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ON THE NUMERICAL-ANALYTIC INVESTIGATION OF PARAMETRIZED PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS

ПРО ЧИСЕЛЬНО-АНАЛІТИЧНЕ ВИВЧЕННЯ ПАРАМЕТРИЗОВАНИХ ЗАДАЧ З НЕЛІНІЙНИМИ ГРАНИЧНИМИ УМОВАМИ

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We consider a parametrized boundary-value problem containing an unknown parameter both in the nonlinear ordinary differential equations and in the nonlinear boundary conditions. By using a suitable change of variables, we reduce the original problem to a family of those with linear boundary conditions plus some nonlinear algebraic determining equations. We construct a numerical-analytic scheme suitable for studying the solutions of the transformed boundary-value problem.

Розглядається параметризована гранична задача, що містить невідомий параметр у нелінійних звичайних диференціальних рівняннях і в нелінійних граничних умовах. Використовуючи відповідну заміну змінних, початкову задачу зведено до сім'ї задач з лінійними граничними умовами та деяких нелінійних алгебраїчних визначальних рівнянь. Побудовано чисельно-аналітичну схему, яку можна використовувати для вивчення розв'язків перетвореної граничної задачі.

1. Introduction. The parametrized boundary-value problems (PBVPs) were studied earlier mostly in the case when the parameters are contained only in the differential equation (see, e.g., [1, 2]). The boundary-value problems with parameters both in the nonlinear differential equations and in the linear boundary conditions were investigated in [3 – 8] by using the so-called numerical-analytic method based upon successive approximations [3, 8]. According to the basic idea of the mentioned method, the given boundary-value problem (BVP) is replaced by a problem for a "perturbed" differential equation containing some new artificially introduced parameter, whose numerical value should be determined later. The solution of the modified problem is sought for in the analytic form by successive iterations with all iterations depending upon both the artificially introduced parameter and the parameter contained in the given BVP.

As to the way how the modified problem is constructed, it is essential that the form of the "perturbation term", depending on the original differential equation and boundary condition, yields a certain system of algebraic or transcendental "determining equations", that give the numerical values both for the artificially introduced parameters and for the parameters of the given BVP.

By studying these determining equations, one can establish existence results for the original PBVP. The numerical-analytic technique described above was used in different types of parametrized BVPs. Namely, in [3, 8] were studied the following two-point PBVPs:

$$\frac{dx}{dt} = f(t,x), \;\; t \in [0,T], \; x, f \in \mathbb{R}^n,$$
$$Ax(0) + \lambda Cx(T) = d, \;\; \det C \neq 0, \; \lambda \in \mathbb{R},$$
$$x_1(0) = x_{10},$$

the PBVPs with nonfixed right boundary:

$$\frac{dx}{dt} = f(t,x), \;\; t \in [0,\lambda], \; x, f \in \mathbb{R}^n,$$
$$Ax(0) + Cx(\lambda) = d, \;\; \det C \neq 0, \; \lambda \in (0,T],$$
$$x_1(0) = x_{10},$$

$$\frac{dx}{dt} = f(t,x), \;\; t \in [0,\lambda_2], \; x, f \in \mathbb{R}^n,$$
$$\lambda_1 Ax(0) + Cx(\lambda_2) = d, \;\; \det C \neq 0, \; \lambda_1 \in \mathbb{R}, \; \lambda_2 \in (0,T],$$
$$x_1(0) = x_{10}, \; x_2(0) = x_{20},$$

and the PBVP of the form

$$\frac{dx}{dt} = f(t,x), \;\; t \in [0,T], \; x, f \in \mathbb{R}^n,$$
$$\lambda_1 Ax(0) + \lambda_2 Cx(T) = d, \;\; \det C \neq 0, \; \lambda_1, \lambda_2 \in \mathbb{R},$$
$$x_1(0) = x_{10}, \; x_2(0) = x_{20}.$$

The paper [4] deals with the two-point PBVP

$$\frac{dx}{dt} = f(t,x) + \lambda_1 g(t,x), \;\; t \in [0,T], \; x, f \in \mathbb{R}^n,$$
$$Ax(0) + \lambda_2 Cx(T) = d, \;\; \det C \neq 0, \; \lambda_1, \lambda_2 \in \mathbb{R},$$
$$x_1(0) = x_{10}, \; x_2(0) = x_{20}.$$

In [5, 6] a scheme of the numerical-analytic method of successive approximations was given for studying the solutions of the PBVP

$$\frac{dx}{dt} = f(t, x, \lambda_1), \ \ t \in [0, \lambda_2], \ x, f \in \mathbb{R}^n,$$

$$\lambda_1 Ax(0) + C(\lambda_1)x(\lambda_2) = d(\lambda_2), \ \ \det C \neq 0, \ \lambda_1 \in \mathbb{R}, \ \lambda_2 \in (0, T],$$

$$x_1(0) = x_{10}, \ x_2(0) = x_{20}.$$

In the paper [7] it was studied the three-point PBVP of the form

$$\frac{dx}{dt} = f(t, x, \lambda_1), \ \ t \in [0, \lambda_2], \ x, f \in \mathbb{R}^n,$$

$$Ax(0) + A_1 x(t_1) + Cx(\lambda_2) = d(\lambda_1), \ \ \det C \neq 0, \ \lambda_1 \in \mathbb{R}, \ \lambda_2 \in (0, T],$$

$$x_1(0) = x_{10}, \ x_2(0) = x_{20}.$$

It should be noted, that the PBVPs mentioned above are subjected to linear boundary conditions. In [3, 8, 9] the methodology of the numerical-analytic method was extended in order to make it possible to study the nonlinear two-point BVP of the form

$$\frac{dy}{dt} = f(t, y(t)), \ \ t \in [0, T], \ y, f \in \mathbb{R}^n,$$

$$g(y(0), y(T)) = 0, \ \ g \in \mathbb{R}^n,$$

with nonlinear boundary conditions; with this aim, a general nonlinear change of variable was introduced in the given equation.

In the paper [10], it was suggested to use a simpler substitution, which, as was shown, greately facilitates the application of the numerical-analytic method based upon successive approximations. In particular all the assumptions for the applicability of the method are formulated in terms of the original problem, and not the transformed one. It was established that for the nonlinear PBVPs with separated nonlinear boundary conditions of the form

$$\frac{dx}{dt} = f(t, x(t)), \ \ t \in [0, T], \ x, f \in \mathbb{R}^n,$$

$$x(T) = a(x(0)), \ \ \ a \in \mathbb{R}^n,$$

the numerical-analytic method can be applied without any change of variables.

Similar results were obtained in [11] for problems with separated nonlinear boundary conditions of the form

$$\frac{dx}{dt} = f(t, x(t)), \ \ t \in [0, T], \ x, f \in \mathbb{R}^n,$$

$$x(0) = b(x(T)), \ \ \ b \in \mathbb{R}^n.$$

Following the method from [10, 11], in [12, 13] it was suggested how to construct a numerical-analytic scheme suitable for studying the PBVPs with parameters both in the nonlinear differential equation and in the nonlinear two-point boundary conditions of the forms

$$\frac{dy}{dt} = f(t, y, \lambda_1, \lambda_2), \;\; t \in [0, T], \;\; y, f \in \mathbb{R}^n,$$

$$g(y(0), y(T), \lambda_1, \lambda_2) = 0, \quad \lambda_1 \in [a_1, b_1], \lambda_2 \in [a_2, b_2],$$

$$y_1(0) = y_{10}, \;\; y_2(0) = y_{20},$$

and

$$\frac{dy}{dt} = f(t, y, \lambda_1), \;\; t \in [0, T], \;\; y, f \in \mathbb{R}^n,$$

$$g(y(0), y(T), \lambda_1, \lambda_2) = 0, \quad \lambda_1 \in [a_1, b_1], \lambda_2 \in [a_2, b_2],$$

$$y_1(0) = y_{10}, \;\; y_2(0) = y_{20}.$$

Here we give a possible approach how to handle, by using the numerical-analytic method, some PBVPs with boundary conditions of a more general form than mentioned above.

2. Problem setting. We consider the nonlinear two-point parametrized BVP

$$\frac{dy}{dt} = f(t, y(t), \lambda), \;\; t \in [0, T], \tag{2.1}$$

$$g(y(0), y(T), \lambda) = 0, \tag{2.2}$$

$$y_1(0) = \alpha_1 + \lambda \sum_{j=2}^{n} \alpha_j y_j(0), \tag{2.3}$$

containing the scalar parameter λ both in Eq. (2.1) and in conditions (2.2), (2.3).

Here, we suppose that the function

$$f : [0, T] \times G \times [a, b] \to \mathbb{R}^n, \quad n \geq 2,$$

and

$$g : G \times G \times [a, b] \to \mathbb{R}^n$$

are continuous, where $G \subset \mathbb{R}^n$ is a closed, connected, bounded domain and $\lambda \in J := [a, b]$ is an unknown scalar parameter; $\alpha_1, \alpha_2, ..., \alpha_n$ are given coefficients.

Assume that, for $t \in [0, T]$ and $\lambda \in J$ fixed, the function f satisfies the Lipschitz condition of the form

$$|f(t, u, \lambda) - f(t, v, \lambda)| \leq K |u - v| \tag{2.4}$$

for all $\{u, v\} \subset G$ and some nonnegative constant matrix $K = (K_{ij})_{i,j=1}^n$. In inequality (2.4), as well as in similar relations below, the signs $|\cdot|$, $\le$, $\ge$ are understood componentwise.

The problem is to find the values of the control parameter λ such that the problem (2.1), (2.2) has a classical continuously differentiable solution satisfying the additional condition (2.3). Thus, a solution is the pair $\{y, \lambda\}$ and, therefore, (2.1) – (2.3) is similar, in a sense, to an eigenvalue or a control problem.

3. Construction of an equivalent problem with linear boundary conditions. Let us introduce the substitution

$$y(t) = x(t) + w, \tag{3.1}$$

where $w = \operatorname{col}(w_1, w_2, \ldots, w_n) \in \Omega \subset \mathbb{R}^n$ is an unknown parameter. The domain Ω is chosen so that $D + \Omega \subset G$, whereas the new variable x is supposed to have range in D, the closure of a bounded subdomain of G. Using the change of variables (3.1), the problem (2.1) – (2.3) can be rewritten as

$$\frac{dx}{dt} = f(t, x(t) + w, \lambda), \ \ t \in [0, T], \tag{3.2}$$

$$g(x(0) + w, x(T) + w, \lambda) = 0, \tag{3.3}$$

$$x_1(0) = \alpha_1 + \lambda \sum_{j=2}^{n} \alpha_j [x_j(0) + w_j] - w_1. \tag{3.4}$$

Let us rewrite the boundary conditions (3.3) in the form

$$Ax(0) + Bx(T) + g(x(0) + w, x(T) + w, \lambda) = Ax(0) + Bx(T), \tag{3.5}$$

where A, B are fixed square n-dimentional matrices such that $\det B \neq 0$.

It is natural to determine the artificially introduced parameter w from the system of algebraic determining equations

$$Ax(0) + Bx(T) + g(x(0) + w, x(T) + w, \lambda) = 0. \tag{3.6}$$

Obviously, if (3.6) holds then, from (3.5),

$$Ax(0) + Bx(T) = 0. \tag{3.7}$$

Thus, the essentially nonlinear problem (2.1) – (2.3) with nonlinear boundary conditions turns out to be equivalent to the collection of two-point BVPs

$$\frac{dx}{dt} = f(t, x(t) + w, \lambda), \ \ t \in [0, T], \tag{3.8}$$

$$Ax(0) + Bx(T) = 0, \tag{3.9}$$

$$x_1(0) = \alpha_1 + \lambda \sum_{j=2}^{n} \alpha_j [x_j(0) + w_j] - w_1, \tag{3.10}$$

parametrized by the unknown vector $w \in \mathbb{R}^n$ and considered together with the determining equation (3.6). The essential advantage obtained thereby is that the boundary condition (3.9) is already linear.

By virtue of (3.9), every solution x of the BVP (3.8) – (3.10) satisfies the condition

$$x(T) = -B^{-1}Ax(0). \tag{3.11}$$

Therefore, taking into account (3.11), the determining equation (3.6) can be rewritten as

$$g\left(x(0) + w, -B^{-1}Ax(0) + w, \lambda\right) = 0. \tag{3.12}$$

So, we conclude that the original nonlinear BVP (2.1) – (2.3) is equivalent to the family of BVPs (3.8) – (3.10) with linear conditions (3.9) considered together with the nonlinear system of algebraic determining equations (3.12).

We note that the family of BVPs (3.8) – (3.10) can be studied by using the numerical-analytic method based upon successive approximations developed in [3, 8].

Assume that the given PBVP (2.1) – (2.3) is such that the subset

$$D_\beta := \{y \in \mathbb{R}^n : B(y, \beta(y)) \subset G\}$$

is nonempty,

$$D_\beta \neq \oslash, \tag{3.13}$$

where

$$\begin{aligned} \beta(y) &:= \frac{T}{2}\delta_G(f) + \left|(B^{-1}A + I_n)y\right|, \\ \delta_G(f) &:= \frac{1}{2}\left[\max_{(t,y,\lambda)\in[0,T]\times G\times J} f(t,y,\lambda) - \min_{(t,y,\lambda)\in[0,T]\times G\times J} f(t,y,\lambda)\right], \end{aligned} \tag{3.14}$$

I_n is an n-dimensional unit matrix and $B(y, \beta(y))$ denotes the ball of radius $\beta(y)$ with the center at the point y.

Moreover, we suppose that the spectral radius $r(K)$ of the matrix K in (2.4) satisfies the inequality

$$r(K) < \frac{10}{3T}. \tag{3.15}$$

Let us define the subset $U \subset \mathbb{R}^{n-1}$ such that

$$U := \left\{u = \operatorname{col}(u_2, u_3, \ldots, u_n) \in \mathbb{R}^{n-1} : z \in D_\beta\right\},$$

where

$$z = \operatorname{col}(\alpha_1 + \lambda\sum_{j=2}^{n}\alpha_j[u_j + w_j] - w_1, u_2, u_3, \ldots, u_n). \tag{3.16}$$

Let us connect with the BVP (3.8) – (3.10) the sequence of functions

$$x_{m+1}(t,w,u,\lambda) := z + \int_0^t f\left(s, x_m(s,w,u,\lambda)+w,\lambda\right) ds -$$

$$- \frac{t}{T}\int_0^T f\left(s, x_m(s,w,u,\lambda)+w,\lambda\right) ds -$$

$$- \frac{t}{T}\left[B^{-1}A + I_n\right] z,$$

$$m = 0,1,2,\ldots, \;\; x_0(t,w,u,\lambda) = z \in D_\beta, \tag{3.17}$$

depending on the artificially introduced parameters $w \in \Omega \subset \mathbb{R}^n$, $u \in U \subset \mathbb{R}^{n-1}$ and on the parameter $\lambda \in [a,b]$ contained in the problem (2.1) – (2.3), where the vector z has the form (3.16).

Note that for the initial value of functions $x_m(t,w,u,\lambda)$ at the point $t = 0$, the following equality holds:

$$x_m(0,w,u,\lambda) = z \tag{3.18}$$

for all $m = 0,1,2,\ldots$, and arbitrary $w \in \Omega,\; u \in U,\; \lambda \in [a,b]$.

It can be also verified that all functions in the sequence (3.17) satisfy the linear homogeneous two-point boundary condition (3.9) and the additional condition (3.10) for arbitrary $u \in U$ given by (3.16) and $w \in \Omega,\; \lambda \in [a,b]$.

We propose to solve the PBVP (3.8) – (3.10), together with the determining equation (3.12), sequentially, namely first solve (3.8) – (3.10), and then try to find the values of parameters $w \in \Omega \subset \mathbb{R}^n,\; u \in U \subset \mathbb{R}^{n-1},\; \lambda \in [a,b]$, for which the equation (3.12) can simultaneously be fulfilled.

4. Investigation of the solutions of the transformed problem (3.8) – (3.10). First we establish some results concerning the PBVP (3.8) – (3.10) with a specially modified function in the right-hand side of Eq. (3.8).

Theorem 1. *Let us suppose that the functions* $f : [0,T] \times G \times [a,b] \to \mathbb{R}^n,\; g : G \times G \times [a,b] \to \mathbb{R}^n$ *are continuous and the conditions (2.4), (3.13) – (3.16) are satisfied.*

Then:

1. The sequence of functions (3.17) satisfying the boundary conditions (3.9), (3.10) for arbitrary $u \in U,\; w \in \Omega$*, and* $\lambda \in [a,b]$*, converges uniformly as* $m \to \infty$*, with respect the domain*

$$(t,w,u,\lambda) \in [0,T] \times \Omega \times U \times [a,b], \tag{4.1}$$

to the limit function

$$x^*(t,w,u,\lambda) = \lim_{m\to\infty} x_m(t,w,u,\lambda). \tag{4.2}$$

2. The limit function $x^*(\cdot, w, u, \lambda)$ that has the initial value $x^*(0, w, u, \lambda) = z$ and given by (3.16) is a unique solution of the integral equation

$$x(t) = z + \int_0^t f(s, x(s) + w, \lambda)\, ds -$$

$$- \frac{t}{T} \left[\int_0^T f(s, x(s) + w, \lambda)\, ds + \left(B^{-1}A + I_n\right) z \right], \tag{4.3}$$

i. e., it is a solution of the modified (with regard to (3.8)) integro-differential equation

$$\frac{dx}{dt} = f(t, x + w, \lambda) + \Delta(w, u, \lambda) \tag{4.4}$$

satisfying the same boundary conditions (3.9),(3.10), where

$$\Delta(w, u, \lambda) = -\frac{1}{T} \left[\left(B^{-1}A + I_n\right) z + \int_0^T f(s, x(s) + w, \lambda)\, ds \right]. \tag{4.5}$$

3. The following error estimation holds:

$$|x^*(t, w, u, \lambda) - x_m(t, w, u, \lambda)| \le h(t, w, u, \lambda), \tag{4.6}$$

where

$$h(t, w, u, \lambda) := \frac{20}{9} t \left(1 - \frac{t}{T}\right) Q^{m-1} (I_n - Q)^{-1} [Q\delta_G(t) +$$

$$+ \; K \left|\left(B^{-1}A + I_n\right) z\right|],$$

the vector $\delta_G(t)$ is given by Eq. (3.14) and the matrix $Q = \dfrac{3T}{10} K$.

Proof. We shall prove that, under the assumed conditions, sequence (3.17) is a Cauchy sequence in the Banach space $C([0, T], \mathbb{R}^n)$ equipped with the usual uniform norm. First, we show that $x_m(t, w, u, \lambda) \in D$ for all $(t, w, u, \lambda) \in [0, T] \times \Omega \times U \times [a, b]$ and $m \in \mathbb{N}$. Indeed, using the estimate

$$\left| \int_0^t \left[f(\tau) - \frac{1}{T} \int_0^T f(s) ds \right] d\tau \right| \le$$

$$\le \frac{1}{2} \alpha_1(t) \left[\max_{t \in [0,T]} f(t) - \min_{t \in [0,T]} f(t) \right] \tag{4.7}$$

of Lemma 2.3 from [8] or its generalization in Lemma 4 from [11], relation (3.17), for $m = 0$, implies that

$$|x_1(t,w,u,\lambda) - z| \leq \left| \int_0^t \left[f(t, z+w, \lambda) - \frac{1}{T} \int_0^T f(s, z+w, \lambda) ds \right] dt \right| +$$

$$+ \left| \left[\left(B^{-1}A + I_n \right) z \right] \right| \leq \alpha_1(t)\delta_G(f) + \beta_1(z) \leq \beta(z) \tag{4.8}$$

where

$$\alpha_1(t) = 2t\left(1 - \frac{t}{T}\right), \ |\alpha_1(t)| \leq \frac{T}{2}, \tag{4.9}$$

$$\beta_1(z) = \left| \left[B^{-1}A + I_n \right] z \right|. \tag{4.10}$$

Therefore, by virtue of (3.13), (3.14), (4.8), we conclude that $x_1(t,w,u,\lambda) \in D$ whenever $(t,w,u,\lambda) \in [0,T] \times \Omega \times U \times [a,b]$. By induction, one can easily establish that all functions (3.17) are also contained in the domain D for all $m = 1, 2, \ldots, \ t \in [0,T], \ w \in \Omega, \ u \in U, \ \lambda \in [a,b]$. Now, consider the difference of functions

$$x_{m+1}(t,w,u,\lambda) - x_m(t,w,u,\lambda) = \int_0^t \left[f(s, x_m(s,w,u,\lambda) + w, \lambda) - \right.$$

$$\left. - \ f(s, x_{m-1}(s,w,u,\lambda) + w, \lambda) \right] ds -$$

$$- \frac{t}{T} \int_0^T \left[f(s, x_m(s,w,u,\lambda) + w, \lambda) - \right.$$

$$\left. - \ f(s, x_{m-1}(s,w,u,\lambda) + w, \lambda) \right] ds, \tag{4.11}$$

and introduce the notation

$$d_m(t,w,u,\lambda) := \left| x_m(t,w,u,\lambda) - x_{m-1}(t,w,u,\lambda) \right|, \ m = 1, 2, \ldots \ . \tag{4.12}$$

By virtue of identity (4.12) and the Lipschitz condition (2.4), we have

$$d_{m+1}(t,w,u,\lambda) \leq K \left[\left(1 - \frac{t}{T}\right) \int_0^t d_m(s,w,u,\lambda) ds + \frac{t}{T} \int_0^T d_m(s,w,u,\lambda) ds \right] \tag{4.13}$$

for every $m = 0, 1, 2, \ldots$. According to (4.8),

$$d_1(t, w, u, \lambda) = |x_1(t, w, u, \lambda) - z| \leq \alpha_1(t)\delta_G(f) + \beta_1(z), \tag{4.14}$$

where $\beta_1(z)$ is given by (4.10).

Now we need the following estimate Lemma 2.4 from [8]:

$$\alpha_{m+1}(t) \leq \left(\frac{3}{10}T\right)\alpha_m(t),\ \alpha_{m+1}(t) \leq \left(\frac{3}{10}T\right)^m \overline{\alpha}_1(t), \tag{4.15}$$

obtained for the sequence of functions

$$\alpha_{m+1}(t) = \left(1 - \frac{t}{T}\right)\int\limits_0^t \alpha_m(s)ds + \frac{t}{T}\int\limits_t^T \alpha_m(s)ds,\ m = 0, 1, 2, \ldots,$$

$$\alpha_0(t) = 1,\ \alpha_1(t) = 2t\left(1 - \frac{t}{T}\right), \tag{4.16}$$

where $\overline{\alpha_1}(t) = \dfrac{10}{9}\alpha_1(t)$.

In view of (4.14), (4.16), for $m = 1$ it follows from (4.13) that

$$d_2(t, w, u, \lambda) \leq K\delta_G(f)\left[\left(1 - \frac{t}{T}\right)\int\limits_0^t \alpha_1(s)ds + \frac{t}{T}\int\limits_t^T \alpha_1(s)ds\right] +$$

$$+\ K\beta_1(z)\left[\left(1 - \frac{t}{T}\right)\int\limits_0^t ds + \frac{t}{T}\int\limits_t^T ds\right] \leq$$

$$\leq K\left[\alpha_2(t)\delta_G(f) + \alpha_1(t)\beta_1(z)\right].$$

By induction, we can easily obtain

$$d_{m+1}(t, w, u, \lambda) \leq K^m[\alpha_{m+1}(t)\delta_G(f) + \alpha_m(t)\beta_1(z)], m = 0, 1, 2, \ldots, \tag{4.17}$$

where $\alpha_{m+1}(t),\ \alpha_m(t)$ are calculated according to (4.16), and $\delta_G(f)$, $\beta_1(z)$ are given by (3.14) and (4.10). By virtue of the second estimate from (4.15), we have from (4.17) that

$$d_{m+1}(t, w, u, \lambda) \leq \overline{\alpha}_1(t)\left[\left(\frac{3}{10}TK\right)^m \delta_G(f) + K\left(\frac{3}{10}TK\right)^{m-1}\beta_1(z)\right] =$$

$$= \overline{\alpha}_1(t)\left[Q^m\delta_G(f) + KQ^{m-1}\beta_1(z)\right], \tag{4.18}$$

for all $m = 1, 2, \ldots$, where the matrix

$$Q = \frac{3}{10} TK. \tag{4.19}$$

Therefore, in view of (4.18),

$$\begin{aligned}
&\left|x_{m+j}(t, w, u, \lambda) - x_m(t, w, u, \lambda)\right| \leq \\
&\leq \left|x_{m+j}(t, w, u, \lambda) - x_{m+j-1}(t, w, u, \lambda)\right| + \\
&+ \left|x_{m+j-1}(t, w, u, \lambda) - x_{m+j-2}(t, w, u, \lambda)\right| + \ldots \\
&\ldots + \left|x_{m+1}(t, w, u, \lambda) - x_m(t, w, u, \lambda)\right| = \sum_{i=1}^{j} d_{m+i}(t, w, u, \lambda) \leq \\
&\leq \overline{\alpha}_1(t) \left[\sum_{i=1}^{j} \left(Q^{m+i}\delta_G(f) + KQ^{m+i-1}\beta_1(z)\right)\right] = \\
&= \overline{\alpha}_1(t) \left[Q^m \sum_{i=0}^{j-1} Q^i \delta_G(f) + KQ^m \sum_{i=0}^{j-1} Q^i \beta_1(z)\right].
\end{aligned} \tag{4.20}$$

Since, due to (3.15), the maximal eigenvalue of the matrix Q of the form (4.19) does not exceed one, we have

$$\sum_{i=0}^{j-1} Q^i \leq (I_n - Q)^{-1}$$

and

$$\lim_{m\to\infty} Q^m = [0].$$

Therefore we can conclude from (4.20) that, according to the Cauchy criteria, the sequence $x_m(t, w, u, \lambda)$ of the form (3.17) uniformly converges in the domain (4.1) and, hence, the assertion (4.2) follows.

Since all functions $x_m(t, w, u, \lambda)$ of the sequence (3.17) satisfy the boundary conditions (3.9), (3.10), the limit function $x^*(t, w, u, \lambda)$ also satisfies these conditions. Passing to the limit as $m \to \infty$ in equality (3.17), we show that the limit function satisfies the integral equation (4.3). It is also obvious from (4.3) that

$$x^*(T, w, u, \lambda) = B^{-1}Az, \tag{4.21}$$

which means that $x^*(t, w, u, \lambda)$ is a solution of the integral equation (4.3) as well as a solution of the integro-differential equation (4.4). Estimate (4.6) is an immediate consequence of (4.20).

The theorem is proved.

Now we show that, in view of Theorem 1, the PBVP (3.8) – (3.10) can be formally interpreted as a family of initial value problems for differential equations with an "additively forced" member in the right-hand side. Namely, consider the Cauchy problem

$$\frac{dx(t)}{dt} = f(t, x(t) + w, \lambda) + \mu, \; t \in [0, T], \tag{4.22}$$

$$x(0) = z = \operatorname{col}(\alpha_1 + \lambda \sum_{j=2}^{n} \alpha_j [x_j(0) + w_j] - w_1, u_2, u_3, \ldots, u_n), \tag{4.23}$$

where $\mu \in \mathbb{R}^n, \; z \in D_\beta, \; w \in \Omega, \; \lambda \in [a, b]$ are parameters.

Theorem 2. *Under the conditions of Theorem 1, the solution $x = x(t, w, u, \lambda)$ of the initial value problem (4.22), (4.23) satisfies the boundary conditions (3.9), (3.10) if and only if*

$$\mu = \Delta(w, u, \lambda), \tag{4.24}$$

where $\Delta : \Omega \times U \times [a, b] \to \mathbb{R}^n$ is the mapping defined by (4.5).

Proof. According to Picard – Lindelöf existence theorem it is easy to show that the Lipschitz condition (2.4) implies that the initial value problem (4.22), (4.23) has a unique solution for all

$$(\mu, w, u, \lambda) \in \mathbb{R}^n \times \Omega \times U \times [a, b].$$

It follows from the proof of Theorem 1 that, for every fixed

$$(w, u, \lambda) \in \Omega \times U \times [a, b], \tag{4.25}$$

the limit function (4.2) of the sequence (3.17) satisfies the integral equation (4.3) and, in addition, $x^*(t, w, u, \lambda) = \lim_{m \to \infty} x_m(t, w, u, \lambda)$ satisfies the boundary conditions (3.9), (3.10). This implies immediately that the function $x = x^*(t, w, u, \lambda)$ of the form (4.2) is a unique solution of the initial value problem

$$\frac{dx(t)}{dt} = f(t, x(t) + w, \lambda) + \Delta(w, u, \lambda), \; t \in [0, T], \tag{4.26}$$

$$x(0) = \operatorname{col}(\alpha_1 + \lambda \sum_{j=2}^{n} \alpha_j [x_j(0) + w_j] - w_1, u_2, u_3, \ldots, u_n), \tag{4.27}$$

where $\Delta(w, u, \lambda)$ is given by (4.5). Hence, (4.26), (4.27) coincide with (4.22), (4.23) corresponding to

$$\mu = \Delta(w, u, \lambda) = -\frac{1}{T} \left[\left(B^{-1} A + I_n \right) z + \int_0^T f\left(s, x(s) + w, \lambda\right) ds \right]. \tag{4.28}$$

The fact that the function (4.2) is not a solution of (4.22), (4.23) for any other value of μ, not equal to that in (4.28), is obvious, e.g., from (4.24).

The theorem is proved.

The following statement shows how the solution $x = x^*(t, w, u, \lambda)$ of the modified PBVP (4.3), (3.9), (3.10) relates to the solution of the unperturbed BVP (3.8) – (3.10).

Theorem 3. *If the conditions of Theorem 1 are satisfied, then, the function $x^*(t, w, u^*, \lambda^*)$ is a solution of the PBVP (3.8) – (3.10) if and only if the triplet*

$$\{w, u^*, \lambda^*\} \in \Omega \times U \times [a, b] \tag{4.29}$$

satisfies the system of determining equations

$$\left[B^{-1}A + I_n\right] z + \int\limits_o^T f\left(s, x^*(s, w, u, \lambda) + w, \lambda\right) ds = 0, \tag{4.30}$$

where z is given by (4.27) and w is considered as a parameter.

Proof. It suffices to apply Theorem 2 and notice that the differential equation in (4.26) coincedes with (3.8) if and only if the triplet (4.29) satisfies the equation

$$\Delta(w, u^*, \lambda^*) = 0, \tag{4.31}$$

i.e., when the relation (4.30) holds, where w is considered as a parameter, $w \in \Omega$.

The theorem is proved.

It now becomes clear how one should choose the value $w = w^*$ of the artificially introduced parameter w in (3.1) in order for the function

$$y^*(t) = x^*(t, w^*, u^*, \lambda^*) + w^* \tag{4.32}$$

to be a solution of the original PBVP (2.1) – (2.3).

Theorem 4. *If the conditions of Theorem 1 are satisfied, then, for function (4.32) to be a solution of the given PBVP (2.1) – (2.3) it is necessary and sufficient that the triplet*

$$\{w^*, u^*, \lambda^*\} \tag{4.33}$$

satisfy the system of algebraic determining equations

$$g(z + w, -B^{-1}Az + w, \lambda) = 0, \tag{4.34}$$

where

$$z := \operatorname{col}(\alpha_1 + \lambda^* \sum_{j=2}^{n} \alpha_j [u_j^* + w_j^*] - w_1^*, u_2^*, u_3^*, \ldots, u_n^*), \tag{4.35}$$

and the pair $\{u^*, \lambda^*\}$ is a solution of the system (4.30), parametrized by w.

Proof. It was established in Section 3 that the PBVP (2.1) – (2.3) is equivalent to the family of BVPs (3.8) – (3.10) considered together with the determining equation (3.12). The vector parameter z in (4.35) can be interpreted as an initial value at $t = 0$ of a possible solution of the problem (3.8) – (3.10). Therefore, Eq. (3.12) can be rewritten in the form (4.34). Taking into account the change of variables (3.1) and the equivalence of (2.1) – (2.3) to (3.8) – (3.10) (3.12), we notice that the function $y^*(t)$ in (4.32) coincides with the solution of the PBVP (2.1) – (2.3) if and only if $w = w^*$ satisfies the equation (4.34).

Corollary 1. *Under the conditions of Theorem 1, the function $y^*(t)$ of the form (4.32), (4.2) will be a solution of the PBVP (2.1) – (2.3) if and only if the triplet (4.33) satisfies the system of determining equations*

$$\left[B^{-1}A + I_n\right] z + \int\limits_0^T f\left(s, x^*(s, w, u, \lambda) + w, \lambda\right) ds = 0,$$

$$g(z + w, -B^{-1}Az + w, \lambda) = 0, \tag{4.36}$$

$$z = \mathrm{col}(\alpha_1 + \lambda \sum_{j=2}^{n} \alpha_j [u_j + w_j] - w_1, u_2, u_3, \ldots, u_n),$$

that contains $2n$ scalar algebraic equations, where $x^*(t, w, u, \lambda)$ is given by (4.2).

Proof. It suffices to apply Theorem 3 and Theorem 4.

Remark 1. In practice, it is natural to fix some natural m and, instead of (4.36), to consider the "approximate determining system"

$$\left[B^{-1}A + I_n\right] z + \int\limits_0^T f\left(s, x_m(s, w, u, \lambda) + w, \lambda\right) ds = 0,$$

$$g(z + w, -B^{-1}Az + w, \lambda) = 0, \tag{4.37}$$

$$z = \mathrm{col}(\alpha_1 + \lambda \sum_{j=2}^{n} \alpha_j [u_j + w_j] - w_1, u_2, u_3, \ldots, u_n).$$

In the case when system (4.37) has an isolated root, say

$$w = w_m,\ u = u_m,\ \lambda = \lambda_m, \tag{4.38}$$

in some open subdomain of

$$\Omega \times U \times [a, b],$$

one can prove that under certain additional conditions, the exact determining system (4.36) is also solvable,

$$w = w^*,\ u = u^*,\ \lambda = \lambda^*.$$

Hence, the given nonlinear PBVP (2.1) – (2.3) has a solution of form (4.32), such that

$$x^*(t = 0) = \mathrm{col}(\alpha_1 + \lambda^* \sum_{j=2}^{n} \alpha_j [u_j^* + w_j^*] - w_1^*, u_2^*, u_3^*, \ldots, u_n^*) \in D_\beta,$$

$$w^* \in \Omega,\ \lambda^* \in [a, b],\ u^* \in U,\ y^* \in G.$$

Furthermore, the function

$$y_m(t) := x_m(t, w_m, u_m, \lambda_m) + w_m,\ t \in [0, T], \tag{4.39}$$

can be regarded as the "m-th approximation" to the exact solution, $y^*(t) = x^*(t, w^*, u^*, \lambda^*) + w^*$ (see estimate (4.6)). To prove solvability of system (4.36), one can use some topological degree techniques (cf. Theorem 3.1 in [8, p. 43]) or the methods oriented to the solution of nonlinear equations in Banach spaces developed in [14] (see, e.g., Theorem 19.2 in [14, p. 281]). Here, we do not consider this problem in more detail.

Remark 2. If we choose in (3.5), (3.7) the matrix A to be the zero matrix, then the PBVP (3.8) – (3.10) is reduced to the parametrized initial value problem

$$\frac{dx}{dt} = f(t, x(t) + w, \lambda),\ \ t \in [0, T], \tag{4.40}$$

$$x(T) = 0, \tag{4.41}$$

with the additional condition (3.10). In this case, instead of successive approximations (3.17) we obtain

$$x_{m+1}(t, w, u, \lambda) := z + \int_0^t f(s, x_m(s, w, u, \lambda) + w, \lambda)\, ds -$$

$$- \frac{t}{T} \int_0^T f(s, x_m(s, w, u, \lambda) + w, \lambda)\, ds - \frac{t}{T} z, \tag{4.42}$$

$$m = 0, 1, 2, \ldots,\ \ x_0(t, w, u, \lambda) = z \in D_\beta,$$

where $z = \mathrm{col}(\alpha_1 + \lambda \sum_{j=2}^{n} \alpha_j [u_j + w_j] - w_1, u_2, u_3, \ldots, u_n)$, and the system of determining equations (4.36) is transformed into the system

$$z + \int_0^T f(s, x^*(s, w, u, \lambda) + w, \lambda)\, ds = 0,$$

$$g(z + w, w, \lambda) = 0, \tag{4.43}$$

$$z = \mathrm{col}(\alpha_1 + \lambda \sum_{j=2}^{n} \alpha_j [u_j + w_j] - w_1, u_2, u_3, \ldots, u_n).$$

In this case Theorem 3 guarantees existence of a solution of the parametrized Cauchy problem (4.40), (4.41) with the additional condition (3.10) on the interval $[0, T]$.

Remark 3. If one can obtain the solution $x = \widetilde{x}^0(t, w, \lambda)$ of the parametrized initial value problem (4.40), (4.41) on the interval $[0, T]$, e.g., by Picard's iterations,

$$\widetilde{x}^0(t, w, \lambda) = \lim_{m \to \infty} \widetilde{x}_m(t, w, \lambda) =$$

$$= \lim_{m \to \infty} \int_T^t f(s, \widetilde{x}_{m-1}(t, w, \lambda))\, ds, \tag{4.44}$$

$m = 1, 2, \ldots,\ \widetilde{x}_0(t, w, \lambda) = z$, then to find values of the parameters

$$w = w^0,\ \lambda = \lambda^0, \tag{4.45}$$

for which the function

$$y^0(t) = \widetilde{x}^0(t, w, \lambda) + w^0 \tag{4.46}$$

will be a solution of the original PBVP (2.1) – (2.3), we should solve, according to (3.12), (3.4), the determining system

$$\begin{gathered} g(\widetilde{x}^0(0, w, \lambda) + w, w, \lambda) = 0, \\ \widetilde{x}_1^0(0, w, \lambda) = \alpha_1 + \lambda \sum_{j=2}^{n} \alpha_j [\widetilde{x}_j^0(0, w, \lambda) + w_j] - w_1 \end{gathered} \tag{4.47}$$

that contains $(n+1)$ equations with respect to $(n+1)$ the unknown values $w = \mathrm{col}(w_1, w_2, \ldots, w_n)$ and λ.

We apply the above techniques to the following PBVP.

Example 1. Consider the second order parametrized two-point BVP

$$\frac{d^2y}{dt^2} - \frac{t}{8}\frac{dy}{dt} + \frac{\lambda^2}{2}\left(\frac{dy}{dt}\right)^2 + \frac{1}{2}y(t) = \frac{9}{32} + \frac{t^2}{16},\ t \in [0,1], \tag{4.48}$$

$$y(0) = \left[\frac{dy(1)}{dt}\right]^2, \tag{4.49}$$

$$\frac{dy(0)}{dt} = \frac{dy(1)}{dt} - y(1) - \frac{\lambda^2}{16}, \tag{4.50}$$

satisfying the additional condition

$$y(0) = \frac{1}{16} - \frac{\lambda}{4}\frac{dy(0)}{dt}. \tag{4.51}$$

By setting $y_1 := y$ and $y_2 := \frac{dy}{dt}$, the PBVP (4.48) – (4.51) can be rewritten in the form of system (2.1) – (2.3),

$$\frac{dy_1}{dt} = y_2,$$

$$\frac{dy_2}{dt} = \frac{9}{32} + \frac{t^2}{16} + \frac{t}{8}y_2 - \frac{\lambda^2}{2}y_2^2 - \frac{1}{2}y_1, \tag{4.52}$$

$$y_1(0) = [y_2(1)]^2,$$

$$y_2(0) = y_2(1) - y_1(1) - \frac{\lambda^2}{16}, \tag{4.53}$$

$$y_1(0) = \frac{1}{16} - \frac{\lambda}{4}y_2(0). \tag{4.54}$$

Suppose that the PBVP (4.52) – (4.54) is considered in the domain

$$(t, y, \lambda) \in [0,1] \times G \times [-1,1], \tag{4.55}$$

$$G := \left\{(y_1, y_2) : |y_1| \le 1,\ |y_2| \le \frac{3}{4}\right\}.$$

One can verify that for the PBVP (4.52) – (4.54), conditions (3.3), (3.13) and (3.15) are fulfiled in the domain (4.55) with the matrices

$$A := B := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, K := \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{7}{8} \end{bmatrix}.$$

Indeed, from the Perron theorem it is known that the greatest eigenvalue $\lambda_{\max}(K)$ of the matrix K, in virtue of the nonnegativity of its elements, is real, nonnegative, and computations show that

$$\lambda_{\max}(K) \le \frac{21}{16}.$$

Moreover the vectors $\delta_G(f)$ and $\beta(y)$ in (3.14) satisfy

$$\delta_G(f) \le \begin{bmatrix} \frac{3}{4} \\ \frac{5}{4} \end{bmatrix}, \quad \beta(y) := \frac{T}{2}\delta_G(f) + \left|\left(B^{-1}A + I_2\right) y\right| \le \begin{bmatrix} \frac{3}{8} \\ \frac{5}{8} \end{bmatrix} + 2\,|y|\,.$$

Substitution (3.1) takes the given system of differential equations (4.52) and the additional conditions (4.54) to the following form:

$$\begin{aligned} \frac{dx_1(t)}{dt} &= x_2(t) + w_2, \\ \frac{dx_2(t)}{dt} &= \frac{9}{32} + \frac{t^2}{16} + \frac{t}{8}(x_2(t) + w_2) - \\ &\quad - \frac{\lambda^2}{2}\left(x_2(t) + w_2)^2 - \frac{1}{2}(x_1(t) + w_1\right), \end{aligned} \tag{4.56}$$

and

$$x_1(0) = \frac{1}{16} - \frac{\lambda}{4}(x_2(0) + w_2) - w_1. \tag{4.57}$$

Thus we reduce the essentially nonlinear PBVP (4.52) – (4.54) to a collection of two-point BVPs of view (3.8) – (3.10), namely, to the system (4.56) which is considered under the linear two-point boundary condition

$$x(0) + x(1) = 0, \tag{4.58}$$

together with an additional condition (4.57) and an algebraic determining system of equations of form (3.12),

$$x_1(0) + w_1 = (x_2(1) + w_2)^2,$$

$$x_2(0) + w_2 = x_2(1) + w_2 - (x_1(1) + w_1) - \frac{\lambda^2}{16}.$$

Taking into account that, according to (3.11),

$$x(1) = \mathrm{col}(x_1(1), x_2(1)) = -B^{-1}Ax(0) = \mathrm{col}(-x_1(0), -x_2(0)),$$

the determining system obtained above can be rewritten in the form

$$x_1(0) + w_1 = (-x_2(0) + w_2)^2,$$
$$2x_2(0) = x_1(0) - w_1 - \frac{\lambda^2}{16}. \tag{4.59}$$

Due to the equality (3.16), in our case

$$z = \mathrm{col}(z_1, z_2) = \mathrm{col}\left(\frac{1}{16} - \frac{\lambda}{4}(u_2 + w_2) - w_1, u_2\right), \tag{4.60}$$

and the components of the iteration sequence (3.17) for the PBVP (4.56), under the linear boundary conditions (4.57), have the form

$$x_{m+1,1}(t, w, u, \lambda) = \left[\frac{1}{16} - \frac{\lambda}{4}(u_2 + w_2) - w_1\right] + \\ + \int_0^t [x_{m,2}(s, w, u, \lambda) + w_2]\, ds - \\ - t\int_0^t [x_{m,2}(s, w, u, \lambda) + w_2]\, ds - 2t\left[\frac{1}{16} - \frac{\lambda}{4}(u_2 + w_2) - w_1\right], \tag{4.61}$$

$$x_{m+1,2}(t, w, u, \lambda) = u_2 + \int_0^t \left[\frac{9}{32} + \frac{s^2}{16} + \frac{s}{8}(x_{m,2}(s, w, u, \lambda) + w_2) - \\ - \frac{\lambda^2}{2}(x_{m,2}(s, w, u, \lambda) + w_2)^2 - \frac{1}{2}(x_{m,1}(s, w, u, \lambda) + w_1)\right] ds - \\ - t\int_0^1 \left[\frac{9}{32} + \frac{s^2}{16} + \frac{s}{8}(x_{m,2}(s, w, u, \lambda) + w_2) - \\ - \frac{\lambda^2}{2}(x_{m,2}(s, w, u, \lambda) + w_2)^2 - \frac{1}{2}(x_{m,1}(s, w, u, \lambda) + w_1)\right] ds - 2tu_2, \tag{4.62}$$

where $m = 0, 1, 2, \ldots,$ and

$$x_0(t, w, u, \lambda) = z = \mathrm{col}\left(\frac{1}{16} - \frac{\lambda}{4}(u_2 + w_2) - w_1, u_2\right). \tag{4.63}$$

Using equalities (3.18) and (4.60), the determining equations (4.59), which are independent on the number of the iterations, can be rewritten in the form

$$\frac{1}{16} - \frac{\lambda}{4}(u_2 + w_2) = (w_2 - u_2)^2,$$
$$2u_2 = \frac{1}{16} - \frac{\lambda}{4}(u_2 + w_2) - 2w_1 - \frac{\lambda^2}{16}. \tag{4.64}$$

The system of approximate determining equations depending on the number of iterations, which is given by the first equation in the system (4.37) together with (4.60), is written in the component form as

$$2\left[\frac{1}{16} - \frac{\lambda}{4}(u_2 + w_2) - w_1\right] + \int_0^1 [x_{m,2}(s,w,u,\lambda) + w_2]\,ds = 0,$$

$$2u_2 + \int_0^1 \left[\frac{9}{32} + \frac{s^2}{16} + \frac{s}{8}(x_{m,2}(s,w,u,\lambda) + w_2) - \right.$$
$$\left. - \frac{\lambda^2}{2}(x_{m,2}(s,w,u,\lambda) + w_2)^2 - \frac{1}{2}(x_{m,1}(s,w,u,\lambda) + w_1)\right] ds = 0. \tag{4.65}$$

Thus, for every $m \geq 1$, we have four equations (4.64), (4.65) in four unknowns w_1, w_2, u_2, and λ. Note that, in our case, we can decrease the number of the unknown values as follows.

Obviously, from the first equation of (4.64), we have

$$\lambda = \frac{1 - 16(w_2 - u_2)^2}{4(u_2 + w_2)}. \tag{4.66}$$

Considering the auxiliarly equations (4.64) in the given domain, we find that

$$\frac{1}{16} - \frac{\lambda}{4}(u_2 + w_2) = (w_2 - u_2)^2,$$

$$\frac{1}{16} - \frac{\lambda}{4}(u_2 + w_2) = 2u_2 + 2w_1 + \frac{\lambda^2}{16},$$

from which

$$w_1 = \frac{(w_2 - u_2)^2}{2} - u_2 - \frac{\lambda^2}{32},$$

or, by using (4.66), we obtain

$$w_1 = \frac{(w_2 - u_2)^2}{2} - u_2 - \frac{1}{32}\left[\frac{1 - 16\,(w_2 - u_2)^2}{4\,(u_2 + w_2)}\right]^2. \tag{4.67}$$

So, by solving the determining system (4.59), which is independent on the number of iterations, we have already determined λ and w_1 in (4.66) and (4.67) as functions of two other unknowns w_2 and u_2.

For finding the rest of unknown values of w_2 and u_2 in each step of iterations (4.61) and (4.62), one should use the approximate determining equations (4.65). From (4.61) and (4.62), as a result of the first iteration ($m = 1$), we get

$$\begin{aligned} x_{1,1}(t, w, u, \lambda) &= \frac{1}{16} - \frac{1}{4}\lambda u_2 - \frac{1}{4}\lambda w_2 - w_1 - \frac{1}{8}t + \frac{1}{2}\lambda t u_2 + \frac{1}{2}\lambda t w_2 + 2tw_1, \\ x_{1,2}(t, w, u, \lambda) &= u_2 + \frac{1}{48}t^3 + \frac{1}{16}t^2 u_2 + \frac{1}{16}t^2 w_2 - \frac{1}{48}t - \frac{33}{16}u_2 t - \frac{1}{16}w_2 t. \end{aligned} \tag{4.68}$$

The system (4.65), as follows from the first iteration (4.68), now has the form

$$\frac{1}{256}\frac{-32u_2^2 + 64u_2w_2 - 32w_2^2 + 1 + 768u_2^3 + 256w_2^3 + 512u_2^4 + 512w_2^4 + 1792u_2^2w_2}{(u_2 + w_2)^2} +$$

$$+ \frac{1}{256}\frac{1280u_2w_2^2 - 1024u_2^3w_2 + 1024u_2^2w_2^2 - 1024u_2w_2^3}{(u_2 + w_2)^2} = 0,$$

$$\frac{13}{48} + \frac{33}{16}u_2 + \frac{1}{16}w_2 - \frac{1}{32}\frac{(-1 + 16u_2^2 - 32u_2w_2 + 16w_2^2)^2u_2^2}{(u_2 + w_2)^2} -$$

$$- \frac{1}{16}\frac{(-1 + 16u_2^2 - 32u_2w_2 + 16w_2^2)^2u_2w_2}{(u_2 + w_2)^2} - \frac{1}{32}\frac{(-1 + 16u_2^2 - 32u_2w_2 + 16w_2^2)^2w_2^2}{(u_2 + w_2)^2} -$$

$$- \frac{1}{32}\frac{(-1 + 16u_2^2 - 32u_2w_2 + 16w_2^2)u_2 - (-1 + 16u_2^2 - 32u_2w_2 + 16w_2^2)w_2}{u_2 + w_2} = 0,$$

whose solution, in the given domain, is

$$w_{1,2} \approx 0,1245396563, \qquad u_{1,2} \approx -0,1339659756.$$

Note that there are other solutions in other domains. From (4.66) and (4.67) one can easily obtain the values

$$\lambda_1 \approx 1,835355492, \qquad w_{1,1} \approx 0,06211202021.$$

Therefore, the first approximation to the first and second components of the solution, according to (4.39), have the form

$$y_{1,1}(t) \approx x_{1,1}(t, w_{1,1}, w_{1,2}, u_{1,2}, \lambda_1) + w_{1,2} =$$

$$= 0,06682516175 - 0,009426283100t,$$

$$y_{1,2}(t) = x_{1,2}(t, w_{1,1}, w_{1,2}, u_{1,2}, \lambda_1) + w_{1,2} =$$

$$= -\ 0,009426319300 + 0,02083333333t^3 -$$

$$-\ 0,5891449560 \cdot 10^{-3}t^2 + 0,2476877629t.$$

Proceeding analogously for the fourth approximation ($m = 2$) in (4.61) and (4.62) we find

$$x_{4,1}(t, w, u, \lambda) = 0,06250000000 - 0,2500000000\lambda u_2 - 0,2500000000\lambda w_2 - w_1 -$$

$$-\ 0,1194062104 \cdot 10^{-11}t^{16} + 0,1800897703 \cdot 10^{-12}t^{15} -$$

$$-\ 0,2408329197 \cdot 10^{-10}t^{14} + 0,7189018645 \cdot 10^{-12}t^{13} +$$

$$+\ 0,5101736163 \cdot 10^{-8}t^{12} + 0,1618135334 \cdot 10^{-9}t^{11} -$$

$$-\ 0,2527520491t + 0,1283575086t^2 + 0,9709047280 \cdot 10^{-7}t^{10} -$$

$$-\ 0,00001805766167t^3 + 0,1428247920 \cdot 10^{-5}t^5 -$$

$$-\ 0,0002260468395t^6 - 0,0003540640278t^4 +$$

$$+\ 0,2944072199 \cdot 10^{-8}t^9 - 0,6946488839 \cdot 10^{-5}t^8 -$$

$$-\ 0,1877914607 \cdot 10^{-5}t^7 + 0,5000000000t\lambda u_2 +$$

$$+\ 0,5000000000t\lambda w_2 + 2w_1 t,$$

and

$$x_{4,2}(t, w, u, \lambda) = 0,4976866454 \cdot 10^{-13}\lambda^2 t^{18} - 0,1404778946 \cdot 10^{-12}t^{17} -$$

$$-\ 2tu_2 + u_2 - 0,007465543200t - 0,06250000000w_2 t +$$

$$+\ 0,01379906533t^3 + 0,3312081224 \cdot 10^{-9}t^{10} - 0,00556655867t^2 +$$

$$+\ 0,1103300827 \cdot 10^{-7}t^{11} - 0,2809717397 \cdot 10^{-11}t^{15} +$$

$$+\ 0,8344396643\cdot 10^{-13}t^{14}+0,5886618649\cdot 10^{-9}t^{13}+$$

$$+\ 0,2110426996\cdot 10^{-13}t^{16}-0,2173620444\cdot 10^{-6}t^{8}-$$

$$-\ (0,2237083782t^{7}+0,9720803613t^{4})10^{-4}+0,2533009667\cdot 10^{-6}t^{6}-$$

$$-\ 0,6307951402\cdot 10^{-6}t^{9}-0,0006468017252t^{5}-0,2222484346\times$$

$$\times\ 10^{-21}\lambda^2t^{29}+0,1854113403\cdot 10^{-10}t^{12}-0,1134099269\cdot 10^{-15}\lambda^2t^{23}-$$

$$-\ 0,4954982044\cdot 10^{-14}\lambda^2t^{21}+0,1567434632\cdot 10^{-17}\lambda^2t^{25}-$$

$$-\ 0,9551537601\cdot 10^{-17}\lambda^2t^{22}-0,8702276884\cdot 10^{-19}\lambda^2t^{24}+$$

$$+\ 0,3890544470\cdot 10^{-22}\lambda^2t^{28}+0,4121325604\cdot 10^{-19}\lambda^2t^{27}+$$

$$+\ 0,1720306960\cdot 10^{-23}\lambda^2t^{30}-0,5887109399\cdot 10^{-23}\lambda^2t^{31}-$$

$$-\ 0,1731456709\cdot 10^{-15}\lambda^2t^{20}+0,1287651362\cdot 10^{-12}\lambda^2t^{19}-$$

$$-\ 0,4931604453\cdot 10^{-20}\lambda^2t^{26}+0,1044769886\cdot 10\lambda^2t^{10}+$$

$$+\ 0,1877914608\cdot 10^{-5}\lambda^2t^{7}w_2-0,2944072199\cdot 10^{-8}\lambda^2t^{9}w_2-$$

$$-\ 0,1800897703\cdot 10^{-12}\lambda^2t^{15}w_2+0,2408329197\cdot 10^{-10}t^{14}\lambda^2w_2-$$

$$-\ 0,5101736162\cdot 10^{-8}\lambda^2t^{12}w_2+0,1194062104\cdot 10^{-11}\lambda^2t^{16}w_2-$$

$$-\ 0,7189018646\cdot 10^{-12}\lambda^2t^{12}w_2-0,9709047280\cdot 10^{-7}\lambda^2t^{10}w_2+$$

$$+\ 0,00007289581339\lambda^2t^{5}-0,01098605843\lambda^2t^{3}-0,00004145889033\times$$

$$\times\ \lambda^2t^{4}+0,06250000000t^{2}w_2+0,01629035143t^{2}\lambda^2+0,4935812452\times$$

$$\times\ 10^{-4}\lambda t^{7}+0,1371991985\cdot 10^{-5}\lambda^2t^{9}-0,355733585010^{-11}\lambda^2t^{15}+$$

$$+\ 0,8319872882\cdot 10^{-11}\lambda^2t^{17}-0,5291889647\cdot 10^{-10}\lambda^2t^{14}-$$

$$-\ 0,8357541416\cdot 10^{-9}\lambda^2t^{12}+0,8180703847\cdot 10^{-12}\lambda^2t^{16}-$$

$$-\ 0,6907514911\cdot 10^{-8}\lambda^2t^{13}-0,4676981841\cdot 10^{-6}\lambda^2t^{8}-$$

$$- 0,00002900681601\lambda^2 t^6 - 0,1133979816 \cdot 10^{-6}\lambda^2 t^{11} -$$

$$- 0,1618135334 \cdot 10^{-9}\lambda^2 t^{11} w_2 + 0,6946488838 \cdot 10^{-5}\lambda^2 t^8 w_2 +$$

$$+ 0,0002260468395\lambda^2 t^6 w_2 - 0,1428247920 \cdot 10^{-5}\lambda^2 t^5 w_2 +$$

$$+ 0,00001805766166\lambda^2 t^3 w_2 + 0,0003540640278\lambda^2 t^4 w_2 -$$

$$- 0,1283575086\lambda^2 t^2 w_2 - 0,005356874774\lambda^2 t + 0,1277520491\lambda^2 t w_2.$$

The determining system (4.65) for the fourth approximation is

$$0,1{\cdot}10^{-9}\frac{-1241618464u_2^2 + 2516763073u_2w_2 - 1241618464w_2^2}{(u_2+w_2)^2} +$$

$$+ 0,1\cdot 10^{-9}\frac{39062500 + 0,2\cdot 10^{11}u_2^3 + 0,1\cdot 10^{11}w_2^3 + 0,2\cdot 10^{11}u_2^4}{(u_2+w_2)^2} +$$

$$+ 0,1\cdot 10^{-9}\frac{0,2\cdot 10^{11}w_2^4 + 0,5\cdot 10^{11}u_2^2w_2 + 0,4\cdot 10^{11}u_2w_2^2}{(u_2+w_2)^2} +$$

$$+ 0,1\cdot 10^{-9}\frac{-0,4\cdot 10^{11}u_2^3w_2 0,4\cdot 10^{11}u_2^2w_2^2 - 0,4\cdot 10^{11}u_2w_2^3}{(u_2+w_2)^2} = 0,$$

$$- 0,2\cdot 10^{-13}\frac{-0,1444008049\cdot 10^{14}u_2^2 - 0,3405148352\cdot 10^{14}u_2w_2}{(u_2+w_2)^2} -$$

$$- 0,2\cdot 10^{-13}\frac{-0,1287758049\cdot 10^{14}w_2^2 + 0,24\cdot 10^{16}w_2^4u_2^2}{(u_2+w_2)^2} -$$

$$- 0,2\cdot 10^{-13}\frac{0,4\cdot 10^{15}w_2^6 + 0,4\cdot 10^{15}w_2^2u_2^4 - 0,2682091646\cdot 10^{13}u_2w_2^4}{(u_2+w_2)^2} -$$

$$- 0,2\cdot 10^{-13}\frac{0,4023137469\cdot 10^{13}u_2^2w_2^3 + 0,6705229115\cdot 10^{12}w_2^5}{(u_2+w_2)^2} -$$

$$- 0,2\cdot 10^{-13}\frac{0,6705229115\cdot 10^{12}w_2u_2^4 - 0,4040095731\cdot 10^{11}}{(u_2+w_2)^2} -$$

$$- 0,2\cdot 10^{-13}\frac{2619230123w_2 - 0,125\cdot 10^{15}u_2^3 - 0,3208815364\cdot 10^{13}w_2^3}{(u_2+w_2)^2} -$$

$$- \ 0,2 \cdot 10^{-13} \frac{0,2157354929 \cdot 10^{13} u_2^4 - 0,4784264507 \cdot 10^{14} w_2^4}{(u_2 + w_2)^2} -$$

$$- \ 0,2 \cdot 10^{-13} \frac{-0,2532088154 \cdot 10^{15} u_2^2 w_2 - 0,1310823693 \cdot 10^{15} u_2 w_2^2}{(u_2 + w_2)^2} -$$

$$- \ 0,2 \cdot 10^{-13} \frac{0,4137058029 \cdot 10^{14} u_2^3 w_2 - 0,1370558704 \cdot 10^{15} u_2^2 w_2^2}{(u_2 + w_2)^2} -$$

$$- \ 0,2 \cdot 10^{-13} \frac{0,1413705803 \cdot 10^{15} u_2 w_2^3 - 0,16 \cdot 10^{16} w_2^3 u_2^3}{(u_2 + w_2)^2} -$$

$$- \ 0,2 \cdot 10^{-13} \frac{-0,2682091646 \cdot 10^{13} u_2^3 w_2^2 - 0,16 \cdot 10^{16} w_2^5 u_2}{(u_2 + w_2)^2} = 0.$$

Solving numerically the system (4.65), taking into account (4.66), (4.67), we obtain the following values of the parameters:

$$w_{4,2} \approx 0,1262581431, \qquad u_{4,2} \approx -0,1245642413,$$

$$\lambda_4 \approx -0,9725912388, \qquad w_{4,1} \approx 0,1264593375.$$

The fourth approximation of the first and second components of the solution of PBVP (4.52) – (4.54) then have the form

$$y_{4,1}(t) \approx x_{4,1}(t, w_{4,1}, w_{4,2}, u_{4,2}, \lambda_4) + w_{4,1} \approx$$

$$\approx 0,0629118685 - 0,1194062104 \cdot 10^{-11} t^{16} + 0,1800897703 \cdot 10^{-12} t^{15} -$$

$$- \ 0,2408329197 \cdot 10^{-10} t^{14} + 0,7189018645 \cdot 10^{-12} t^{13} +$$

$$+ \ 0,5101736163 \cdot 10^{-8} t^{12} + 0,1618135334 \cdot 10^{-9} t^{11} -$$

$$- \ 0,657111111 \cdot 10^{-3} t + 0,1283575086 t^2 + 0,970904728 \cdot 10^{-7} t^{10} -$$

$$- \ 0,1805766167 \cdot 10^{-4} t^3 + 0,142824792 \cdot 10^{-5} t^5 - 0,2260468395 \cdot 10^{-3} t^6 -$$

$$- \ 0,3540640278 \cdot 10^{-3} t^4 + 0,2944072199 \cdot 10^{-8} t^9 -$$

$$- \ 0,6946488839 \cdot 10^{-5} t^8 - 0,1877914607 \cdot 10^{-5} t^7,$$

$$y_{4,2}(t) \approx x_{4,2}(t, w_{4,1}, w_{4,2}, u_{4,2}, \lambda_4) + w_{4,2} \approx$$

$$\approx 0,7729570392 \cdot 10^{-11}t^{17} - 0,4664970935 \cdot 10^{-20}t^{26} +$$

$$+ 0,1627296358 \cdot 10^{-23}t^{30} - 0,5568815281 \cdot 10^{-23}t^{31} +$$

$$+ 0,4707785788 \cdot 10^{-13}t^{18} - 0,210232288 \cdot 10^{-21}t^{29} -$$

$$- 0,1072782738 \cdot 10^{-15}t^{23} - 0,4687084587 \cdot 10^{-14}t^{21} +$$

$$+ 0,1482689269 \cdot 10^{-17}t^{25} - 0,9035121474 \cdot 10^{-17}t^{22} -$$

$$- 0,8231777126 \cdot 10^{-19}t^{24} + 0,3680197195 \cdot 10^{-22}t^{28} +$$

$$+ 0,3898500851 \cdot 10^{-19}t^{27} - 0,1637843282 \cdot 10^{-15}t^{20} +$$

$$+ 0,1218032840 \cdot 10^{-12}t^{19} + 0,2439622186t + 0,16939018 \cdot 10^{-2} +$$

$$+ 0,34091389 \cdot 10^{-2}t^{3} - 0,138165455 \cdot 10^{-8}t^{10} +$$

$$+ 0,2404195214 \cdot 10^{-2}t^{2} - 0,9625329172 \cdot 10^{-7}t^{11} -$$

$$- 0,6196229775 \cdot 10^{-11}t^{15} - 0,4709801276 \cdot 10^{-10}t^{14} -$$

$$- 0,5945475256 \cdot 10^{-8}t^{13} + 0,9375536582 \cdot 10^{-12}t^{16} +$$

$$+ 0,1698583801 \cdot 1^{-6}t^{8} + 0,245429592 \cdot 10^{-4}t^{7} -$$

$$- 0,1880355733 \cdot 10^{-6}t^{6} - 0,9413888196 \cdot 10^{-4}t^{4} +$$

$$+ 0,6666667229 \cdot 10^{-6}t^{9} - 0,5780176957 \cdot 10^{-3}t^{5} -$$

$$- 0,1381336598 \cdot 10^{-8}t^{12}.$$

As is seen in Figures 1, 2 and 3, 4, the graphs of the exact solution

$$\left\{ y^*(t) = \frac{t^2}{8} + \frac{1}{16}, \ \lambda = \lambda^* = 1 \right\}$$

and especially the fourth approximation almost coincide, whereas the deviation of their derivatives does not exceed $0,001$.

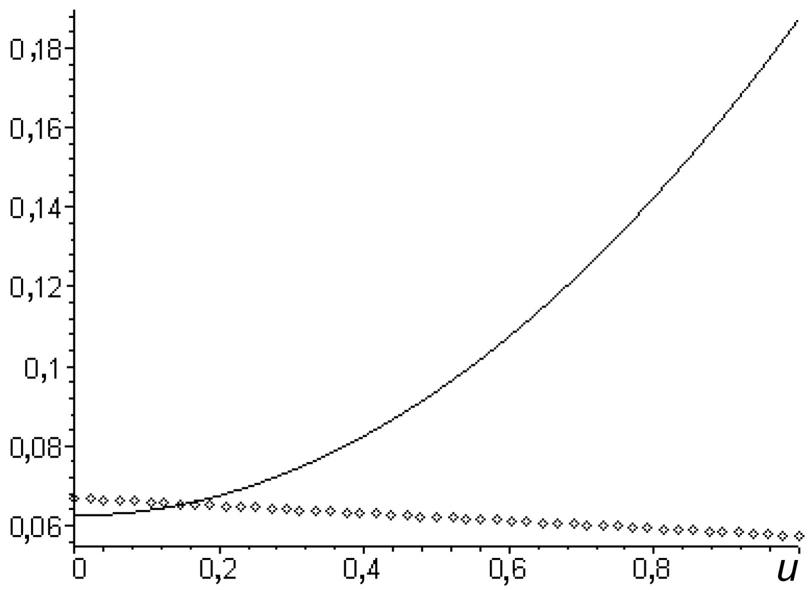

Fig. 1. The first components of the exact solution (solid line) and its first approximation (dotted line).

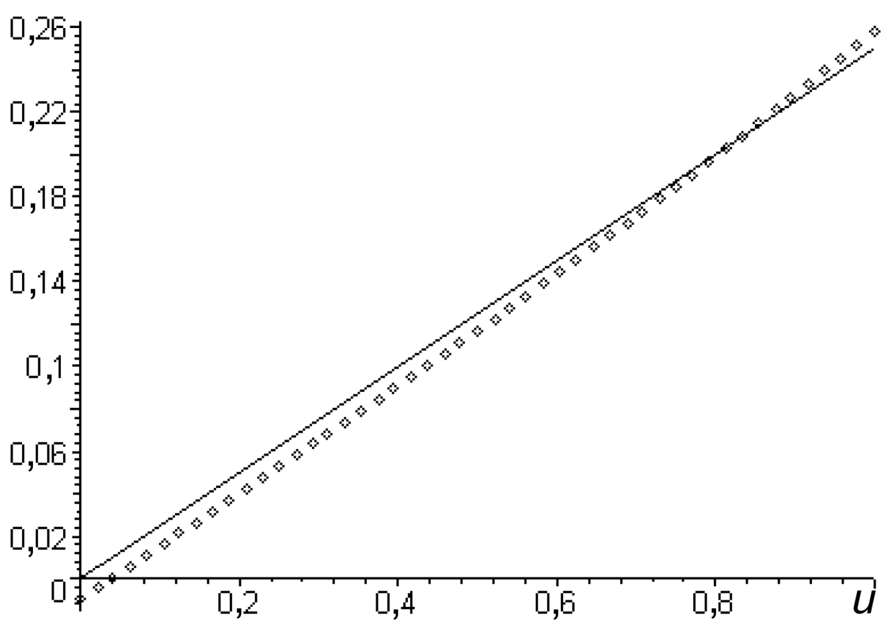

Fig. 2. The second components of the exact solution (solid line) and its first approximation (dotted line).

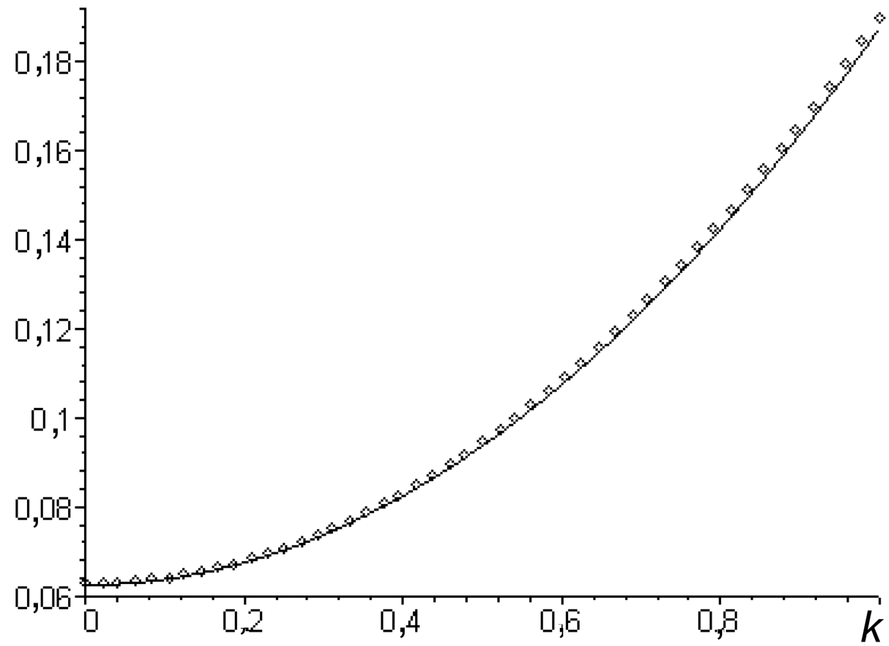

Fig. 3. The first components of the exact solution (solid line) and its fourth approximation (dotted line).

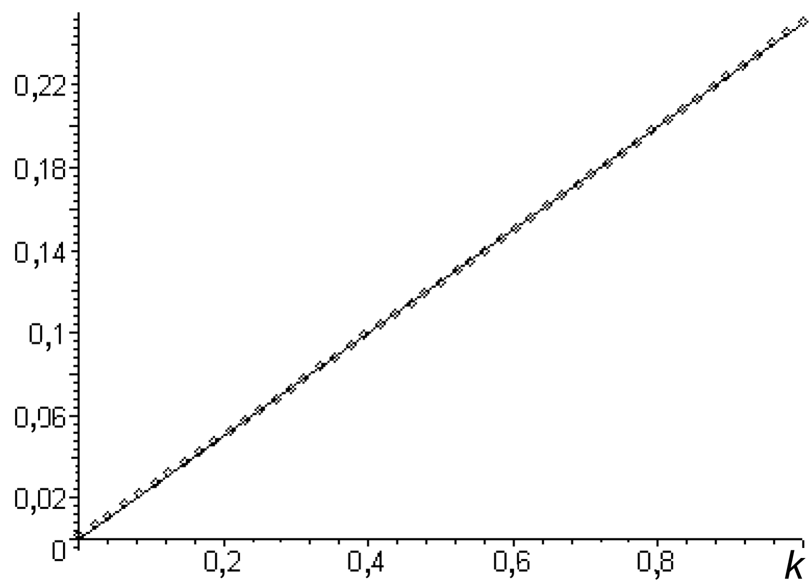

Fig. 4. The second components of the exact solution (solid line) and its fourth approximation (dotted line).

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