCHY PROBLEM FOR TWO-DIMENSIONAL SYSTEMS OF LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH MONOTONE OPERATORS*
ПРО ЗАДАЧУ КОШІ ДЛЯ ДВОВИМІРНИХ СИСТЕМ ЛІНІЙНИХ ФУНКЦІОНАЛЬНО-ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ З МОНОТОННИМИ ОПЕРАТОРАМИ

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We establish new efficient conditions sufficient for the unique solvability of the Cauchy problem for two-dimensional systems of linear functional differential equations with monotone operators.

Знайдено нові ефективні умови, що є достатніми для існування єдиного розв'язку задачі Коші для двовимірних систем лінійних функціонально-диференціальних рівнянь з монотонними операторами.

1. Introduction and rotation. On the interval $[a, b]$, we consider two-dimensional differential system

$$u_i'(t) = \sigma_{i1}\,\ell_{i1}(u_1)(t) + \sigma_{i2}\,\ell_{i2}(u_2)(t) + q_i(t), \qquad i = 1, 2, \tag{1.1}$$

with the initial conditions

$$u_1(a) = c_1, \qquad u_2(a) = c_2\,, \tag{1.2}$$

where $\ell_{ik} : C([a, b]; \mathbb{R}) \to L([a, b]; \mathbb{R})$ are linear nondecreasing operators, $\sigma_{ik} \in \{-1, 1\}$, $q_i \in L([a, b]; \mathbb{R})$, and $c_i \in \mathbb{R}$, $i, k = 1, 2$. By a solution of the problem (1.1), (1.2) we understand an absolutely continuous vector function $u = (u_1, u_2)^T : [a, b] \to \mathbb{R}^2$ satisfying (1.1) almost everywhere on $[a, b]$ and verifying also the initial conditions (1.2).

The problem of solvability of the Cauchy problem for linear functional differential equations and their systems has been studied by many authors (see, e.g., [1–6] and references therein). There are a lot of interesting results but only a few efficient conditions is known at present. Furthermore, most of them are available for the one-dimensional case only or for systems with the so-called Volterra operators (see, e.g., [2, 3, 5, 7–9]). Let us mention that the efficient conditions guaranteeing the unique solvability of the initial value problem for n-dimensional systems of linear functional differential equations are given, e.g., in [4, 10–13].

In this paper, we establish new efficient condition sufficient for the unique solvability of the problem (1.1), (1.2) with $\sigma_{11} = 1$ and $\sigma_{22} = 1$. The cases where $\sigma_{11}\sigma_{22} = -1$ and $\sigma_{11} = \sigma_{22} = -1$ are studied in [14] and [15], respectively.

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The integral conditions given in Theorems 2.1 and 2.2 are optimal in a certain sense which is shown by counter-examples constructed in the last part of the paper.

The following notation is used throughout the paper:

(1) $\mathbb{R}$ is the set of all real numbers, $\mathbb{R}_+ = [0, +\infty[$;

(2) $C([a,b];\mathbb{R})$ is the Banach space of continuous functions $u : [a,b] \to \mathbb{R}$ equipped with the norm

$$\|u\|_C = \max\Big\{|u(t)| : t \in [a,b]\Big\};$$

(3) $L([a,b];\mathbb{R})$ is the Banach space of Lebesgue integrable functions $h : [a,b] \to \mathbb{R}$ equipped with the norm

$$\|h\|_L = \int_a^b |h(s)|ds;$$

(4) $L\big([a,b];\mathbb{R}_+\big) = \Big\{h \in L([a,b];\mathbb{R}) : h(t) \geq 0 \text{ for a.a. } t \in [a,b]\Big\}$;

(5) an operator $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ is said to be nondecreasing if the inequality

$$\ell(u_1)(t) \leq \ell(u_2)(t) \quad \text{for a.a.} \quad t \in [a,b]$$

holds for every functions $u_1, u_2 \in C([a,b];\mathbb{R})$ such that

$$u_1(t) \leq u_2(t) \quad \text{for} \quad t \in [a,b];$$

(6) $\mathcal{P}_{ab}$ is the set of linear nondecreasing operators $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$.

In what follows, the equalities and inequalities with integrable functions are understood to hold almost everywhere.

2. Main results. In this section, we present the main results of the paper. The proofs are given later, in Section 3. Theorems formulated below contain the efficient conditions sufficient for the unique solvability of the problem (1.1), (1.2) with $\sigma_{11} = 1$ and $\sigma_{22} = 1$. Recall that the operators ℓ_{ik} are supposed to be linear and nondecreasing, i.e., such that $\ell_{ik} \in \mathcal{P}_{ab}$ for $i,k = 1,2$.

Put

$$A_{ik} = \int_a^b \ell_{ik}(1)(s)ds \quad \text{for} \quad i,k = 1,2. \tag{2.1}$$

At first, we consider the case where $\sigma_{12}\sigma_{21} > 0$.

Theorem 2.1. *Let $\sigma_{11} = 1$, $\sigma_{22} = 1$, and $\sigma_{12}\sigma_{21} > 0$. Let, moreover,*

$$A_{11} < 1, \qquad A_{22} < 1, \tag{2.2}$$

and

$$A_{12}\,A_{21} < (1 - A_{11})(1 - A_{22}), \tag{2.3}$$

where the numbers A_{ik}, $i,k = 1,2$, are defined by (2.1). Then the problem (1.1), (1.2) has a unique solution.

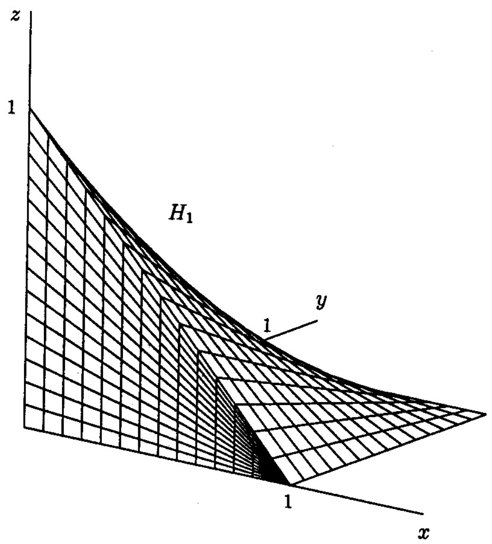

Fig. 2.1

Remark 2.1. Neither one of the strict inequalities (2.2) and (2.3) can be replaced by the nonstrict one (see Examples 4.1 and 4.2).

Remark 2.2. Let H_1 be the set of triplets $(x, y, z) \in \mathbb{R}^3_+$ satisfying

$$x < 1, \quad y < 1, \quad z < (1 - x)(1 - y)$$

(see Fig. 2.1). According to Theorem 2.1, the problem (1.1), (1.2) is uniquely solvable if $\ell_{ik} \in \mathcal{P}_{ab}$, $i, k = 1, 2$, are such that

$$\left(\int_a^b \ell_{11}(1)(s)ds\,, \int_a^b \ell_{22}(1)(s)ds\,, \int_a^b \ell_{12}(1)(s)ds \int_a^b \ell_{21}(1)(s)ds \right) \in H_1\,.$$

Remark 2.3. It should be noted that Theorem 2.1 can be derived as a consequence of Corollary 1.3.1 given in [4]. However, we shall prove this theorem using the technique common for both theorems given in this paper.

Remark 2.4. It follows from Corollary 3.2 of [16] that if $\sigma_{11} = 1, \sigma_{22} = 1, \sigma_{12}\sigma_{21} > 0$, and

$$A_{11} + A_{12} < 1, \qquad A_{21} + A_{22} < 1, \tag{2.4}$$

where the numbers A_{ik}, $i, k = 1, 2$, are defined by (2.1), then the problem (1.1), (1.2) has a unique solution $(u_1, u_2)^T$. Moreover, this solution satisfies

$$u_1(t) \geq 0, \quad \sigma_{12}u_2(t) \geq 0 \quad \text{for} \quad t \in [a, b]$$

provided that $c_1 \geq 0, \sigma_{12}c_2 \geq 0$, and

$$q_1(t) \geq 0, \quad \sigma_{12}q_2(t) \geq 0 \quad \text{for} \quad t \in [a, b].$$

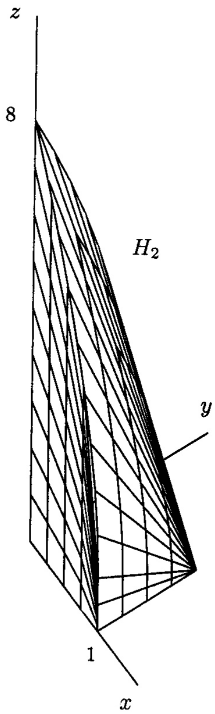

Fig. 2.2

On the other hand, if the assumption (2.4) is weakened to the assumptions (2.2), (2.3) then the problem (1.1), (1.2) has still a unique solution but no information about the sign of this solution is guaranteed in general.

Now we consider the case where $\sigma_{12}\sigma_{21} < 0$.

Theorem 2.2. *Let* $\sigma_{11} = 1$, $\sigma_{22} = 1$, *and* $\sigma_{12}\sigma_{21} < 0$. *Let, moreover, the condition (2.2) be satisfied and*

$$A_{12}A_{21} < 4\sqrt{(1-A_{11})(1-A_{22})} + \left(\sqrt{1-A_{11}} + \sqrt{1-A_{22}}\right)^2, \tag{2.5}$$

where the numbers A_{ik}, $i,k = 1,2$, *are defined by (2.1). Then the problem (1.1), (1.2) has a unique solution.*

Remark 2.5. The strict inequalities (2.2) in Theorem 2.2 cannot be replaced by the nonstrict ones (see Example 4.1). Furthermore, the strict inequality (2.5) cannot be replaced by the nonstrict one provided $A_{11} = A_{22}$ (see Example 4.3).

Remark 2.6. Let H_2 be the set of triplets $(x, y, z) \in \mathbb{R}^3_+$ satisfying

$$x < 1, \quad y < 1, \quad z < 4\sqrt{(1-x)(1-y)} + \left(\sqrt{1-x} + \sqrt{1-y}\right)^2$$

(see Fig. 2.2). According to Theorem 2.2, the problem (1.1), (1.2) is uniquely solvable if $\ell_{ik} \in \mathcal{P}_{ab}$, $i,k = 1,2$, are such that

$$\left(\int_a^b \ell_{11}(1)(s)ds\,,\ \int_a^b \ell_{22}(1)(s)ds\,,\ \int_a^b \ell_{12}(1)(s)ds \int_a^b \ell_{21}(1)(s)ds\right) \in H_2\,.$$

At last, we give consequences of Theorems 2.1 and 2.2 for the system with argument deviations,

$$\begin{aligned} u_1'(t) &= h_{11}(t)u_1\big(\tau_{11}(t)\big) + \sigma_1 h_{12}(t)u_2\big(\tau_{12}(t)\big) + q_1(t), \\ u_2'(t) &= \sigma_2 h_{21}(t)u_1\big(\tau_{21}(t)\big) + h_{22}(t)u_2\big(\tau_{22}(t)\big) + q_2(t), \end{aligned} \tag{2.6}$$

where $h_{ik} \in L\big([a,b];\mathbb{R}_+\big)$, $\tau_{ik} : [a,b] \to [a,b]$ are measurable functions, $\sigma_i \in \{-1,1\}$, and $q_i \in L\big([a,b];\mathbb{R}\big)$, $i,k = 1,2$.

Corollary 2.1. *Let $\sigma_1\sigma_2 > 0$ and let the conditions (2.2) and (2.3) be fulfilled, where*

$$A_{ik} = \int_a^b h_{ik}(s)ds \quad \textit{for} \quad i,k = 1,2. \tag{2.7}$$

Then the problem (2.6), (1.2) has a unique solution.

Corollary 2.2. *Let $\sigma_1\sigma_2 < 0$ and let the conditions (2.2) and (2.5) be fulfilled, where the numbers A_{ik}, $i,k = 1,2$, are defined by (2.7). Then the problem (2.6), (1.2) has a unique solution.*

3. Proofs of the main results. In this section, we shall prove the statements formulated above. Recall that the numbers A_{ik}, $i,k = 1,2$, are defined by (2.1).

It is well-known from the general theory of boundary-value problems for functional differential equations (see, e.g., [4, 11, 17, 18]) that the following lemma is true.

Lemma 3.1. *The problem (1.1), (1.2) is uniquely solvable if and only if the corresponding homogeneous problem*

$$u_i'(t) = \sigma_{i1}\,\ell_{i1}(u_1)(t) + \sigma_{i2}\,\ell_{i2}(u_2)(t), \qquad i = 1,2, \tag{3.1}$$

$$u_1(a) = 0, \qquad u_2(a) = 0, \tag{3.2}$$

has only the trivial solution.

In order to simplify the discussion in the proofs, we formulate the following obvious lemma.

Lemma 3.2. $(u_1,u_2)^T$ *is a solution of the problem (3.1), (3.2) if and only if* $(u_1,-u_2)^T$ *is a solution of the problem*

$$v_i'(t) = (-1)^{i-1}\sigma_{i1}\,\ell_{i1}(v_1)(t) + (-1)^{i}\sigma_{i2}\,\ell_{i2}(v_2)(t), \qquad i = 1,2, \tag{3.3}$$

$$v_1(a) = 0, \qquad v_2(a) = 0\,. \tag{3.4}$$

Lemma 3.3 ([19], Remark 1.1). *Let $\ell \in \mathcal{P}_{ab}$ be such that*

$$\int_a^b \ell(1)(s)ds < 1.$$

Then every absolutely continuous function $u : [a,b] \to \mathbb{R}$ such that

$$u'(t) \geq \ell(u)(t) \quad \textit{for} \quad t \in [a,b], \qquad u(a) \geq 0,$$

satisfies $u(t) \geq 0$ for $t \in [a,b]$.

Now we are in a position to prove the main results.

Proof of Theorem 2.1. According to Lemmas 3.1 and 3.2, in order to prove the theorem it is sufficient to show that the system

$$u_i'(t) = \ell_{i1}(u_1)(t) + \ell_{i2}(u_2)(t), \qquad i = 1,2, \tag{3.5}$$

has only the trivial solution satisfying (3.2).

Suppose that, on the contrary, $(u_1, u_2)^T$ is a nontrivial solution of the problem (3.5), (3.2). If the inequality

$$u_i(t) \geq 0 \quad \text{for} \quad t \in [a,b] \tag{3.6}$$

holds for some $i \in \{1,2\}$ then, by virtue of (2.2), the assumption $\ell_{3-i\,i} \in \mathcal{P}_{ab}$, and Lemma 3.3, we get

$$u_{3-i}(t) \geq 0 \quad \text{for} \quad t \in [a,b]. \tag{3.7}$$

Consequently, the functions u_1 and u_2 satisfy one of the following alternatives.

(a) Both functions u_1 and u_2 do not change their signs. Then, without loss of generality, we can assume that (3.6) holds for $i = 1,2$.

(b) Both functions u_1 and u_2 change their signs.

Put

$$M_i = \max\big\{u_i(t) : t \in [a,b]\big\}, \qquad i = 1,2, \tag{3.8}$$

and choose $\alpha_i \in [a,b]$, $i = 1,2$, such that

$$u_i(\alpha_i) = M_i \quad \text{for} \quad i = 1,2. \tag{3.9}$$

Obviously, in both cases (a) and (b), we have

$$M_1 \geq 0, \quad M_2 \geq 0, \quad M_1 + M_2 > 0. \tag{3.10}$$

The integration of (3.5) from a to α_i, in view of (3.8) – (3.10), and the assumptions $\ell_{i1}, \ell_{i2} \in \mathcal{P}_{ab}$, yield

$$M_i = \int_a^{\alpha_i} \ell_{i1}(u_1)(s)ds + \int_a^{\alpha_i} \ell_{i2}(u_2)(s)ds \le$$

$$\le M_1 \int_a^{\alpha_i} \ell_{i1}(1)(s)ds + M_2 \int_a^{\alpha_i} \ell_{i2}(1)(s)ds \le$$

$$\le M_1 A_{i1} + M_2 A_{i2}, \quad i = 1, 2. \tag{3.11}$$

By virtue of (2.2) and (3.10), we get from (3.11) that

$$0 \le M_i\big(1 - A_{ii}\big) \le M_{3-i} A_{i\,3-i}, \qquad i = 1, 2. \tag{3.12}$$

Using (2.2) and (3.10) once again, (3.12) implies $M_1 > 0$, $M_2 > 0$, and

$$(1 - A_{11})(1 - A_{22}) \le A_{12}A_{21},$$

which contradicts (2.3).

The contradiction obtained proves that the problem (3.5), (3.2) has only the trivial solution.

Proof of Theorem 2.2. According to Lemmas 3.1 and 3.2, in order to prove the theorem it is sufficient to show that the system

$$u_1'(t) = \ell_{11}(u_1)(t) + \ell_{12}(u_2)(t), \tag{3.13}$$

$$u_2'(t) = -\ell_{21}(u_1)(t) + \ell_{22}(u_2)(t) \tag{3.14}$$

has only the trivial solution satisfying (3.2).

Suppose that, on the contrary, $(u_1, u_2)^T$ is a nontrivial solution of the problem (3.13), (3.14), (3.2). It is clear that u_1 and u_2 satisfy one of the following.

(a) One of the functions u_1 and u_2 is of a constant sign. According to Lemma 3.2, we can assume without loss of generality that $u_1(t) \ge 0$ for $t \in [a, b]$.

(b) Both functions u_1 and u_2 change their signs.

Case (a): $u_1(t) \ge 0$ for $t \in [a, b]$. In view of (2.2) and the assumption $\ell_{21} \in \mathcal{P}_{ab}$, Lemma 3.3 yields $u_2(t) \le 0$ for $t \in [a, b]$. Now, by virtue of (2.2) and the assumption $\ell_{12} \in \mathcal{P}_{ab}$, Lemma 3.3 again implies $u_1(t) \le 0$ for $t \in [a, b]$. Consequently, $u_1 \equiv 0$ and Lemma 3.3 once again results in $u_2 \equiv 0$, which is a contradiction.

Case (b): u_1 and u_2 change their signs. For $i = 1, 2$, we put

$$M_i = \max\big\{u_i(t) : t \in [a, b]\big\}, \quad m_i = -\min\big\{u_i(t) : t \in [a, b]\big\}. \tag{3.15}$$

Choose $\alpha_i, \beta_i \in [a, b]$, $i = 1, 2$, such that the equalities

$$u_1(\alpha_1) = M_1\,, \qquad u_1(\beta_1) = -m_1 \tag{3.16}$$

and

$$u_2(\alpha_2) = M_2\,, \qquad u_2(\beta_2) = -m_2 \tag{3.17}$$

are satisfied. Obviously,

$$M_i > 0, \quad m_i > 0 \quad \text{for} \quad i = 1, 2. \tag{3.18}$$

Furthermore, for $i, k = 1, 2$, we denote

$$B_{ik} = \int\limits_a^{\min\{\alpha_i,\beta_i\}} \ell_{ik}(1)(s)ds, \qquad D_{ik} = \int\limits_{\min\{\alpha_i,\beta_i\}}^{\max\{\alpha_i,\beta_i\}} \ell_{ik}(1)(s)ds. \tag{3.19}$$

It is clear that

$$B_{ik} + D_{ik} \le A_{ik} \quad \text{for} \quad i, k = 1, 2. \tag{3.20}$$

According to Lemma 3.2, we can assume without loss of generality that $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$. The integrations of (3.13) from a to α_1 and from α_1 to β_1, in view of (3.15), (3.16), (3.19), and the assumptions $\ell_{11}, \ell_{12} \in \mathcal{P}_{ab}$, result in

$$M_1 = \int\limits_a^{\alpha_1} \ell_{11}(u_1)(s)ds + \int\limits_a^{\alpha_1} \ell_{12}(u_2)(s)ds \le$$

$$\le M_1 \int\limits_a^{\alpha_1} \ell_{11}(1)(s)ds + M_2 \int\limits_a^{\alpha_1} \ell_{12}(1)(s)ds = M_1 B_{11} + M_2 B_{12}$$

and

$$M_1 + m_1 = -\int\limits_{\alpha_1}^{\beta_1} \ell_{11}(u_1)(s)ds - \int\limits_{\alpha_1}^{\beta_1} \ell_{12}(u_2)(s)ds \le$$

$$\le m_1 \int\limits_{\alpha_1}^{\beta_1} \ell_{11}(1)(s)ds + m_2 \int\limits_{\alpha_1}^{\beta_1} \ell_{12}(1)(s)ds = m_1 D_{11} + m_2 D_{12}\,.$$

The last relations, by virtue of (2.2) and (3.18), imply

$$0 < \frac{M_1}{M_2}(1 - B_{11}) + \frac{m_1}{m_2}(1 - D_{11}) + \frac{M_1}{m_2} \le B_{12} + D_{12} \le A_{12}\,. \tag{3.21}$$

On the other hand, the integrations of (3.14) from a to α_2 and from α_2 to β_2, using (3.15), (3.17), (3.19), and the assumptions $\ell_{21}, \ell_{22} \in \mathcal{P}_{ab}$, give

$$M_2 = -\int_a^{\alpha_2} \ell_{21}(u_1)(s)ds + \int_a^{\alpha_2} \ell_{22}(u_2)(s)ds \le$$

$$\le m_1 \int_a^{\alpha_2} \ell_{21}(1)(s)ds + M_2 \int_a^{\alpha_2} \ell_{22}(1)(s)ds = m_1 B_{21} + M_2 B_{22}$$

and

$$M_2 + m_2 = \int_{\alpha_2}^{\beta_2} \ell_{21}(u_1)(s)ds - \int_{\alpha_2}^{\beta_2} \ell_{22}(u_2)(s)ds \le$$

$$\le M_1 \int_{\alpha_2}^{\beta_2} \ell_{21}(1)(s)ds + m_2 \int_{\alpha_2}^{\beta_2} \ell_{22}(1)(s)ds = M_1 D_{21} + m_2 D_{22}\,.$$

The last relations, by virtue of (2.2) and (3.18), yield

$$0 < \frac{M_2}{m_1}(1 - B_{22}) + \frac{m_2}{M_1}(1 - D_{22}) + \frac{M_2}{M_1} \le B_{21} + D_{21} \le A_{21}\,. \tag{3.22}$$

Now, it follows from (3.21) and (3.22) that

$$A_{12}A_{21} \ge \frac{M_1}{m_1}(1 - B_{11})(1 - B_{22}) + \frac{m_2}{M_2}(1 - B_{11})(1 - D_{22}) + 1 - B_{11} +$$

$$+ \frac{M_2}{m_2}(1 - D_{11})(1 - B_{22}) + \frac{m_1}{M_1}(1 - D_{11})(1 - D_{22}) + \frac{m_1 M_2}{m_2 M_1}(1 - D_{11}) +$$

$$+ \frac{M_2 M_1}{m_1 m_2}(1 - B_{22}) + 1 - D_{22} + \frac{M_2}{m_2}\,. \tag{3.23}$$

Using the relation

$$x + y \ge 2\sqrt{xy} \quad \text{for} \quad x \ge 0,\ y \ge 0,$$

it is easy to verify that

$$\frac{M_1}{m_1}(1 - B_{11})(1 - B_{22}) + \frac{m_1}{M_1}(1 - D_{11})(1 - D_{22}) \ge$$

$$\ge 2\sqrt{(1 - B_{11})(1 - B_{22})(1 - D_{11})(1 - D_{22})} \ge$$

$$\ge 2\sqrt{(1 - B_{11} - D_{11})(1 - B_{22} - D_{22})} \ge 2\sqrt{(1 - A_{11})(1 - A_{22})}\,,$$

$$\frac{m_1 M_2}{m_2 M_1}(1-D_{11})+\frac{M_2 M_1}{m_1 m_2}(1-B_{22})\geq 2\,\frac{M_2}{m_2}\sqrt{(1-D_{11})(1-B_{22})}\,, \tag{3.24}$$

$$\frac{M_2}{m_2}(1-D_{11})(1-B_{22})+2\,\frac{M_2}{m_2}\sqrt{(1-D_{11})(1-B_{22})}+\frac{M_2}{m_2}=$$

$$=\frac{M_2}{m_2}\left(\sqrt{(1-D_{11})(1-B_{22})}+1\right)^2,$$

and

$$\frac{m_2}{M_2}(1-B_{11})(1-D_{22})+\frac{M_2}{m_2}\left(\sqrt{(1-D_{11})(1-B_{22})}+1\right)^2\geq$$

$$\geq 2\sqrt{(1-B_{11})(1-D_{22})}\left(\sqrt{(1-D_{11})(1-B_{22})}+1\right)\geq$$

$$\geq 2\sqrt{(1-B_{11}-D_{11})(1-B_{22}-D_{22})}+2\sqrt{(1-B_{11})(1-D_{22})}\geq$$

$$\geq 2\sqrt{(1-A_{11})(1-A_{22})}+2\sqrt{(1-B_{11})(1-D_{22})}\,. \tag{3.25}$$

Therefore, by virtue of (3.24), (3.25), (3.23) implies

$$A_{12}A_{21}\geq$$

$$\geq 4\sqrt{(1-A_{11})(1-A_{22})}+1-B_{11}+2\sqrt{(1-B_{11})(1-D_{22})}+1-D_{22}\geq$$

$$\geq 4\sqrt{(1-A_{11})(1-A_{22})}+\left(\sqrt{1-A_{11}}+\sqrt{1-A_{22}}\right)^2,$$

which contradicts (2.5).

The contradictions obtained in (a) and (b) prove that the problem (3.13), (3.14), (3.2) has only the trivial solution.

Proof of Corollary 2.1. The validity of the corollary follows immediately from Theorem 2.1.

Proof of Corollary 2.2. The validity of the corollary follows immediately from Theorem 2.2.

4. Counter-examples. In this part, the counter-examples are constructed verifying that the results obtained above are optimal in a certain sense.

Example 4.1. Let $\sigma_{ik}\in\{-1,1\}$, $h_{ik}\in L\big([a,b];\mathbb{R}_+\big)$, $i,k=1,2$, be such that

$$\sigma_{11}=1,\qquad \int\limits_a^b h_{11}(s)ds\geq 1.$$

It is clear that there exists $t_0 \in]a,b]$ such that

$$\int_a^{t_0} h_{11}(s)ds = 1.$$

Let the operators $\ell_{ik} \in \mathcal{P}_{ab}$, $i,k = 1,2$, be defined by

$$\ell_{ik}(v)(t) \stackrel{\text{df}}{=} h_{ik}(t)v\big(\tau_{ik}(t)\big) \quad \text{for} \quad t \in [a,b],\ v \in C([a,b];\mathbb{R}), \tag{4.1}$$

where $\tau_{11}(t) = t_0$, $\tau_{12}(t) = a$, $\tau_{21}(t) = a$, and $\tau_{22}(t) = a$ for $t \in [a,b]$. Put

$$u(t) = \int_a^t h_{11}(s)ds \quad \text{for} \quad t \in [a,b].$$

It is easy to verify that $(u,0)^T$ is a nontrivial solution of the problem (1.1), (1.2) with $q_i \equiv 0$ and $c_i = 0$, $i = 1,2$.

An analogous example can be constructed for the case where

$$\sigma_{22} = 1, \qquad \int_a^b h_{22}(s)ds \geq 1.$$

This example shows that the constant 1 in the right-hand side of the inequalities in (2.2) is optimal and cannot be weakened.

Example 4.2. Let $\sigma_{ik} = 1$ for $i,k = 1,2$ and let $h_{ik} \in L\big([a,b];\mathbb{R}_+\big)$, $i,k = 1,2$, be such that

$$\int_a^b h_{11}(s)ds < 1, \quad \int_a^b h_{22}(s)ds < 1, \tag{4.2}$$

and

$$\int_a^b h_{12}(s)ds \int_a^b h_{21}(s)ds \geq \left(1 - \int_a^b h_{11}(s)ds\right)\left(1 - \int_a^b h_{22}(s)ds\right).$$

It is clear that there exists $t_0 \in]a,b]$ such that

$$\int_a^{t_0} h_{12}(s)ds \int_a^{t_0} h_{21}(s)ds = \left(1 - \int_a^{t_0} h_{11}(s)ds\right)\left(1 - \int_a^{t_0} h_{22}(s)ds\right).$$

Let the operators $\ell_{ik} \in \mathcal{P}_{ab}$, $i, k = 1, 2$, be defined by (4.1), where $\tau_{ik}(t) = t_0$ for $t \in [a, b]$, $i, j = 1, 2$. Put

$$u_1(t) = \int_a^t h_{11}(s)ds + \frac{1 - \int_a^{t_0} h_{11}(s)ds}{\int_a^{t_0} h_{12}(s)ds} \int_a^t h_{12}(s)ds \quad \text{for} \quad t \in [a, b],$$

$$u_2(t) = \int_a^t h_{21}(s)ds + \frac{\int_a^{t_0} h_{21}(s)ds}{1 - \int_a^{t_0} h_{22}(s)ds} \int_a^t h_{22}(s)ds \quad \text{for} \quad t \in [a, b].$$

It is easy to verify that $(u_1, u_2)^T$ is a nontrivial solution of the problem (1.1), (1.2) with $q_i \equiv 0$ and $c_i = 0$, $i = 1, 2$.

This example shows that the strict inequality (2.3) in Theorem 2.1 cannot be replaced by the nonstrict one.

Example 4.3. Let $\sigma_{11} = 1, \sigma_{12} = 1, \sigma_{21} = -1$, and $\sigma_{22} = 1$. Let $\alpha \in [0, 1[$ and $h_{12}, h_{21} \in L([a, b]; \mathbb{R}_+)$ be such that

$$\int_a^b h_{12}(s)ds \int_a^b h_{21}(s)ds \geq 8(1 - \alpha).$$

It is clear that there exist $t_0 \in]a, b]$ and $t_1, t_2 \in]a, t_0[$ such that

$$\int_a^{t_0} h_{12}(s)ds \int_a^{t_0} h_{21}(s)ds = 8(1 - \alpha)$$

and

$$\int_a^{t_1} h_{12}(s)ds = \frac{1}{4} \int_a^{t_0} h_{12}(s)ds, \qquad \int_a^{t_2} h_{21}(s)ds = \frac{1}{2} \int_a^{t_0} h_{21}(s)ds.$$

Furthermore, we choose $h_{11}, h_{22} \in L([a, b]; \mathbb{R}_+)$ with the properties

$$h_{11}(t) = 0 \quad \text{for} \quad t \in [a, t_1] \cup [t_0, b], \qquad h_{22}(t) = 0 \quad \text{for} \quad t \in [t_2, b],$$

and

$$\int_a^b h_{11}(s)ds = \int_a^b h_{22}(s)ds = \alpha.$$

Let the operators $\ell_{ik} \in \mathcal{P}_{ab}$, $i,k = 1,2$, be defined by (4.1), where $\tau_{11}(t) = t_0$, $\tau_{22}(t) = t_2$ for $t \in [a,b]$, and

$$\tau_{12}(t) = \begin{cases} t_0 & \text{for} \quad t \in [a,t_1[, \\ t_2 & \text{for} \quad t \in [t_1,b], \end{cases} \qquad \tau_{21}(t) = \begin{cases} t_1 & \text{for} \quad t \in [a,t_2[, \\ t_0 & \text{for} \quad t \in [t_2,b]. \end{cases}$$

Put

$$u_1(t) = \begin{cases} \int\limits_{t_2}^{t_0} h_{21}(s)ds \int\limits_a^t h_{12}(s)ds & \text{for} \quad t \in [a,t_1[, \\ 1-\alpha-2\int\limits_{t_1}^{t} h_{11}(s)ds - \int\limits_{t_2}^{t_0} h_{21}(s)ds \int\limits_{t_1}^{t} h_{12}(s)ds & \text{for} \quad t \in [t_1,b], \end{cases}$$

$$u_2(t) = \begin{cases} -(1-\alpha)\int\limits_a^t h_{21}(s)ds - \int\limits_{t_2}^{t_0} h_{21}(s)ds \int\limits_a^t h_{22}(s)ds & \text{for} \quad t \in [a,t_2[, \\ -\int\limits_{t_2}^{t_0} h_{21}(s)ds + 2\int\limits_{t_2}^{t} h_{21}(s)ds & \text{for} \quad t \in [t_2,b]. \end{cases}$$

It is easy to verify that $(u_1,u_2)^T$ is a nontrivial solution of the problem (1.1), (1.2) with $q_i \equiv 0$ and $c_i = 0$, $i = 1,2$.

This example shows that the strict inequality (2.5) in Theorem 2.2 cannot be replaced by the nonstrict one provided $A_{11} = A_{22}$.

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