UDC 517.9

OPTIMAL EXISTENCE THEORY FOR SINGLE AND MULTIPLE POSITIVE PERIODIC SOLUTIONS TO FUNCTIONAL DIFFERENTIAL EQUATIONS*

ОПТИМАЛЬНА ТЕОРІЯ ІСНУВАННЯ ЄДИНОГО І КРАТНИХ ДОДАТНИХ ПЕРІОДИЧНИХ РОЗВ'ЯЗКІВ ФУНКЦІОНАЛЬНО-ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ

D. Jiang

Northeast Normal Univ.
Changchun 130024, P. R. China
e-mail: daqingjiang@vip.163.com.

D. O'Regan

Nat. Univ. Ireland
Galway, Ireland
e-mail: donal.oregan@nuigalway.ie

R. P. Agarwal

Florida Inst. Technology
Melbourne, Florida 32901-6975, USA

This paper deals with a new optimal existence theory for single and multiple positive periodic solutions to functional differential equations by employing a fixed point theorem in cones. We illustrate our theory by examining several biomathematical models. The paper improves and extends previous results in the literature.

З використанням теореми про нерухому точку в конусах розглянуто оптимальну теорію існування єдиного і кратних додатних періодичних розв'язків функціонально-диференціальних рівнянь. Теорію проілюстровано прикладами кількох математичних моделей, що використовуються в біології. Отримані результати покращують і узагальнюють попередні результати.

1. Introduction. The purpose of the present paper is to present optimal existence conditions for single and multiple positive periodic solutions for the general functional differential equation

$$\dot{y}(t) = -a(t)y(t) + g(t, y(t-\tau(t))) \tag{1.1}$$

where $a(t) \in C(\mathbf{R}, (0,\infty))$, $\tau(t) \in C(\mathbf{R},\mathbf{R})$, $g \in C(\mathbf{R}\times[0,\infty),[0,\infty))$, and $a(t), \tau(t), g(t,y)$ are all ω-periodic functions; here $\omega > 0$ is a constant.

It is well known that the functional differential equation (1.1) includes many mathematical ecological equations. For example, see the Hematopoiesis model [1 – 3]

$$\dot{y}(t) = -a(t)y(t) + b(t)e^{-\beta(t)y(t-\tau(t))}; \tag{1.2}$$

* The work was supported by NNSF of China (project 10171010).

and the more general model of blood cell production [1, 3 – 5]

$$\dot{y}(t) = -a(t)y(t) + b(t)\frac{1}{1 + y(t - \tau(t))^n}, \quad n > 0, \tag{1.3}$$

$$\dot{y}(t) = -a(t)y(t) + b(t)\frac{y(t - \tau(t))}{1 + y(t - \tau(t))^n}, \quad n > 0; \tag{1.4}$$

and also the more general Nicholson's blowflies model [1, 3, 6 – 8]

$$\dot{y}(t) = -a(t)y(t) + b(t)y(t - \tau(t))e^{-\beta(t)y(t-\tau(t))}. \tag{1.5}$$

To our knowledge, there are only a few papers on the existence of positive periodic solutions for Eq. (1.1), even for (1.2) – (1.5). The systems (1.2), (1.3) and (1.5) have been investigated in [2, 4, 6]. In these papers estimates of solutions are obtained and also it is shown that the solutions are uniformly bounded and uniformly-ultimately bounded. In addition a group of conditions are given to guarantee the existence of one positive ω-periodic solution for Eq. (1.2), (1.3) and (1.5) by applying the Yoshizawa theorem [9].

Very recently, the authors in [3] have considered the existence of one positive periodic solution for the general functional differential equation

$$\dot{y}(t) = -a(t)y(t) + b(t)f(t, y(t - \tau(t))) \tag{1.6}$$

where $a(t),\ b(t) \in C(\mathbf{R}, (0, \infty)),\ \tau(t) \in C(\mathbf{R}, \mathbf{R}),\ f \in C(\mathbf{R}\times[0, \infty), [0, \infty))$, and $a(t), b(t), \tau(t)$, $f(t, y)$ are all ω-periodic functions; here $\omega > 0$ is a constant. The main results in [3] are as follows.

Theorem A. *Eq. (1.6) has at least one ω-periodic positive solution, provided the following condition holds:*

$$\lim_{u\downarrow 0}\ \min_{t\in[0,\omega]} \frac{f(t, u)}{u} = \infty \quad \textit{and} \quad \lim_{u\uparrow\infty}\ \max_{t\in[0,\omega]} \frac{f(t, u)}{u} = 0 \quad (sublinear).$$

Theorem B. *Assume that*

B_1) $\min_{t\in[0,\omega]}\{b(t) - a(t)\} > 0$;

B_2) there exists a $\varepsilon_0 > 0$ such that $f(t, u)$ is increasing in $0 \le u \le \varepsilon_0$. Then Eq. (1.6) has at least one ω-periodic positive solution, provided the following condition holds:

$$\lim_{u\downarrow 0}\ \min_{t\in[0,\omega]} \frac{f(t, u)}{u} = 1 \quad \textit{and} \quad \lim_{u\uparrow\infty}\ \max_{t\in[0,\omega]} \frac{f(t, u)}{u} = 0.$$

The proofs of Theorems A and B are based on an application of the norm-type compression theorem in cones due to Krasnoselskii (see [10, 11]).

Motivated by the work above, in this paper we shall present a new optimal existence theory for single and multiple positive periodic solutions of Eq. (1.1).

Let

$$\sigma = e^{-\int\limits_0^\omega a(\xi)d\xi}. \tag{1.7}$$

In this paper, we have the following hypotheses:

H$_1$) $\liminf_{u\downarrow 0} \dfrac{g(t,u)}{u} > a(t)$ and $\liminf\limits_{u\uparrow\infty} \dfrac{g(t,u)}{u} > a(t)$;

H$_2$) $\limsup_{u\downarrow 0} \dfrac{g(t,u)}{u} < a(t)$ and $\limsup\limits_{u\uparrow\infty} \dfrac{g(t,u)}{u} < a(t)$;

H$_3$) there is a $p > 0$ such that $\sigma p \le u \le p$ implies

$$g(t,u) < a(t)p, \quad 0 \le t \le \omega;$$

H$_4$) there is a $p > 0$ such that $\sigma p \le u \le p$ implies

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Remark 1. If there is a $p > 0$ such that $\sigma p \le u \le p$ implies

$$g(t,u) < a(t)u, \quad 0 \le t \le \omega,$$

then H$_3$) holds.

2. Main results. First of all, notice that to find a ω-periodic solution of Eq. (1.1) is equivalent to finding a ω-periodic solution of the integral equation

$$y(t) = \int\limits_t^{t+\omega} G(t,s)g(s,y(s-\tau(s)))ds, \tag{2.1}$$

where

$$G(t,s) := \frac{\exp\left(\int\limits_t^s a(\xi)d\xi\right)}{\exp\left(\int\limits_0^\omega a(\xi)d\xi\right) - 1}. \tag{2.2}$$

One can see, for $s \in [t, t+\omega]$, that

$$A \overset{\text{df}}{=} G(t,t) \le G(t,s) \le G(t,t+\omega) \overset{\text{df}}{=} B, \tag{2.3}$$

where

$$A = \frac{1}{\exp\left(\int\limits_0^\omega a(\xi)d\xi\right) - 1}, \quad B = \frac{\exp\left(\int\limits_0^\omega a(\xi)d\xi\right)}{\exp\left(\int\limits_0^\omega a(\xi)d\xi\right) - 1}.$$

Thus $\sigma = A/B$, where σ is as in (1.7).

Let

$$X = \{y(t) : y(t) \in C(\mathbf{R}, \mathbf{R}), y(t+\omega) = y(t)\}, \tag{2.4}$$

and define

$$\|y\| = \sup_{t\in[0,\omega]} \{|y(t)| : y \in X\}.$$

Then X with the norm $\|\cdot\|$ is a Banach space.

By using (2.1), (2.2), we know for every positive ω-periodic solution of Eq. (1.1), one has

$$\|y\| \leq B \int\limits_0^\omega g(s, y(s-\tau(s)))ds,$$

and

$$y(t) \geq A \int\limits_0^\omega g(s, y(s-\tau(s)))ds,$$

so we have

$$y(t) \geq \frac{A}{B}\|y\| = \sigma\|y\|. \tag{2.5}$$

The following theorems are our main results.

Theorem 1. *Assume that H_1) and H_3) are satisfied. Then Eq. (1.1) has at least two ω-periodic positive solutions y_1 and y_2 such that*

$$0 < \|y_1\| < p < \|y_2\|.$$

Corollary 1. *The conclusion of Theorem 1 remains valid if H_1), is replaced by:*

H_1^*) $\liminf_{u\downarrow 0} \dfrac{g(t,u)}{u} = \infty$ *and* $\liminf\limits_{u\uparrow\infty} \dfrac{g(t,u)}{u} = \infty.$

Theorem 2. *Assume that H_2) and H_4) are satisfied. Then Eq. (1.1) has at least two ω-periodic positive solutions y_1 and y_2 such that*

$$0 < \|y_1\| < p < \|y_2\|.$$

Corollary 2. *The conclusion of Theorem 2 remains valid if H_2) is replaced by:*

H_2^*) $\limsup_{u\downarrow 0} \dfrac{g(t,u)}{u} = 0$ *and* $\limsup\limits_{u\uparrow\infty} \dfrac{g(t,u)}{u} = 0.$

Theorem 3. *Eq. (1.1) has at least one ω-periodic positive solution, provided one of the following conditions holds:*

$i)$ $\liminf_{u\downarrow 0} \dfrac{g(t,u)}{u} > a(t)$ *and* $\limsup\limits_{u\uparrow\infty} disp \dfrac{g(t,u)}{u} < a(t);$

$ii)$ $\limsup_{u\downarrow 0} \dfrac{g(t,u)}{u} < a(t)$ *and* $\liminf\limits_{u\uparrow\infty} \dfrac{g(t,u)}{u} > a(t).$

Corollary 3. *Eq. (1.1) has at least one ω-periodic positive solution, provided one of the following conditions holds:*

$i)$ $\liminf_{u\downarrow 0} \dfrac{g(t,u)}{u} = \infty$ *and* $\limsup\limits_{u\uparrow\infty} \dfrac{g(t,u)}{u} = 0$ $(sublinear);$

$ii)$ $\limsup_{u\downarrow 0} \dfrac{g(t,u)}{u} = 0$ *and* $\liminf\limits_{u\uparrow\infty} \dfrac{g(t,u)}{u} = \infty$ $(superlinear).$

Remark 2. Theorem 3 extends and improves Theorems A and B in [3].

Remark 3. Note that if $g(t,u) = a(t)u$, then the existence of positive ω-periodic solutions for linear problem

$$\dot{y}\ (t) = -a(t)y(t) + a(t)y(t-\tau(t))$$

cannot be guaranteed. As a result the conditions in Theorems 1 – 3 are optimal.

3. Proof of main results. First, we state the fixed point theorem in cones which will be used in this section.

Lemma 1[10]. *Let $X = (X, ||\cdot||)$ be a Banach space and let K be a cone in X. Also, r, R are constants with $0 < r < R$. Suppose $\Phi : \bar{\Omega}_R \cap K \to K($ here $\Omega_R = \{x \in X, ||x|| < R\})$ be a continuous and completely continuous operator such that*

i) $x \neq \lambda\Phi x, \quad for\ \lambda \in [0,1]\ and\ x \in K \cap \partial\Omega_r,$

and

ii) *there exists $\psi \in K\backslash\{0\}$ such that $x \neq \Phi x + \delta\psi$ for $x \in K \cap \partial\Omega_R$ and $\delta \geq 0$.*

Then Φ has a fixed point in $K \cap \{x \in X : r < ||x|| < R\}$.

Remark 4. In Theorem 1, if i) and ii) are replaced by

i) $^*\, x \neq \lambda\Phi x, \quad for\ \lambda \in [0,1]\ and\ x \in K \cap \partial\Omega_R,$

and

ii) * there exists $\psi \in K\backslash\{0\}$ such that $x \neq \Phi x + \delta\psi$ for $x \in K \cap \partial\Omega_r$ and $\delta \geq 0$.

Then Φ has a fixed point in $K \cap \{x \in X : r < ||x|| < R\}$.

Let X be as in (2.4). Define an operator on X as follows:

$$y = \Phi y \tag{3.1}$$

where Φ is defined by

$$(\Phi y)(t) = \int\limits_{t}^{t+\omega} G(t,s) g(s, y(s - \tau(s))) ds, \tag{3.2}$$

for $y \in X$. Clearly, Φ is a continuous and completely continuous operator on X.

Let

$$K = \{y \in X : y(t) \geq 0 \quad \text{and} \quad y(t) \geq \sigma \|y\|\};$$

here σ is as in (1.7). It is not difficult to verify that K is a cone in X.

Lemma 2. $\Phi(K) \subseteq K$.

Proof. For any $y \in K$, we have

$$\|\Phi y\| \leq B \int\limits_{0}^{\omega} g(s, y(s - \tau(s))) ds,$$

and

$$(\Phi y)(t) \geq A \int\limits_{0}^{\omega} g(s, y(s - \tau(s))) ds.$$

Thus we have

$$(\Phi y)(t) \geq \frac{A}{B} \|\Phi y\| = \sigma \|\Phi y\|,$$

i. e., $\Phi y \in K$. This completes the proof of Lemma 2.

Proof of Theorem 1. Suppose that $\mathrm{H}_1)$ and $\mathrm{H}_3)$ hold. By using the first inequality of $\mathrm{H}_1)$, i.e., $\liminf_{u \downarrow 0} \dfrac{g(t,u)}{u} > a(t)$, one can find $0 < r < p$ and $\varepsilon > 0$ such that

$$g(t,u) \geq a(t)(1+\varepsilon)u, \quad \text{whenever} \quad 0 \leq u \leq r. \tag{3.3}$$

Thus, if $y \in K$ with $\|y\| = r$, then $y(t) \geq \sigma r$.

Let $\psi \equiv 1$ and we now prove that

$$y \neq \Phi y + \delta \psi \quad \text{for} \quad y \in K \cap \partial \Omega_r \quad \text{and} \quad \delta \geq 0. \tag{3.4}$$

If not, there exist $y_0 \in K \cap \partial \Omega_r$ and $\delta_0 \geq 0$ such that

$$y_0 = \Phi y_0 + \delta_0 \psi.$$

Let $\mu = \min_{t\in\mathbf{R}} y_0(t)$. Then for $t \in \mathbf{R}$ we have

$$y_0(t) = (\Phi y_0)(t) + \delta_0 = \\ = \int_t^{t+\omega} G(t,s)g(s, y_0(s-\tau(s)))ds + \delta_0 \geq \\ \geq \int_t^{t+\omega} G(t,s)a(s)(1+\varepsilon)y_0(s-\tau(s))ds \geq \\ \geq \mu(1+\varepsilon)\int_t^{t+\omega} G(t,s)a(s)ds = \mu(1+\varepsilon),$$

and this implies $\mu \geq \mu(1+\varepsilon)$, a contradiction.

Next, by using the inequality in H_3), we prove that

$$y \neq \lambda\Phi y \quad \text{for} \quad y \in K \cap \partial\Omega_p \text{ and } 0 \leq \lambda \leq 1. \tag{3.5}$$

If not, there exist $y_0 \in K \cap \partial\Omega_p$ and $0 \leq \lambda_0 \leq 1$ such that

$$y_0 = \lambda_0 \Phi y_0.$$

Clearly, $\lambda_0 > 0$. Thus, $\|y_0\| = p$ and $\sigma p \leq y_0(t) \leq p$ for $t \in \mathbf{R}$, so we have

$$g(t, y_0(t-\tau(t))) < a(t)p, \quad t \in \mathbf{R}. \tag{3.6}$$

Then we obtain

$$y_0(t) = \lambda_0(\Phi y_0)(t) = \\ = \lambda_0 \int_t^{t+\omega} G(t,s)g(s, y_0(s-\tau(s)))ds < \\ < \int_t^{t+\omega} G(t,s)a(s)pds = p,$$

and this implies $||y_0|| = p < p$, a contradiction.

In view of (3.4) and (3.5), by Lemma 1, we see that Φ has a fixed point $y_1 \in K$ and $r < \|y_1\| < p$. Thus $y_1(t) \geq \sigma r > 0$, which means that $y_1(t)$ is a ω-periodic positive solution of (1.1).

Next, by using the second inequality of H_1), i. e., $\liminf_{u\uparrow\infty} \dfrac{g(t,u)}{u} > a(t)$, one can find $r_1 > p$ and $\varepsilon > 0$ such that

$$g(t,u) \geq a(t)(1+\varepsilon)u, \quad \text{whenever} \quad u \geq r_1. \tag{3.7}$$

Let $R = \dfrac{r_1}{\sigma}$, so we have

$$u(t) \geq \sigma\|u\| = \sigma R = r_1 \quad \text{for} \quad u \in K \cap \partial\Omega_R. \tag{3.8}$$

Thus, if $y \in K$ with $\|y\| = R$, then $y(t) \geq \sigma R = r_1$.

Let $\psi \equiv 1$ and we now prove that

$$y \neq \Phi y + \delta\psi \quad \text{for} \quad y \in K \cap \partial\Omega_R \quad \text{and} \quad \delta \geq 0. \tag{3.9}$$

If not, there exist $y_0 \in K \cap \partial\Omega_R$ and $\delta_0 \geq 0$ such that

$$y_0 = \Phi y_0 + \delta_0\psi.$$

Let $\mu = \min_{t\in\mathbf{R}} y_0(t)$. Then for $t \in \mathbf{R}$ we have

$$\begin{aligned} y_0(t) &= (\Phi y_0)(t) + \delta_0 = \\ &= \int_t^{t+\omega} G(t,s)g(s,y_0(s-\tau(s)))ds + \delta_0 \geq \\ &\geq \int_t^{t+\omega} G(t,s)a(s)(1+\varepsilon)y_0(s-\tau(s))ds \geq \\ &\geq \mu(1+\varepsilon)\int_t^{t+\omega} G(t,s)a(s)ds = \mu(1+\varepsilon), \end{aligned}$$

and this implies $\mu \geq \mu(1+\varepsilon)$, a contradiction.

In view of (3.5) and (3.9), by Lemma 1, we see that Φ has a fixed point $y_2 \in K$ and $p < \|y_2\| < R$. Thus $y_2(t) \geq \sigma p > 0$, which means that $y_2(t)$ is a ω-periodic positive solution of (1.1).

This completes the proof of Theorem 1.

Proof of Theorem 2. Suppose that H_2) and H_4) hold. By using the first inequality of H_2), i.e., $\limsup_{u\downarrow 0} \dfrac{g(t,u)}{u} < a(t)$, one can find $0 < r < p$ and $0 < \varepsilon < 1$ such that

$$g(t,u) \leq a(t)(1-\varepsilon)u, \quad \text{whenever} \quad 0 \leq u \leq r. \tag{3.10}$$

Thus, if $y \in K$ with $\|y\| = r$, then $y(t) \geq \sigma r$. We want to show that

$$y \neq \lambda \Phi y \quad \text{for} \quad y \in K \cap \partial\Omega_r \text{ and } 0 \leq \lambda \leq 1. \tag{3.11}$$

If not, there exist $y_0 \in K \cap \partial\Omega_r$ and $0 \leq \lambda_0 \leq 1$ such that

$$y_0 = \lambda_0 \Phi y_0.$$

Clearly, $\lambda_0 > 0$. Then we have

$$\begin{aligned} y_0(t) &= \lambda_0(\Phi y_0)(t) = \\ &= \lambda_0 \int\limits_t^{t+\omega} G(t,s) g(s, y_0(s-\tau(s))) ds \leq \\ &\leq \int\limits_t^{t+\omega} G(t,s) a(s)(1-\varepsilon) y_0(s-\tau(s)) ds \leq \\ &\leq (1-\varepsilon)||y_0|| \int\limits_t^{t+\omega} G(t,s) a(s) ds = \\ &= (1-\varepsilon)||y_0||, \end{aligned}$$

and this implies $||y_0|| \leq (1-\varepsilon)||y_0||$, a contradiction.

Next, by using the inequality in H_4), and letting $\psi \equiv 1$ we now prove that

$$y \neq \Phi y + \delta\psi \quad \text{for} \quad y \in K \cap \partial\Omega_p \text{ and } \delta \geq 0. \tag{3.12}$$

If not, there exist $y_0 \in K \cap \partial\Omega_p$ and $\delta_0 \geq 0$ such that

$$y_0 = \Phi y_0 + \delta_0 \psi.$$

Thus, $\|y_0\| = p$ and $\sigma p \leq y_0(t) \leq p$ for $t \in \mathbf{R}$, so we have

$$g(t, y_0(t-\tau(t))) > a(t) y_0(t-\tau(t)), \quad t \in \mathbf{R}. \tag{3.13}$$

Let $\mu = \min_{t\in\mathbf{R}} y_0(t)$. Then for $t \in \mathbf{R}$ we have

$$y_0(t) = (\Phi y_0)(t) + \delta_0 =$$

$$= \int\limits_t^{t+\omega} G(t,s)g(s, y_0(s-\tau(s)))ds + \delta_0 >$$

$$> \int\limits_t^{t+\omega} G(t,s)a(s)y_0(s-\tau(s))ds \geq$$

$$\geq \mu \int\limits_t^{t+\omega} G(t,s)a(s)ds = \mu,$$

and this implies $\mu > \mu$, a contradiction.

In view of (3.11) and (3.12), by Lemma 1, we see that Φ has a fixed point $y_1 \in K$ and $r < \|y_1\| < p$. Thus $y_1(t) \geq \sigma r > 0$, which means that $y_1(t)$ is a ω-periodic positive solution of (1.1).

Next, by using the second inequality of H$_2$), i.e., $\limsup_{u\uparrow\infty} \dfrac{g(t,u)}{u} < a(t)$, one can find $r_1 > p$ and $0 < \varepsilon < 1$ such that

$$g(t,u) \leq a(t)(1+\varepsilon)u, \quad \text{whenever} \quad u \geq r_1. \tag{3.14}$$

Let $R = \dfrac{r_1}{\sigma}$, so we have,

$$u(t) \geq \sigma\|u\| = \sigma R = r_1 \quad \text{for} \quad u \in K \cap \partial\Omega_R. \tag{3.15}$$

Thus, if $y \in K$ with $\|y\| = R$, then $y(t) \geq \sigma R = r_1$.

Essentially the same reasoning as before (the details are left to the reader) yields

$$y \neq \lambda\Phi y \quad \text{for} \quad y \in K \cap \partial\Omega_R \quad \text{and} \quad 0 \leq \lambda \leq 1. \tag{3.16}$$

In view of (3.12) and (3.16), by Lemma 1, we see that Φ has a fixed point $y_2 \in K$ and $p < \|y_2\| < R$. Thus $y_2(t) \geq \sigma p > 0$, which means that $y_2(t)$ is a ω-periodic positive solution of (1.1).

This completes the proof Theorem 2.

Proof of Theorem 3. Essentially the same reasoning as in the proof of Theorems 1 and 2 establishes the result.

Remark 5. Essentially the same reasoning as in this paper establishes (the details are left to the reader) the existence of single and multiple positive periodic solutions for the general Volterra integro-differential equation (see [12], which has results similar to those in [3])

$$\dot{y}(t) = -a(t)y(t) + \int\limits_{-\infty}^{0} K(r)g(t, y(t+r))dr$$

where $a(t) \in C(R, (0, \infty))$, $g \in C(R \times [0, \infty), [0, \infty))$, and $a(t)$, $g(t, y)$ are all ω-periodic functions; here $\omega > 0$ is a constant. Moreover, $K(r) \in C((-\infty, 0], [0, \infty))$ and $\int_{-\infty}^{0} K(r)dr = 1$.

4. Examples. In this section, we apply the main result obtained in the previous section to examples modelling biological phenomena.

It follows from Theorem 3 and Corollary 3 that the following results hold.

Corollary 4. *Assume that*

H_1) $a(t),\ b(t) \in C(\mathbf{R}, (0, \infty))$, $\beta(t) \in C(\mathbf{R}, (0, \infty))$, $\tau(t) \in C(\mathbf{R}, \mathbf{R})$, $a(t), b(t), \tau(t)$ *and* $\beta(t)$ *are all* ω*-periodic functions; here* $\omega > 0$ *is a constant.*

Then Eq(1.2) has at least one ω*-periodic positive solution.*

Corollary 5. *Assume that*

H_1) $a(t),\ b(t) \in C(\mathbf{R}, (0, \infty))$, $\tau(t) \in C(\mathbf{R}, \mathbf{R})$, $a(t), b(t)$ *and* $\tau(t)$ *are all* ω*-periodic functions; here* $\omega > 0$ *is a constant.*

Then Eq. (1.3) has at least one ω*-periodic positive solution.*

Corollary 6. *Assume that*

H_1) $a(t),\ b(t) \in C(\mathbf{R}, (0, \infty))$, $\tau(t) \in C(\mathbf{R}, \mathbf{R})$, $a(t), b(t)$ *and* $\tau(t)$ *are all* ω*-periodic functions; here* $\omega > 0$ *is a constant;*

H_2) $b(t) > a(t)$ *for* $t \in [0, \omega]$.

Then Eq. (1.4) has at least one ω*-periodic positive solution.*

Corollary 7. *Assume that*

H_1) $a(t),\ b(t) \in C(\mathbf{R}, (0, \infty))$, $\beta(t) \in C(\mathbf{R}, (0, \infty))$, $\tau(t) \in C(\mathbf{R}, \mathbf{R})$, $a(t), b(t), \tau(t)$ *and* $\beta(t)$ *are all* ω*-periodic functions; here* $\omega > 0$ *is a constant;*

H_2) $b(t) > a(t)$ *for* $t \in [0, \omega]$.

Then Eq. (1.5) has at least one ω*-periodic positive solution.*

Corollary 4 and Corollary 5 can be checked easily. For Corollary 6 and Corollary 7, notice

$$\lim_{u \downarrow 0} \frac{g(t, u)}{u} = b(t) > a(t) \quad \text{and} \quad \lim_{u \uparrow \infty} \frac{g(t, u)}{u} = 0 < a(t),$$

so the result follows from Theorem 3.

Example 1. Consider the equation

$$\dot{y}(t) = -a(t)y(t) + b(t)[y^a(t - \tau(t)) + y^b(t - \tau(t))], \ \ 0 < a < 1 < b, \tag{4.1}$$

where $a(t),\ b(t) \in C(\mathbf{R}, (0, \infty))$, $\tau(t) \in C(\mathbf{R}, \mathbf{R})$, and $a(t), b(t), \tau(t)$ are all ω-periodic functions; here $\omega > 0$ is a constant.

Applying Theorem 1, we will show that Eq. (4.1) has two ω-periodic positive solutions provided

$$\max_{t \in [0, \omega]} \frac{b(t)}{a(t)} < \sup_{x \in (0, \infty)} \frac{x}{x^a + x^b}. \tag{4.2}$$

Set $g(t,u) = b(t)(u^a + u^b)$, then

$$\lim_{u\downarrow 0} \frac{g(t,u)}{u} = \infty \quad \text{and} \quad \lim_{u\uparrow\infty} \frac{g(t,u)}{u} = \infty,$$

so H_1) holds. Set

$$T(x) := \frac{x}{x^a + x^b}, \ \ x > 0,$$

then $T(0+) = 0, T(\infty) = 0$, and

$$T(p) = \sup_{x\in(0,\infty)} T(x), \ \ p = \left(\frac{1-a}{b-1}\right)^{\frac{1}{b-a}}.$$

Then for $\sigma p \leq u \leq p$, we have

$$g(t,u) \leq b(t)(p^a + p^b) \leq$$

$$\leq a(t)(p^a + p^b) \max_{t\in[0,\omega]} \frac{b(t)}{a(t)} <$$

$$< a(t)(p^a + p^b)T(p) = \ a(t)p,$$

so H_3) holds. Then the result follows from Theorem 1 (or Corollary 1).

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Received 10.06.2003