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IMPULSIVE SEMILINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

ІМПУЛЬСНІ НАПІВЛІНІЙНІ ФУНКЦІОНАЛЬНІ ДИФЕРЕНЦІАЛЬНІ РІВНЯННЯ

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In this paper the Leray – Schauder nonlinear alternative combined with the semigroup theory is used to investigate the existence of mild solutions for first order impulsive semilinear functional differential equations in Banach spaces.

Використовується нелінійна альтернатива Лере – Шаудера разом з теорією напівгруп для вивчення питання існування нежорстких розв'язків імпульсних напівлінійних функціональних диференціальних рівнянь першого порядку в банахових просторах.

1. Introduction. This paper is concerned with the existence of mild solutions for the impulsive semilinear functional differential equation of the form:

$$y' - A(t)y = f(t, y_t), \;\; t \in J = [0, b], \; t \neq t_k, \;\; k = 1, \ldots, m, \tag{1}$$

$$y(t_k^+) = I_k(y(t_k^-)), \;\; k = 1, \ldots, m, \tag{2}$$

$$y(t) = \phi(t), \; t \in [-r, 0], \tag{3}$$

where $f : J \times C([-r, 0], E) \longrightarrow E$ is a given map, $\phi \in C([-r, 0], E)$, $A(t), t \in J$, a linear closed operator from a dense subspace $D(A(t))$ of E into E and E a real Banach space with the norm $|\cdot|$, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = b$, $I_k \in C(E, E)$, $k = 1, 2, \ldots, m$, $y(t_k^-)$ and $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t = t_k$, respectively.

For any continuous function y defined on $[-r,b]-\{t_1,\ldots,t_m\}$ and any $t\in J$, we denote by y_t the element of $C([-r,0],E)$ defined by

$$y_t(\theta)=y(t+\theta),\ \ \theta\in[-r,0].$$

Here $y_t(\cdot)$ represents the history of the state from time $t-r$, up to the present time t.

Many evolution processes and phenomena experience changes of state abrupt through short-term perturbations. Since the durations of the perturbations are negligible in comparison with the duration of each process, it is quite natural to assume that these perturbations act in terms of impulses. See the monographs of Lakshmikantham et al. [1], and Samoilenko and Perestyuk [2] and the papers of Erbe et al. [3], Kirane and Rogovchenko [4], Liu et al. [5] and Liu and Zhang [6] various properties of their solutions are studied and a large bibliography is proposed. For the existence of Caratheodory solutions to the problem (1) – (3) with $A\equiv 0$ we refer the reader to the paper of Benchohra et al. [7].

This paper will be divided into three sections. In Section 2 we will recall briefly some basic definitions and preliminary results which will be used throughout Section 3. In Section 3 we establish an existence theorem for (1) – (3). Our approach is based on the nonlinear alternative of Leray – Schauder type [8] combined with the semigroup theory [9]. In our results we do not assume any type of monotonicity condition on $I_k, k=1,\ldots,m$, which is usually the situation in the literature, see for instance, [3 – 5]. This paper extends to the semilinear and the functional cases some results obtained by Frigon and O'Regan in [10].

2. Preliminaries. We will briefly recall some basic definitions and preliminary results that we will use in the sequel. Let E be a real Banach. $B(E)$ denotes the Banach space of bounded linear operators from E into E.

A measurable function $y:J\longrightarrow E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. For properties of the Bochner integral, we refer to Yosida [11].

$L^1(J,E)$ denotes the Banach space of functions $y:J\longrightarrow E$ which are Bochner integrable normed by

$$\|y\|_{L^1}=\int\limits_0^b|y(t)|dt\quad\text{for all}\quad y\in L^1(J,E).$$

Let X be a Banach space. A map $G:X\to X$ is said to be completely continuous if it is continuous and $G(B)$ is relatively compact for every bounded subset $B\subseteq X$.

Definition 2.1. *The map $f:J\times C([-r,0],E)\longrightarrow E$ is said to be an L^1-Caratheodory if*

(i) $t\longmapsto f(t,u)$ is measurable for each $u\in C([-r,0],E)$;

(ii) $u\longmapsto f(t,u)$ is continuous for almost all $t\in J$;

(iii) for each $q>0$, there exists $h_q\in L^1(J,\mathbb{R}_+)$ such that $|f(t,u)|\le h_q(t)$ for all $\|u\|\le q$ and for almost all $t\in J$.

In order to define the mild solution of (1) – (3) we shall consider the following space

$$\Omega=\{y\ :\ [-r,b]\longrightarrow E: y_k\in C(J_k,E), k=0,\ldots,m,\ \text{and there exist}$$

$$y(t_k^-)\ \text{and}\ y(t_k^+), k=1,\ldots,m,\ \text{with}\ y(t_k^-)=y(t_k)\}$$

which is a Banach space with the norm

$$\|y\|_\Omega = \max\{\|y_k\|_\infty, k = 0, \ldots, m\},$$

where y_k is the restriction of y to $J_k = (t_k, t_{k+1}],\ k = 0, \ldots, m$.

Before we give the definition of a mild solution of problem (1) – (3), we make the following assumption on $A(t)$ in (1):

(H_1) $A(\cdot) : t \longrightarrow A(t),\ t \in J$, is continuous such that

$$A(t)y = \lim_{h \to 0+} \frac{T(t+h,t)y - y}{h}, \quad y \in D(A(t)),$$

where $T(t,s) \in B(E)$ for each $(t,s) \in \gamma := \{(t,s); 0 \le s \le t < b\}$, satisfying

(i) $T(t,t) = I$ (I is the identity operator in E),

(ii) $T(t,s)T(s,r) = T(t,r)$ for $0 \le r \le s \le t < b$,

(iii) the mapping $(t,s) \longmapsto T(t,s)y$ is strongly continuous in γ for each $y \in E$; moreover there exists a positive constant M such that for any $(t,s) \in \gamma,\ ||T(t,s)|| \le M$.

Definition 2.2. *A function $y \in \Omega$ is said to be a mild solution of (1) – (3) (see [5]) if*

$$y(t) = \begin{cases} \phi(t), \quad t \in [-r, 0]; \\ T(t,0)\phi(0) + \displaystyle\int_0^t T(t,s) f(s, y_s) ds, \quad t \in [0, t_1]; \\ I_k(y(t_k^-)) + \displaystyle\int_{t_k}^t T(t,s) f(s, y_s) ds, \quad t \in J_k,\ k = 1, \ldots, m. \end{cases}$$

Our main result is based on the following:

Lemma 2.1 (Nonlinear Alternative [8]). *Let X be a Banach space with $C \subset X$ convex. Assume U is a relatively open subset of C with $0 \in U$ and $G : \overline{U} \longrightarrow C$ is a compact map. Then either,*

(i) G has a fixed point in $\overline{U}$; or

(ii) there is a point $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda G(u)$.

Remark 2.1. By $\overline{U}$ and ∂U we denote the closure of U and the boundary of U respectively.

3. Main result. We are now in a position to state and prove our existence result for the IVP (1) – (3).

Theorem 3.1. *Let $t_0 = 0,\ t_{m+1} = b$. Suppose:*

(H_2) *$f : J \times C([-r,0], E) \longrightarrow E$ is an L^1-Caratheodory map;*

(H_3) *there exist a continuous nondecreasing function $\psi : [0, \infty) \longrightarrow (0, \infty)$ and $p \in L^1(J, \mathbb{R}_+)$ such that*

$$|f(t,u)| \le p(t)\psi(\|u\|) \text{ for a.e. } t \in J \text{ and each } u \in C([-r,0], E)$$

with

$$\int_{t_{k-1}}^{t_k} p(s)ds < \int_{N_{k-1}}^{\infty} \frac{d\tau}{\psi(\tau)}, \quad k = 1, \dots, m+1;$$

here $N_0 = M\|\phi\|$ *and for* $k = 2, \dots, m+1$ *we have*

$$N_{k-1} = \sup_{y \in [-M_{k-2}, M_{k-2}]} |I_{k-1}(y)|, \;\; M_{k-2} = \Gamma_{k-1}^{-1}\left(M \int_{t_{k-2}}^{t_{k-1}} p(s)ds \right)$$

with

$$\Gamma_l(z) = \int_{N_{l-1}}^{z} \frac{d\tau}{\psi(\tau)}, \quad z \geq N_{l-1}, \;\; l \in \{1, \dots, m+1\};$$

(H_4) *for each bounded set* $B \subset C([-r, b], E)$ *and for each* $t \in J$ *the set*

$$\left\{ I_k(y(t_k^-)) + \int_{t_k}^{t} T(t,s) f(s, y_s) ds : y \in B \right\}$$

is relatively compact in E, $k = 0, \dots, m$.

Then the problem (1)–(3) has at least one mild solution $y \in \Omega$.

Remark 3.1. (i) (H_3) is satisfied if for example $\psi(u) = u + 1$, for $u > 0$.

(ii) If $T(t,s), (t,s) \in \gamma$, is compact, then (H_4) is satisfied.

(iii) The condition (H_4) is also satisfied if the function f maps bounded sets into relatively compact sets in $C(J, E)$.

(iv) If the Banach space E is finite dimentional, then the condition (H_4) is automatically satisfied.

Proof of the theorem. The proof is given in several steps.

Step 1. Consider the problem (1)–(3) on $[-r, t_1]$

$$y' - A(t)y = f(t, y_t), \;\; \text{a.e. } t \in J_0, \tag{4}$$

$$y(t) = \phi(t), \; t \in [-r, 0]. \tag{5}$$

We shall show that the possible mild solutions of (4), (5) are *a priori* bounded, that is there exists a constant b_0 such that, if $y \in \Omega$ is a mild solution of (4), (5), then

$$\sup\{|y(t)| : t \in [-r, 0] \cup J_0\} \leq b_0.$$

Let y be a (possible) solution to (4), (5). Then for each $t \in [0, t_1]$

$$y(t) - T(t,0)\phi(0) = \int_0^t T(t,s)f(s,y_s)ds.$$

From (H_3) we get

$$|y(t)| \le M\|\phi\| + M\int_0^t p(s)\psi(\|y_s\|)ds, \; t \in [0, t_1].$$

We consider the function μ_0 defined by

$$\mu_0(t) = \sup\{|y(s)| : -r \le s \le t\}, \;\; 0 \le t \le t_1.$$

Let $t^* \in [-r, t]$ be such that $\mu_0(t) = |y(t^*)|$. If $t^* \in [0, t_1]$, by the previous inequality we have for $t \in [0, t_1]$

$$\mu_0(t) \le M\|\phi\| + M\int_0^t p(s)\psi(\mu_0(s))ds.$$

If $t^* \in [-r, 0]$ then $\mu_0(t) = \|\phi\|$ and the previous inequality holds since $M \ge 1$.

Let us take the right-hand side of the above inequality as $v_0(t)$, then we have

$$v_0(0) = M\|\phi\| = N_0, \quad \mu_0(t) \le v_0(t), \quad t \in [0, t_1],$$

and

$$v_0'(t) = Mp(t)\psi(\mu_0(t)), \quad t \in [0, t_1].$$

Using the nondecreasing character of ψ we get

$$v_0'(t) \le Mp(t)\psi(v_0(t)), \quad t \in [0, t_1].$$

This implies for each $t \in [0, t_1]$ that

$$\int_{N_0}^{v_0(t)} \frac{d\tau}{\psi(\tau)} \le M\int_0^{t_1} p(s)ds.$$

In view of (H_3), we obtain

$$|v_0(t^*)| \le \Gamma_1^{-1}\left(M\int_0^{t_1} p(s)ds\right) := M_0.$$

Since for every $t \in [0, t_1]$, $\|y_t\| \leq \mu_0(t)$, we have

$$\sup_{t\in[-r,t_1]} |y(t)| \leq \max(\|\phi\|, M_0) = b_0.$$

We transform this problem into a fixed point problem. A mild solution to (4), (5) is a fixed point of the operator $G : C([-r, t_1], E) \longrightarrow C([-r, t_1], E)$ defined by

$$G(y)(t) = \begin{cases} \phi(t), & t \in [-r, 0]; \\ T(t,0)\phi(0) + \int\limits_0^t T(t,s) f(s, y_s) ds, & t \in J_0. \end{cases}$$

We shall show that G satisfies the assumptions of Lemma 2.2.

Claim 1. *G sends bounded sets into bounded sets in $C(J_0, E)$.*

Let $B_q := \{y \in C(J_0, E) : \|y\|_\infty = \sup_{t\in J_0} |y(t)| \leq q\}$ be a bounded set in $C_0(J_0, E)$ and $y \in B_q$. Then for each $t \in J_0$

$$G(y)(t) = T(t,0)\phi(0) + \int\limits_0^t T(t,s) f(s, y_s) ds.$$

Thus for each $t \in J_0$ we get

$$|G(y)(t)| \leq M\|\phi\| + M \int\limits_0^t |f(s, y_s)| ds \leq$$

$$\leq M\|\phi\| + M\psi(q)\|p\|_{L^1}.$$

Claim 2. *G sends bounded sets in $C(J_0, E)$ into equicontinuous sets.*

Let $r_1, r_2 \in J_0$, $r_1 < r_2$, B_q be a bounded set in $C_0(J_0, E)$ as in Claim 1 and $y \in B_q$. Then

$$|G(y)(r_2) - G(y)(r_1)| \leq |(T(r_2,0)\phi(0) - T(r_1,0)\phi(0)| +$$

$$+ \left| \int\limits_0^{r_2} [T(r_2,s) - T(r_1,s)] f(s, y_s) ds \right| + \left| \int\limits_{r_1}^{r_2} T(r_1,s) f(s, y_s) ds \right| \leq$$

$$\le |(T(r_2,0)\phi(0) - T(r_1,0)\phi(0)| +$$

$$+ \int_0^{r_2} |T(r_2,s) - T(r_1,s)| h_q(s)ds + M \int_{r_1}^{r_2} h_q(s)ds.$$

The right-hand side is independent of $y \in B_q$ and tends to zero as $r_2 - r_1 \to 0$.

The equicontinuity for the cases $r_1 < r_2 \le 0$ and $r_1 \le 0 \le r_2$ are obvious.

Claim 3. *G is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \longrightarrow y$ in $C(J_0, E)$. Then there is an integer q such that $\|y_n\|_\infty \le q$ for all $n \in \mathbb{N}$ and $\|y\|_\infty \le q$, so $y_n \in B_q$ and $y \in B_q$.

We have then by the dominated convergence theorem

$$\|G(y_n) - G(y)\|_\infty \le M \sup_{t \in J_0} \left[\int_0^t |f(s, y_{ns}) - f(s, y_s)| ds \right] \longrightarrow 0, \text{ as } n \to \infty.$$

Thus G is continuous.

Set

$$U = \{y \in C([-r, t_1], E) : \|y\|_\infty < b_0 + 1\}.$$

As a consequence of Claims 1, 2 and 3 and (H_4) together with the Arzela – Ascoli theorem we can conclude that $G : \overline{U} \longrightarrow C([-r, t_1], E)$ is a completely continuous map.

From the choice of U there is no $y \in \partial U$ such that $y = \lambda G(y)$ for any $\lambda \in (0, 1)$.

As a consequence of Lemma 2.1 we deduce that G has a fixed point $y_0 \in \overline{U}$ which is a mild solution of (4), (5).

Step 2. Consider now the following problem on $J_1 := [t_1, t_2]$:

$$y' - A(t)y = f(t, y_t), \text{ a.e. } t \in J_1, \tag{6}$$

$$y(t_1^+) = I_1(y(t_1^-)). \tag{7}$$

Let y be a (possible) mild solution to (6), (7). Then for each $t \in [t_1, t_2]$

$$y(t) - I_1(y(t_1^-)) = \int_{t_1}^t T(t,s) f(s, y_s) ds.$$

From (H_3) we get

$$|y(t)| \le \sup_{t \in [-r, t_1]} |I_1(y_0(t^-))| + M \int_{t_1}^t p(s)\psi(\|y_s\|) ds, \ t \in [0, t_1].$$

We consider the function μ_1 defined by

$$\mu_1(t) = \sup\{|y(s)| : t_1 \le s \le t\}, \ \ t_1 \le t \le t_2.$$

Let $t^* \in [t_1, t]$ be such that $\mu_1(t) = |y(t^*)|$. Then we have for $t \in [t_1, t_2]$

$$\mu_1(t) \le N_1 + M \int\limits_{t_1}^{t} p(s)\psi(\mu_1(s))ds.$$

Let us take the right-hand side of the above inequality as $v_1(t)$, then we have

$$v_1(t_1) = N_1, \quad \mu_1(t) \le v_1(t), \quad t \in [t_1, t_2],$$

and

$$v_1'(t) = Mp(t)\psi(\mu_1(t)), \quad t \in [t_1, t_2].$$

Using the nondecreasing character of ψ we get

$$v_1'(t) \le Mp(t)\psi(v_1(t)), \quad t \in [t_1, t_2].$$

This implies for each $t \in [t_1, t_2]$ that

$$\int\limits_{N_1}^{v_1(t)} \frac{d\tau}{\psi(\tau)} \le M \int\limits_{t_1}^{t_2} p(s)ds.$$

In view of (H_3), we obtain

$$|v_1(t^*)| \le \Gamma_1^{-1} \left(M \int\limits_{t_1}^{t_2} p(s)ds \right) := M_1.$$

Since for every $t \in [t_1, t_2]$, $\|y_t\| \le \mu_1(t)$, we have

$$\sup_{t \in [t_1, t_2]} |y(t)| \le M_1.$$

A mild solution to (6), (7) is a fixed point of the operator $G : C(J_1, E) \longrightarrow C(J_1, E)$ defined by

$$G(y)(t) := I_1(y(t_1^-)) + \int\limits_{t_1}^{t} T(t,s)f(s, y_s)ds.$$

Set

$$U = \{y \in C(J_1, E) : \|y\|_\infty < M_1 + 1\}.$$

As in Step 1 we can show that $G : \overline{U} \longrightarrow C(J_1, E)$ is a completely continuous map.

From the choice of U there is no $y \in \partial U$ such that $y = \lambda G(y)$ for any $\lambda \in (0, 1)$.

As a consequence of Lemma 2.1 we deduce that G has a fixed point $y_1 \in \overline{U}$ which is a mild solution of (6), (7).

Step 3. Continue this process and construct solutions $y_k \in C(J_k, E), k = 2, \ldots, m$, to

$$y' - A(t)y = f(t, y_t), \text{ a.e. } t \in J_k, \tag{8}$$

$$y(t_k^+) = I_k(y(t_k^-)). \tag{9}$$

Then

$$y(t) = \begin{cases} y_0(t), & t \in [-r, t_1]; \\ y_1(t), & t \in (t_1, t_2]; \\ . \\ . \\ . \\ y_{m-1}(t), & t \in (t_{m-1}, t_m]; \\ y_m(t), & t \in (t_m, b], \end{cases}$$

is a mild solution of (1) – (3).

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