Hydrodynamic theory of magnets with strong exchange interaction

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A microscopic approach to the description of multisublattice magnets with strong exchange interaction is proposed. Low-frequency dynamics of such magnets is characterized by the appearance of an additional dynamical variable, i.e., the orthogonal matrix of rotation, which corresponds to the total breaking of spin invariance [broken $SO(3)$ symmetry]. The structure of the source that breaks the symmetry of the equilibrium Gibbs distribution is established. The quasiaverage representation is generalized to weakly anisotropic, locally equilibrium states. The thermodynamics of such states is constructed. The method of reduced description is formulated and in its framework the hydrodynamic equations for the density of total spin and the matrix of rotation are obtained. The spectra of spin waves are found and the number of Goldstone and activation modes is determined. Two-sublattice ferrimagnet is considered as a special case of the magnet with broken $SO(3)$ symmetry, which corresponds to the special dependence of thermodynamic functions from the matrix of rotation.

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Introduction

In the present work we study the low-frequency dynamics of the multisublattice magnet with strong exchange interaction. It is known that high-frequency processes in magnets can be described on the basis of the Landau-Lifshitz equation [1,2]. The use of this equation in the low-frequency case (hydrodynamic limit) for the multisublattice magnets is not well justified since the sublattice spins are not approximate integrals of motion because of the strong intersublattice exchange interaction. In Ref. 3 it was shown that reduced description arises in the investigation of the low-frequency dynamics of the multisublattice magnet with exchange interaction. Reduced description parameters are the density of total spin $s_\alpha(\mathbf{x})$ and the orthogonal matrix of rotation $a_{\alpha\beta}(\mathbf{x})$, which characterizes the orientation of the rigid complex of the sublattice spins formed as a result of the exchange interaction. The appearance of the matrix of rotation as an additional dynamic variable corresponds to the total symmetry breaking relative to spin rotations [broken $SO(3)$ symmetry]. Thus, low-frequency dynamics of the multisublattice magnet with exchange interaction is accompanied by the appearance of the states with

spontaneously broken spin invariance. This description needs the attraction of nontraditional reduced description parameters, which are connected with the matrix of rotation $a_{\alpha\beta}$. Effective method for the study of such states is a Hamiltonian approach [4–7]. In its framework the dynamics of the magnet with total symmetry breaking relative to spin rotations [8] and the dynamics of the ferrimagnet [9] were considered. The idea of spontaneous symmetry breaking of the statistical equilibrium state [10–12] has been also used for disordered magnetic systems of the «spin glass» type [13–15]. Note that in Ref. 16 on the basis of this concept and with the use of the analogy between the «easy-plane» magnetic systems and the superfluid systems the equations of motion have been formulated for uniaxial magnets with spontaneous symmetry breaking relative to the spin rotations around the anisotropy axis. This symmetry breaking is a special case of the total spontaneous symmetry breaking.

We shall consider in the microscopic approach the thermodynamics and hydrodynamics of the magnetic systems with strong exchange interaction in the presence of weak anisotropy on the basis of the quasiaverages [10] and the reduced description [17] methods. Standard quasiaverages, which apply for

the description of equilibrium states, are generalized in the case of weakly anisotropic, locally equilibrium states. Locally equilibrium Gibbs distribution is constructed on the basis of consideration of the local unitary transformation which corresponds to the broken symmetry relative to the spin rotations [see Eq. (2.8)]. Performing this transformation on the source in Gibbs distribution, we introduce the matrix of rotation $a(\mathbf{x})$ for the locally equilibrium states. Weak anisotropy permits us to consider the total spin as an approximate integral of motion and in the sense of the main approximation of anisotropy (see the text for details) the corresponding term with the spin is included in the exponent of the Gibbs distribution.

To construct the hydrodynamic equations on the basis of reduced description method we introduce the matrix of rotation $b(\mathbf{x}, \hat{\rho})$ as a functional of the nonequilibrium statistical operator $\hat{\rho}$, using the concept of the system order parameter operator. The connection between the matrix of rotation $a(\mathbf{x})$ in locally equilibrium Gibbs distribution and the matrix of rotation $b(\mathbf{x}, \hat{\rho})$, which is the functional of the nonequilibrium statistical operator $\hat{\rho}$ is established. The equations of motion for the density of total spin $s_\alpha(\mathbf{x})$ and the matrix of rotation $a_{\alpha\beta}(\mathbf{x})$ [Eqs. (3.17) and (3.18)] are found. These equations describe the low-frequency dynamics of the multisublattice magnet with strong exchange interaction and weak anisotropy. The structure of the spectrum of spin waves is determined.

It is shown that weakly anisotropic ferrimagnet represents a special case of the magnet with total symmetry breaking, which corresponds to a special dependence of the matrix of rotation.

1. The order parameter

In the microscopic approach to the magnetic systems the basic operators, from which all other operators are constructed, are the operators of site spins $s_n(l)$ of atoms (l is the number of the site, and n is the number of the crystal sublattice). The Hamiltonian H and the statistical operator $\hat{\rho}$ are constructed from these operators only, $H = H(\hat{s})$, $\hat{\rho} = \hat{\rho}(\hat{s})$. An arbitrary physical quantity $\hat{c}$ of the magnet is also the operator functional of the site spins $\hat{c} = \hat{c}(\hat{s})$. We switch from the site representation to the continuum representation: $v_0^{-1} \hat{s}_{n\alpha}(l) \xrightarrow[v_0 \to 0]{} \hat{s}_{n\alpha}(\mathbf{x})$. Here v_0 is the volume of

the unit cell, and $\mathbf{x} \equiv \mathbf{R_l}$ is the position vector which defines the position of lth site. The spin-density operators $\hat{s}_{n\alpha}(\mathbf{x})$ satisfy the commutation relations

$$[\hat{s}_{n\alpha}(\mathbf{x}), \hat{s}_{m\beta}(\mathbf{x}')] = i\varepsilon_{\alpha\beta\gamma}\,\delta_{nm}\,\hat{s}_{m\gamma}(\mathbf{x})\delta(\mathbf{x} - \mathbf{x}') . \quad (1.1)$$

We introduce in the continuum limit the operation of the spatial shifts $[P_k, \ldots]$

$$[P_k, \rho(\hat{s}(\mathbf{x}'))] \equiv i \left.\frac{\partial\rho(\hat{s}(\mathbf{x}' + \mathbf{y}))}{\partial y_k}\right|_{y_k=0} ,$$

$$[P_k, c(\mathbf{x}, \hat{s}(\mathbf{x}'))] \equiv i \left.\frac{\partial c(\mathbf{x}, \hat{s}(\mathbf{x}' + \mathbf{y}))}{\partial y_k}\right|_{y_k=0} , \quad (1.2)$$

where $\rho(\hat{s}(\mathbf{x}'))$ and $c(\mathbf{x}, \hat{s}(\mathbf{x}'))$ are the functionals of $\hat{s}(\mathbf{x})$. In accordance with this definition

$$c(\mathbf{x}, \hat{s}(\mathbf{x}' + \mathbf{y})) = e^{-i\mathbf{Py}}c(\mathbf{x}, \hat{s}(\mathbf{x}'))\,e^{i\mathbf{Py}} . \quad (1.3)$$

Realization of the operator P_k in the classical case in terms of the spin densities in the framework of the Hamiltonian approach is given in Refs. 8 and 9. For our purposes satisfaction of the relationships (1.2) and (1.3) is sufficient in the quantum case and therefore we shall not solve the problem of concrete realization of the spatial shift operator P_k in terms of the spin density operators. Defining the translationally invariant operator $\hat{c}(\mathbf{x}) \equiv \hat{c}(\mathbf{x}, \hat{s}(\mathbf{x}'))$ by the relationship $\hat{c}(\mathbf{x} - \mathbf{y}, \hat{s}(\mathbf{x}' + \mathbf{y})) = \hat{c}(\mathbf{x}, \hat{s}(\mathbf{x}'))$, we have by virtue of Eq. (1.3)

$$i\,[P_k, \hat{c}(\mathbf{x})] = -\nabla_k\,\hat{c}(\mathbf{x}) . \quad (1.4)$$

In the case of weakly anisotropic magnetic systems the main type of interactions are the exchange interactions. Anisotropic interactions are assumed to be small and can be taken into account by means of the perturbation theory. Disregarding the anisotropy, we can characterize the magnetic system by a set of additive motion integrals $\hat{\gamma}_a = \int d^3x\,\hat{\zeta}_a(\mathbf{x})$ ($a = 0, \alpha$), where $\hat{\gamma}_0 \equiv H = \int d^3x\,\hat{\varepsilon}(\mathbf{x})$ is the Hamiltonian, and $\hat{\gamma}_\alpha \equiv \hat{S}_\alpha = \int d^3x\,\sum_n \hat{s}_{n\alpha}(\mathbf{x})$ is the operator of the total spin. Taking into account the weak anisotropy, we see that the total spin $\hat{S}_\alpha$ is only the approximate integral of motion. Equations of motion for the densities $\hat{\zeta}_a(\mathbf{x})$ have the form

$$\dot{\hat{\zeta}}_a(\mathbf{x}) = i\,[H, \hat{\zeta}_a(\mathbf{x})] , \quad (1.5)$$

and with allowance for the operator identity

$$i\,[\hat{A},\,\hat{b}(\mathbf{x})] = -\,i\,[\hat{B},\,\hat{a}(\mathbf{x})] - \nabla_k\,\hat{b}_k(\mathbf{x}) \qquad (1.6)$$

for arbitrary quasilocal operators $\hat{a}(\mathbf{x})$, $\hat{b}(\mathbf{x})$, where

$$\hat{A} = \int d^3x\,\hat{a}(\mathbf{x})\ ,\quad \hat{B} = \int d^3x\,\hat{b}(\mathbf{x})\ ,$$

$$\hat{b}_k(\mathbf{x}) = i\int d^3x'\,x_k'\int_0^1 d\lambda\,[\hat{a}(\mathbf{x} - (1-\lambda)\mathbf{x'}),\,\hat{b}(\mathbf{x} + \lambda\mathbf{x'})]$$

can be represented in the form

$$\dot{\hat{\varepsilon}}(\mathbf{x}) = -\,\nabla_k\,\hat{q}_k(\mathbf{x})\ , \qquad (1.7)$$

$$\dot{\hat{s}}_\alpha(\mathbf{x}) = -\,i\,[\hat{S}_\alpha\,,\,\hat{\varepsilon}(\mathbf{x})] - \nabla_k\,\hat{j}_{\alpha k}(\mathbf{x})\ .$$

Here $\{\hat{q}_k(\mathbf{x}),\,\hat{j}_{\alpha k}(\mathbf{x})\} \equiv \hat{\zeta}_{ak}(\mathbf{x})$ are the flux density operators of energy and momentum, for which, in accordance with (1.6), we have

$$\hat{q}_k(\mathbf{x}) = \frac{i}{2}\int d^3x'\,x_k'\int_0^1 d\lambda\,[\hat{\varepsilon}(\mathbf{x} - (1-\lambda)\mathbf{x'}),\,\hat{\varepsilon}(\mathbf{x} + \lambda\mathbf{x'})], \qquad (1.8)$$

$$\hat{j}_{\alpha k}(\mathbf{x}) = i\int d^3x'\,x_k'\int_0^1 d\lambda\,[\hat{\varepsilon}(\mathbf{x} - (1-\lambda)\mathbf{x'}),\,\hat{s}_\alpha(\mathbf{x} + \lambda\mathbf{x'})].$$

In the case of the isotropic magnetic systems ($[\hat{S}_\alpha\,,\,\hat{\varepsilon}(\mathbf{x})] = 0$) equations of motion (1.7) have the form of differential conservation laws.

In macroscopic description of magnets the notion of the order parameter of the investigated system has an important role. We shall consider in what follows the magnet with total symmetry breaking relative to spin rotations. It is characterized by the three rotation angles φ_α, which realize a parametrization of the three-dimensional rotation group in spin space, or by the real rotation matrix associated with them, $a_{\alpha\beta}(\varphi)$ ($a\tilde{a} = 1$). Moreover, we shall study the two sublattice ferrimagnet with noncompensated sublattices characterized by the unit vector of antiferromagnetism l_α (or by the two rotation angles). In the case of total symmetry violation the order parameter is the complex spin vector $\Delta_\alpha(\mathbf{x}) =$ $= \mathrm{Tr}\,\rho\hat{\Delta}_\alpha(\mathbf{x}) = \Delta_{1\alpha}(\mathbf{x}) + i\Delta_{2\alpha}(\mathbf{x})$, where $\hat{\Delta}_\alpha(\mathbf{x}) =$

$= \hat{\Delta}_{1\alpha}(\mathbf{x}) + i\hat{\Delta}_{2\alpha}(\mathbf{x})$ is the order parameter operator ($\Delta_1^+ = \Delta_1$, $\Delta_2^+ = \Delta_2$). For the two sublattice ferrimagnet the order parameter is the real vector $\Delta_\alpha(\mathbf{x}) = \mathrm{Tr}\,\rho\hat{\Delta}_\alpha(\mathbf{x})$, where $\hat{\Delta}_\alpha(\mathbf{x})$ is the order parameter operator of the ferrimagnetic system ($\hat{\Delta}^+ = \hat{\Delta}$). The order parameter operator in each cases satisfies the symmetry properties

$$i\,[\hat{s}_\alpha(\mathbf{x}),\,\hat{\Delta}_\beta(\mathbf{x'})] =$$

$$= -\,\varepsilon_{\alpha\beta\gamma}\,\hat{\Delta}_\gamma(\mathbf{x})\,\delta(\mathbf{x} - \mathbf{x'}),\ \ i\,[P_k\,,\,\hat{\Delta}_\alpha(\mathbf{x})] = -\nabla_k\,\hat{\Delta}_\alpha(\mathbf{x}). \qquad (1.9)$$

Note that the order parameter operator is expressed in terms of the spin operators of the sublattices and it is usually chosen in such a way that the order parameter for the normal state is equal to zero. By virtue of relations (1.1) and (1.9) the operators $\hat{s}(\mathbf{x})$ and $\hat{\Delta}(\mathbf{x})$ are transformed under the local spin rotations as vectors

$$U_a^+\hat{s}_\alpha(\mathbf{x})\,U_a = a_{\alpha\beta}(\mathbf{x})\,\hat{s}_\beta(\mathbf{x})\ ,$$

$$U_a^+\hat{\Delta}_\alpha(\mathbf{x})\,U_a = a_{\alpha\beta}(\mathbf{x})\,\hat{\Delta}_\beta(\mathbf{x})\ , \qquad (1.10)$$

$$U_a = \exp\left(-i\int d^3x\,\varphi_\lambda(\mathbf{x})\hat{s}_\lambda(\mathbf{x})\right)$$

[$\varphi_\alpha(\mathbf{x})$ are the local rotation angles]. In accordance with (1.10), the orthogonal rotation matrix $a(\varphi)$ has the form

$$a_{\alpha\beta}(\varphi) = (\exp(-\varepsilon\varphi))_{\alpha\beta} =$$

$$= \delta_{\alpha\beta}\cos\varphi + n_\alpha\,n_\beta\,(1 - \cos\varphi) + \varepsilon_{\alpha\lambda\beta}\,n_\lambda\,\sin\varphi\ , \qquad (1.11)$$

where

$$\varphi_\alpha = n_\alpha\,\varphi\ ,\quad n_\alpha^2 = 1\ ,\quad (\varepsilon\varphi)_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma}\,\varphi_\gamma\ .$$

The infinitesimal characteristics of the unitary transformation U_a is the operator $U_a^+\,\delta U_a$, where δU_a is variation of the unitary transformation U_a due to variation δa of the orthogonal rotation matrix ($\delta a\cdot\tilde{a} = -a\cdot\delta\tilde{a}$). In accordance with (1.10),

$$U_a^+\delta U_a = i\int d^3x\,\delta R_\gamma(\mathbf{x})\hat{s}_\gamma(\mathbf{x})\ ,\quad \delta R_\gamma = \frac{1}{2}\,\varepsilon_{\alpha\beta\gamma}\,(\tilde{a}\delta a)_{\alpha\beta}. \qquad (1.12)$$

[The quantity δR_γ in (1.12) is not variation of some vector R_γ]. Note that in what follows it will be convenient for us to use the formalism of left and right Cartan forms [13,18]

$$\omega_{\alpha k} \equiv \frac{1}{2} \varepsilon_{\alpha\beta\gamma} (\tilde{a}\nabla_k \, a)_{\gamma\beta} \, ,$$

$$\underline{\omega}_{\alpha k} \equiv \frac{1}{2} \varepsilon_{\alpha\beta\gamma} (a\nabla_k \, \tilde{a})_{\beta\gamma} = a_{\alpha\beta} \, \omega_{\beta k} \, . \qquad (1.13)$$

We define the translationally invariant states of the magnet by the relation $\hat{\rho}(s(\mathbf{x}' + \mathbf{y})) = \hat{\rho}(s(\mathbf{x}'))$ or by virtue of (1.2)

$$[P_k \, , \hat{\rho}] = 0 \, . \qquad (1.14)$$

The statistical operators describing the equilibrium state of collinear magnet satisfy this relation. For spiral magnetic ordering the transformation of translations by the vector $\mathbf{x}$ and spin rotation around some axis n_α $(n_\alpha^2 = 1)$ by the angle $\mathbf{px}$ do not change the state of the system:

$$\exp\left[i\mathbf{x}(\hat{\mathbf{P}}-\mathbf{p}(n_\alpha \, \hat{S}_\alpha))\right] \hat{\rho} \exp\left[-i\mathbf{x}(\hat{\mathbf{P}}-\mathbf{p}(n_\alpha \, \hat{S}_\alpha))\right] = \hat{\rho}$$

or

$$[\hat{\rho}, \, P_k - p_k(n_\alpha \, \hat{S}_\alpha)] = 0 \, . \qquad (1.15)$$

The vector p_k is called the vector of the magnetic spiral.

2. Weakly anisotropic locally equilibrium states

In the framework of thermodynamics and hydrodynamics of condensed media in the microscopic approach the concept of quasiaverages [10] plays an important role. In accordance with this concept, the equilibrium average of an arbitrary quasilocal operator $\hat{c}(\mathbf{x})$ of a magnetic system with spontaneously broken symmetry is defined by the formula

$$\langle \hat{c}(\mathbf{x}) \rangle = \lim_{\nu \to 0} \lim_{V \to \infty} \text{Tr} \, \omega_\nu \, \hat{c}(\mathbf{x}) \equiv \text{Tr} \, w\hat{c}(\mathbf{x}) \, , \quad (2.1)$$

$$w_\nu = \exp\{\Omega_\nu - Y_a \, \hat{\gamma}_a - \nu\hat{G}\} \, .$$

Here $Y_a \equiv (Y_0 \, , Y_\alpha)$ are the thermodynamic forces conjugate to the additive motion integrals $\hat{\gamma}_a$ $(Y_0^{-1} \equiv T$ is the temperature, and $-Y_\alpha Y_0^{-1} \equiv h_\alpha$ is the effective field). The source $\hat{G}$ in (2.1) lifts the degeneracy of the statistical equilibrium state and represents itself as a linear functional of the order parameter operator

$$\hat{G} = \int d^3x \, \mathbf{g}(\mathbf{x}, t)\hat{\Delta}(\mathbf{x}) + \text{h.c.} = \hat{G}(t) \, . \qquad (2.2)$$

The specific form of the function $g(\mathbf{x}, t)$ (which can depend on time) is defined by the symmetry properties of the equilibrium state.

In the case of anisotropic magnetic systems the total spin $\hat{S}_\alpha$ stops to be the integral of motion so that in Eq. (2.1) one should set $Y_\alpha = 0$. Allowance for anisotropy in the Hamiltonian H lifts the degeneracy relative to the uniform spin rotations. Therefore, the summand with the source in Eq. (2.1) should be put down. Nevertheless, in the investigation of weakly anisotropic magnetic systems we shall use the statistical operator of the form (2.1), which is a «main» approximation of the weak anisotropy. Here the next elucidations can be done. Since we consider the weak anisotropy, the total spin $\hat{S}_\alpha$ can be assumed to be the approximate motion integral and we include the corresponding summand with spin in the exponent of the Gibbs distribution (2.1).

The source $\hat{G}$ is introduced in Eq. (2.1) for the following purpose. If we take into account the weak anisotropy in the framework of perturbation theory, then as zero approximation we will have the statistical operator which corresponds to the magnetically ordered state with broken symmetry relative to the spin rotations. The source $\hat{G}$ in distribution (2.1) plays a role of the parameter that lifts the degeneracy.

This can be illustrated by using the following example. If we consider a ferromagnet with exchange interaction in the magnetic field, then the term describing the interaction with external field will play the role of the anisotropy and will fix the direction of the magnetic moment in space. When the field goes to zero (weak anisotropy), the source in the Gibbs distribution plays the role of the infinitesimally small anisotropy, which removes the degeneracy and fixes the direction of the moment. Thus, the statistical operator (2.1) describes the weakly anisotropic quasiequilibrium states of the magnetic system in the main approximation of the anisotropy and we shall use it in what follows for generalization of the case of locally equilibrium states.

In the given section we shall study the magnet with total symmetry breaking relative to spin rotations, for which the order parameter has the structure $\hat{\Delta}(\mathbf{x}) = \hat{\Delta}_1(\mathbf{x}) + i\hat{\Delta}_2(\mathbf{x})$ and $\hat{\Delta}_1^+ = \hat{\Delta}_1$, $\hat{\Delta}_2^+ = \hat{\Delta}_2$. We formulate the symmetry properties of the equi-

librium state. For the spiral magnetic ordering in accordance with Eq. (1.15) we have

$$[w, P_k - p_k(n_\alpha \hat{S}_\alpha)] = 0 ,$$

$$[w, Y_0 H + Y_\alpha \hat{S}_\alpha] = 0 . \qquad (2.3)$$

From the Jacobi identity follows the condition of compatibility of Eqs. (2.3)

$$\varepsilon_{\alpha\beta\gamma} n_\alpha Y_\beta [w, \hat{S}_\gamma] = 0 .$$

We thus obtain $n_\alpha = Y_\alpha / |\mathbf{Y}|$. The relations (2.3) allow us to find the function $\mathbf{g}(\mathbf{x}, t)$

$$g(\mathbf{x}, t) = \xi a(\varphi^0) a(\varphi(\mathbf{x}, t)) \equiv \xi a(\mathbf{x}, t) ,$$

$$\varphi_\alpha(\mathbf{x}, t) = n_\alpha(\mathbf{px} - ht) , \qquad (2.4)$$

where φ_α^0 is a uniform rotation; $h_\alpha = hn_\alpha$; and ξ_α is a constant complex vector which we can choose for convenience in the form $\xi = \xi_1 + i\xi_2$, $\xi_1^2 = \xi_2^2 = 1$, $\xi_1 \xi_2 = 0$. From here one can see that the statistical equilibrium state is characterized by the thermodynamic forces Y_a , the spiral vector p_k , and the rotation angles φ_α^0 . The vector ξ_α fixes the reference frame for the rotation angles and is not a thermodynamic parameter. In accordance with definitions (1.13), the left and right forms corresponding to the rotation matrix $a(\mathbf{x}, t)$ in (2.4) are

$$\omega_{\alpha k} = p_k n_\alpha , \quad \underline{\omega}_{\alpha k} = p_k \underline{n}_\alpha , \qquad (2.5)$$

where $\underline{n} = a(\varphi^0)n$.

Note that in the case of collinear magnets introduction of a source in the statistical operator (2.1) in the presence of thermodynamic forces Y_α has no meaning since the term $Y_\alpha \hat{S}_\alpha$ lifts the degeneracy of the statistical equilibrium state.

Let us examine the locally equilibrium states. It is well known [17] that the statistical operator

$$w(Y(\mathbf{x}')) = \exp \{\Omega_\nu - \int d^3x' \, Y_a(\mathbf{x}')\hat{\zeta}_a(\mathbf{x}')\} \quad (2.6)$$

generalizes the Gibbs statistical operator for the normal systems in the case of locally equilibrium states. For the systems with spontaneously broken symmetry the locally equilibrium states are described by the statistical operator

$$w_\nu(Y(\mathbf{x}'), \phi(\mathbf{x}')) =$$

$$= \exp \{\Omega_\nu - \int d^3x' \, Y_a(\mathbf{x}')\hat{\zeta}_a(\mathbf{x}') - \nu U_\phi^+\hat{G} U_\phi\} \equiv$$

$$\equiv U_\phi^+ \underline{w}_\nu U_\phi , \qquad (2.7)$$

where

$$\underline{w}_\nu = \exp \{\Omega_\nu - \int d^3x \, Y_a(\mathbf{x})\hat{\underline{\zeta}}_a(\mathbf{x}) - \nu\hat{G}\} ,$$

$$\hat{\underline{\zeta}}_a(\mathbf{x}) = U_\phi \, \hat{\zeta}_a(\mathbf{x}) U_\phi^+$$

and U_ϕ is the local unitary operator which corresponds to the broken symmetry [see, for example, Eq. (1.10) for symmetry breaking relative to spin rotations], and ϕ is the order parameter phase that enters into the source $\hat{G}$. Introducing the statistical operator (2.7) for the locally equilibrium states based on the calculation of the averages with the statistical operator $w_\nu(Y(\mathbf{x}'), \phi(\mathbf{x}'))$, we can go over to the averages with the statistical operator $\underline{w}_\nu$:

$$\text{Tr } w\hat{a}(\mathbf{x}) = \text{Tr } \underline{w}U_\phi \, \hat{a}(\mathbf{x})U_\phi^+ ,$$

where the operator $U_\phi \, \hat{a}(\mathbf{x})U_\phi^+ \equiv \hat{b}(\mathbf{x})$, as a rule, can be easily found. This operator is of the type $\hat{a}(\mathbf{x})$. In the statistical operator $\underline{w}_\nu$ the source $\hat{G}$ is space uniform and for calculation of the averages $\text{Tr } \underline{w}\hat{b}(\mathbf{x})$ the standard perturbation theory [on gradients of the parameters $Y_a(\mathbf{x})$, $\phi(\mathbf{x})$], which leads to the ordinary quasiaverages, can be used.

Thus, in accordance with (2.1) and (2.7), the statistical operator of the weakly anisotropic, locally equilibrium states of the magnet with total symmetry breaking relative to spin rotations is written in the form

$$w_\nu(Y(\mathbf{x}'), a(\mathbf{x}')) =$$

$$= \exp \{\Omega_\nu - \int d^3x' \, [Y_0(\mathbf{x}')\hat{\varepsilon}(\mathbf{x}') + Y_\alpha(\mathbf{x}')\hat{s}_\alpha(\mathbf{x}')] - \nu\hat{G}_a\} \equiv$$

$$\equiv U_a^+ \underline{w}_\nu U_a , \qquad (2.8)$$

$$\hat{G}_a \equiv U_a^+\hat{G} U_a = \int d^3x' \, (\xi a(x')\hat{\Delta}(x') + \text{h.c.}) ,$$

where

$$\underline{w}_\nu = \exp \{\Omega_\nu - \int d^3x \, [Y_0(x)\hat{\underline{\varepsilon}}(x) + \underline{Y}_\alpha(x)\hat{s}_\alpha(x)] - \nu\hat{G}\},$$

$$\hat{G} = \int d^3x\; \xi\Delta(\mathbf{x}) + \text{h.c.}\;,\qquad (2.9)$$

and $\hat{\underline{\varepsilon}} = U_a\,\hat{\varepsilon}U_a^+$, $\underline{Y} = aY$. Here the thermodynamic forces $Y_a(\mathbf{x})$ and the orthogonal rotation matrix $a_{\alpha\beta}(\mathbf{x})$ are the arbitrary functions of coordinates. In the equilibrium state [see (2.1)] $Y_a(\mathbf{x}) = Y_a$ and the structure of the orthogonal rotation matrix $a_{\alpha\beta}(\mathbf{x})$ is determined by Eq. (2.4). The matrix of rotation $a_{\alpha\beta}(\mathbf{x})$ for the locally equilibrium states is introduced in the distribution (2.8) by the transformation of local spin rotation performed on the source $\hat{G}$.

We obtain now the main thermodynamic identity for the locally equilibrium states and show that the locally equilibrium averages of the densities of additive motion integrals and fluxes corresponding to them can be expressed in the approximation of small inhomogeneities in terms of the locally equilibrium thermodynamic potential. In this connection, it is worthwhile to go over to the statistical operator $\underline{w}_\nu$ which is defined by Eq. (2.9). From (2.9) it follows that the locally equilibrium thermodynamic potential Ω is the functional of the thermodynamic forces $\underline{Y}_a \equiv (Y_0\,,\;\underline{Y}_\alpha)$ and the rotation matrix $a(\mathbf{x})$:

$$\Omega = \int d^3x\;\omega(\mathbf{x},\,\underline{Y}(\mathbf{x}'),\,a(\mathbf{x}'))\;.\qquad(2.10)$$

Here ω is the density of the thermodynamic potential. Varying the potential Ω with respect to the thermodynamic forces Y_0 and $\underline{Y}$, we obtain

$$\delta_{Y_0}\Omega = \int d^3x\;\delta Y_0(\mathbf{x})\,\text{Tr}\,\underline{w}\hat{\varepsilon}(\mathbf{x}) = \int d^3x\;\delta Y_0(\mathbf{x})\,\varepsilon(\mathbf{x})\;,$$

$$\delta_{\underline{Y}}\Omega = \int d^3x\;\delta\underline{Y}(\mathbf{x})\,\text{Tr}\,\underline{w}\hat{s}(\mathbf{x}) = \int d^3x\;\delta\underline{Y}(\mathbf{x})\underline{s}(\mathbf{x})\;,$$

where $\underline{s}(\mathbf{x}) = \text{Tr}\,\underline{w}\hat{s}(\mathbf{x}) = a(\mathbf{x})s(\mathbf{x})$. Varying the potential Ω with respect to the orthogonal matrix a, we have $\delta_a\Omega = \int d^3x\,Y_0(\mathbf{x})\,\text{Tr}\,\underline{w}\hat{\delta\varepsilon}(\mathbf{x})$. By virtue of the explicit form of the operator $\hat{\varepsilon}(\mathbf{x})$ and the relation (1.12) we thus find

$$\delta_a\Omega =$$

$$= i\int d^3x\;d^3x'\;\delta R_\alpha(\mathbf{x})Y_0(\mathbf{x}')\,\text{Tr}\,w\{Y,\,a\}\,[\hat{s}_\alpha(\mathbf{x}),\,\hat{\varepsilon}(\mathbf{x}')].$$
$$(2.11)$$

Under calculation of the trace in Eq. (2.11), by virtue of quasilocality of the operator $\hat{\varepsilon}(\mathbf{x}')$, the points $\mathbf{x}'$ placed near $\mathbf{x}$ give the main contribution. Therefore, expanding the quantity $Y_0(\mathbf{x}')$ near the point $\mathbf{x}$, $Y_0(\mathbf{x}') = Y_0(\mathbf{x}) + (\mathbf{x}' - \mathbf{x})_k\,\partial Y_0/\partial x_k + \ldots$, we obtain in the main approximation

$$\delta_a\Omega =$$

$$= i\int d^3x\;\delta R_\alpha(\mathbf{x})Y_0(\mathbf{x})\,\text{Tr}\,w[\hat{s}_\alpha(\mathbf{x}),\,H] + O(\nabla Y_0)\;.$$
$$(2.12)$$

We thus have

$$\left(\frac{\delta\Omega}{\delta a_{\rho\nu}(\mathbf{x})}\right)_{\underline{Y}} =$$

$$= \frac{i}{2}\,\varepsilon_{\alpha\mu\nu}\,a_{\rho\mu}(\mathbf{x})Y_0(\mathbf{x})\,\text{Tr}\,w[\hat{s}_\alpha(\mathbf{x}),\,H] + O(\nabla Y_0)\;.$$
$$(2.13)$$

Thus, the thermodynamic relationship for the locally equilibrium states takes the form

$$\delta\Omega =$$

$$= \int d^3x\left(\varepsilon(\mathbf{x})\delta Y_0(\mathbf{x}) + \underline{s}_\alpha(\mathbf{x})\delta\underline{Y}_\alpha(\mathbf{x}) + \left(\frac{\delta\Omega}{\delta a_{\alpha\beta}(\mathbf{x})}\right)_{\underline{Y}}\delta a_{\alpha\beta}(\mathbf{x})\right),$$
$$(2.14)$$

where the variational derivative $(\delta\Omega/\delta a_{\alpha\beta}(\mathbf{x}))_{\underline{Y}}$ is determined by Eq. (2.13). If instead of the variables $\underline{Y}_\alpha$ we use the variables $Y_\alpha = \underline{Y}_\beta\,a_{\beta\alpha}$, then relation (2.14) can be rewritten in the form

$$\delta\Omega = \int d^3x\left(\varepsilon(\mathbf{x})\delta Y_0(\mathbf{x}) + s_\alpha(\mathbf{x})\delta Y_\alpha(\mathbf{x}) + \int\right.$$
$$\left. + \left(\frac{\delta\Omega}{\delta a_{\alpha\beta}(\mathbf{x})}\right)_Y \delta a_{\alpha\beta}(\mathbf{x})\right),$$

where

$$\left(\frac{\delta\Omega}{\delta a_{\alpha\beta}}\right)_Y = \left(\frac{\delta\Omega}{\delta a_{\alpha\beta}}\right)_{\underline{Y}} + \left(\frac{\delta\Omega}{\delta\underline{Y}_\alpha}\right)_a Y_\beta\;.$$

We define the entropy density by the expression $\sigma = -\omega + Y_a \zeta_a$. We can then easily show that with an accuracy to ∇Y_0 the following equations are valid:

$$\frac{\delta H}{\delta \sigma} = \frac{1}{Y_0} \, , \quad \frac{\delta H}{\delta s_\alpha} = -\frac{Y_\alpha}{Y_0} \, , \quad \frac{\delta H}{\delta a_{\alpha\beta}} = \frac{1}{Y_0} \frac{\delta \Omega}{\delta a_{\alpha\beta}} \, .$$

$$(2.15)$$

Here $H = \int d^3x \, \varepsilon(\mathbf{x})$. If the density of the thermodynamic potential in the local limit allows expansion on gradients of the parameters $\underline{Y}$, a

$$\omega(\mathbf{x}; \underline{Y}(\mathbf{x}'), a(\mathbf{x}')) =$$

$$= \omega(\underline{Y}(\mathbf{x}), a(\mathbf{x}), \underline{\omega}_k(\mathbf{x})) + O(\nabla \underline{Y}, \nabla \underline{\omega}_k) \, , \quad (2.16)$$

then Eq. (2.14) can be simplified. In the operator identity (1.6) setting $a(\mathbf{x}) = \hat{\varepsilon}(\mathbf{x})$, $\hat{b}(\mathbf{x}) = \hat{s}_\alpha(\mathbf{x})$ and taking into account that for the anisotropic magnetic systems $[\hat{S}_\alpha , \hat{\varepsilon}(\mathbf{x})] \neq 0$, from (2.12) we obtain

$$\delta_a \Omega =$$

$$= \int d^3x \, \delta R_\alpha(\mathbf{x}) Y_0(\mathbf{x}) \{ i \, \mathrm{Tr} \, w[\hat{S}_\alpha , \hat{\varepsilon}(\mathbf{x})] + \nabla_k j_{\alpha k} \} \, ,$$

$$j_{\alpha k} \equiv \mathrm{Tr} \, w \hat{j}_{\alpha k} \, ,$$

where the operator of the spin flux density $\hat{j}_{\alpha k}$ is defined by Eq. (1.8). Since $\nabla_k \delta R_\alpha = -\delta \underline{\omega}_{\beta k} \, a_{\beta\alpha}$, the last relation can be rewritten in the form

$$\delta_a \Omega = i \int d^3x \, \delta R_\alpha(\mathbf{x}) Y_0(\mathbf{x}) \, \mathrm{Tr} \, w[\hat{S}_\alpha , \hat{\varepsilon}(\mathbf{x})] +$$

$$+ \int d^3x \, Y_0(\mathbf{x}) \underline{j}_{\alpha k}(\mathbf{x}) \delta \underline{\omega}_{\alpha k}(\mathbf{x}) \, , \quad \underline{j}_k = a j_k \, . \quad (2.17)$$

From it we find

$$\frac{\partial \omega}{\partial \underline{\omega}_{\alpha k}} = Y_0 \, \underline{j}_{\alpha k} \, . \quad (2.18)$$

In addition,

$$\frac{\partial \omega}{\partial a_{\mu\gamma}} = \frac{i Y_0}{2} \varepsilon_{\alpha\beta\gamma} \, a_{\mu\beta} \, \mathrm{Tr} \, w[\hat{S}_\alpha , \hat{\varepsilon}(\mathbf{x})] \, . \quad (2.19)$$

Therefore, the main thermodynamic identity can be represented in the form

$$d\omega = \varepsilon \, dY_0 + \underline{s}_\alpha \, d\underline{Y}_\alpha + \frac{\partial \omega}{\partial a_{\alpha\beta}} \, da_{\alpha\beta} + Y_0 \, \underline{j}_{\alpha k} \, d\underline{\omega}_{\alpha k} \, ,$$

$$(2.20)$$

or, taking into account the definition of the entropy density σ, we can write

$$d\varepsilon = T \, d\sigma + \underline{h}_\alpha \, ds_\alpha + \frac{\partial \varepsilon}{\partial a_{\alpha\beta}} \, da_{\alpha\beta} + \underline{j}_{\alpha k} \, d\underline{\omega}_{\alpha k} \, .$$

$$(2.21)$$

To find the energy flux density in the locally equilibrium state we use the relation

$$i \, \mathrm{Tr} \, w[\hat{A} + \hat{B}, \, \hat{a}(\mathbf{x}) + \hat{b}(\mathbf{x})] = 0 \, ,$$

where $\hat{a}(\mathbf{x}) = Y_0(\mathbf{x}) \hat{\varepsilon}(\mathbf{x})$ and $\hat{b}(\mathbf{x}) = Y_\alpha(\mathbf{x}) \hat{s}_\alpha(\mathbf{x})$. Taking into account (1.6), we obtain

$$\nabla_k \, Q_k(\mathbf{x}) = 0 \, ,$$

$$Q_k(\mathbf{x}) = -\frac{i}{2} \int d^3x' \, x_k' \times$$

$$\times \int_0^1 d\lambda \, Y_a(\mathbf{x} - (1 - \lambda)\mathbf{x}') \, \mathrm{Tr} \, w[\hat{\zeta}_a(\mathbf{x} - (1 - \lambda)\mathbf{x}') \, ,$$

$$\hat{\zeta}_b(\mathbf{x} + \lambda\mathbf{x}')] Y_b(\mathbf{x} + \lambda\mathbf{x}') \, . \quad (2.22)$$

Ignoring in this expression the gradients ∇Y_0 and ∇Y_α, using (1.8), we find

$$Q_k = Y_0^2 q_k + Y_0 \underline{Y}_\alpha \, \underline{j}_{\alpha k} + O(\nabla \underline{Y}) \, .$$

Because of the arbitrariness of the gradients of the parameters $\underline{Y}$ and $\underline{\omega}_k$, we have $Q_k = 0$. We thus obtain

$$q_k = \underline{h}_\alpha \, \underline{j}_{\alpha k} \, ,$$

and, therefore, in accordance with (2.21),

$$\underline{\zeta}_{a k} = \frac{\partial \varepsilon}{\partial \underline{\omega}_{\alpha k}} \frac{\partial \underline{\zeta}_a}{\partial s_\alpha} \, , \quad a = (0, \alpha) \, . \quad (2.23)$$

3. Reduced description. Hydrodynamic equations

The reduced description method is used in describing the nonequilibrium states of macroscopic systems at the hydrodynamic stage of evolution (small inhomogeneities) [17]. For the weakly anisotropic magnetic systems with total symmetry

breaking relative to spin rotations the reduced description parameters are the densities of the additive motion integrals $\zeta_a(\mathbf{x})$ (with respect to exchange Hamiltonian) and the rotation matrix $b_{\alpha\beta}(\mathbf{x})$ in spin space. To formulate the hydrodynamic equations, it is necessary to introduce the rotation matrix $b_{\alpha\beta}(\mathbf{x})$ as a functional of the nonequilibrium statistical operator $\hat{\rho}$, $b(\mathbf{x}) = b(\mathbf{x}, \hat{\rho})$. This matrix, which characterizes the orientation of the mean value of the order parameter operator $\Delta_\alpha(\mathbf{x}, \hat{\rho}) = \text{Tr } \hat{\rho}\hat{\Delta}_\alpha(\mathbf{x})$ relative to some fixed frame $\mathbf{l}$, $\mathbf{m}$, $\mathbf{l} \times \mathbf{m}$ ($\mathbf{l}^2 = \mathbf{m}^2 = 1$, $\mathbf{lm} = 0$), does not coincide, in general, with the rotation matrix $a_{\alpha\beta}(\mathbf{x})$ that enters into the locally equilibrium Gibbs distribution and into the thermodynamic potential ω. We define the rotation matrix $b_{\alpha\beta}(\mathbf{x}, \hat{\rho})$ as a functional of the nonequilibrium statistical operator $\hat{\rho}$ by the relations [19]

$$\mathbf{l}b(\mathbf{x}, \hat{\rho}) \, \Delta(\mathbf{x}, \hat{\rho}) = 0 \ , \quad \mathbf{m}b(\mathbf{x}, \hat{\rho}) \, \Delta_2(\mathbf{x}, \hat{\rho}) = 0 \ .$$

$$(3.1)$$

By virtue of (3.1) and (1.10) for the rotation matrix $b(\mathbf{x}, \hat{\rho})$ the following equation is valid:

$$b(\mathbf{x}, \, U_c^+\rho U_c) = b(\mathbf{x}, \hat{\rho})c(\mathbf{x}) \ , \qquad (3.2)$$

where $c(\mathbf{x})$ is the arbitrary matrix of local rotation. We take into account the variation of the orthogonal matrix of rotation $\delta b(\mathbf{x}, \hat{\rho})$, which is associated with the variation of the statistical operator $\delta\hat{\rho}$:

$$\delta b(\mathbf{x}, \hat{\rho}) = b(\mathbf{x}, \hat{\rho} + \delta\hat{\rho}) - b(\mathbf{x}, \hat{\rho}) =$$
$$= b(\mathbf{x}, \hat{\rho}) \, \chi(\mathbf{x}, \hat{\rho}, \delta\hat{\rho}) \ . \qquad (3.3)$$

Here the matrix $\chi(\mathbf{x}, \hat{\rho}, \delta\hat{\rho})$ is a linear functional of $\delta\hat{\rho}$ that can be represented in the form

$$\chi_{\alpha\beta}(\mathbf{x}, \hat{\rho}, \delta\hat{\rho}) = \text{Tr } \delta\hat{\rho}\hat{\chi}_{\alpha\beta}(\mathbf{x}, \hat{\rho}) \ .$$

The operator $\hat{\chi}_{\alpha\beta}(\mathbf{x}, \hat{\rho})$ which depends on the initial statistical operator $\hat{\rho}$ obeys, by virtue of the orthogonality condition $b\tilde{b} = 1$, the antisymmetry property $\hat{\chi}_{\alpha\beta}(\mathbf{x}, \hat{\rho}) = -\hat{\chi}_{\beta\alpha}(\mathbf{x}, \hat{\rho})$. Defining the dual quantity $\hat{\chi}_\gamma = \frac{1}{2} \, \varepsilon_{\alpha\beta\gamma} \, \hat{\chi}_{\alpha\beta}$, we represent the variation of the rotation matrix $\delta b(\mathbf{x}, \hat{\rho})$ in the form

$$\delta b_{\alpha\beta}(\mathbf{x}, \hat{\rho}) = b_{\alpha\gamma}(\mathbf{x}, \hat{\rho}) \, \varepsilon_{\gamma\beta\lambda} \, \text{Tr } \delta\hat{\rho}\hat{\chi}_\lambda(\mathbf{x}, \hat{\rho}) \ . \quad (3.4)$$

We attribute the following properties to the operator $\hat{\chi}_\alpha(\mathbf{x}, \hat{\rho})$, whose proof one can find in Ref. 19:

1. The operator $\hat{\chi}_\alpha$ is determined to an accuracy of the transformation $\hat{\chi} \to \hat{\chi}' = \hat{\chi} + c(\hat{\rho})$, where $c(\hat{\rho})$ is an arbitrary c-number functional of the statistical operator $\hat{\rho}$, and is uniquely fixed by the condition $\text{Tr } \hat{\rho}\hat{\chi} \, (\mathbf{x}, \hat{\rho}) = 0$.

2. For the operator $\hat{\chi}_\alpha$ the following equations are valid:

$$i \, \text{Tr } \rho[\hat{s}_\alpha(\mathbf{x}), \, \hat{\chi}_\beta(\mathbf{x}', \hat{\rho})] = \delta_{\alpha\beta} \, \delta(\mathbf{x} - \mathbf{x}') \ ,$$
$$i \, \text{Tr } \rho[P_k \, , \, \hat{\chi}_\alpha(\mathbf{x}, \hat{\rho})] = \omega_{\alpha k}(\mathbf{x}, \hat{\rho}) \ .$$

$$(3.5)$$

The transformation laws relative to spin rotations and spatial translations for the operator $\hat{\chi}_\alpha$ are

$$U_c \, \hat{\chi}_\alpha(\mathbf{x}, \, U_c^+\rho U_c)U_c^+ = \hat{\chi}_\lambda(\mathbf{x}, \hat{\rho})c_{\lambda\alpha}(\mathbf{x}) \ ,$$
$$e^{i\mathbf{Py}} \, \hat{\chi}_\alpha(\mathbf{x}, \, e^{-i\mathbf{Py}} \, \hat{\rho} \, e^{i\mathbf{Py}}) \, e^{-i\mathbf{Py}} = \hat{\chi}_\alpha(\mathbf{x} - \mathbf{y}, \hat{\rho}) \ ,$$

where $c(\mathbf{x})$ is the arbitrary matrix of local rotation.

We formulate now the equation of motion for the orthogonal rotation matrix. Accordingly we choose the variation $\delta\rho$ in the form $\delta\rho = \dot{\rho}\delta t$ and assume that the statistical operator $\hat{\rho}(t)$ satisfies the Liouville equation

$$(3.6)$$
$$\dot{\hat{\rho}} \, (t) = i[\hat{\rho}(t), \, H] \ .$$

As a result, by virtue of (3.4), we obtain the equation

$$\dot{b}_{\alpha\beta}(\mathbf{x}, \hat{\rho}) = ib_{\alpha\lambda}(\mathbf{x}, \hat{\rho}) \, \varepsilon_{\lambda\beta\gamma} \, \text{Tr } \hat{\rho}[H, \, \hat{\chi}_\gamma(\mathbf{x}, \hat{\rho})] \ .$$

$$(3.7)$$

We consider the evolution of a nonequilibrium, spatially inhomogeneous state of the magnet with total symmetry breaking in the range $t \gg \tau_0$ (τ_0 is the relaxation time) at the hydrodynamic stage of evolution. In accordance with the reduced description hypothesis, at these times the nonequilibrium statistical operator is a functional of the reduced description parameters

$$\rho(t) \xrightarrow[t \gg \tau_0]{} \rho(\zeta(\mathbf{x}, t), \, b(\mathbf{x}, t)) \ ,$$

$$(3.8)$$

$$\zeta(\mathbf{x}) = \text{Tr } \hat{\rho}(\zeta, \, b) \, \hat{\zeta}(\mathbf{x}) \ , \quad b(\mathbf{x}) = b(\mathbf{x}, \hat{\rho}(\zeta, \, b)) \ .$$

In these relations the orthogonal rotation matrix $b(\mathbf{x}, \hat{\rho})$ as a functional of the nonequilibrium statistical operator is defined by Eqs. (3.1). In accord-

ance with (1.7) and (3.7) and by virtue of the reduced description hypothesis (3.8), the equations of motion for the reduced description parameters, have the form

$$\dot{\varepsilon}(\mathbf{x}, \hat{\rho}) = -\nabla_k \operatorname{Tr} \hat{\rho}(\zeta, b) \, \hat{q}_k(\mathbf{x}) \; ,$$

$$\dot{s}_\alpha(\mathbf{x}, \hat{\rho}) = i \operatorname{Tr} \hat{\rho}(\zeta, b) \, [H, \hat{s}_\alpha(\mathbf{x})] \; , \qquad (3.9)$$

$$\dot{b}_{\alpha\beta}(\mathbf{x}, \hat{\rho}) = i b_{\alpha\lambda}(\mathbf{x}, \hat{\rho}) \, \varepsilon_{\lambda\beta\gamma} \operatorname{Tr} \hat{\rho}(\zeta, b) \, [H, \hat{\chi}_\gamma(\mathbf{x}, \hat{\rho})] \; .$$

We represent the statistical operator $\hat{\rho}(\zeta, b)$ in the form

$$\hat{\rho}(\zeta, b) = w(Y, a) + \hat{\rho}'(\zeta, b) \; . \qquad (3.10)$$

Here $w(Y, a)$ is the locally equilibrium statistical operator (2.8) and the operator $\hat{\rho}'(\zeta, b)$ determines the dissipative processes. Since we are interested in the main approximation of the spatial gradients, and since we disregard the dissipative processes, we can disregard the contribution of the operator $\hat{\rho}'(\zeta, b)$ in Eqs. (3.9). We can therefore assume that the relation $\operatorname{Tr} \hat{\rho}(\zeta, b)... \approx \operatorname{Tr} w(Y, a)...$ is approximately satisfied. The relation between the densities of the additive motion integrals and the thermodynamic forces is defined by the relation (2.14). We recall that the orthogonal rotation matrix $a(\mathbf{x})$, which enters into the locally equilibrium Gibbs distribution and the thermodynamic potential ω, does not coincide with the rotation matrix $b(\mathbf{x}, w)$.

To find the equation of motion for the spin density $s_\alpha(\mathbf{x})$, we make use of the expression (2.13) for the variational derivative $\delta\Omega/\delta a$ of the thermodynamic potential Ω with respect to the rotation matrix a. Comparing Eqs. (3.9) and (2.13), we represent the equation of motion for the spin density $s_\alpha(\mathbf{x})$ in the form

$$\dot{s}_\alpha = -\frac{1}{Y_0} \, \varepsilon_{\alpha\mu\nu} \, a_{\lambda\mu} \left(\frac{\delta\Omega}{\delta a_{\lambda\nu}} \right)_{\underline{Y}} \; . \qquad (3.11)$$

In Eq. (3.11) the thermodynamic potential $\Omega = \int d^3x \, \omega(\mathbf{x})$ is considered to be a functional of the variables $\underline{Y}_\alpha(\mathbf{x})$, $a_{\alpha\beta}(\mathbf{x})$ under the local dependence of the inverse temperature $Y_0(\mathbf{x})$: $\omega(\mathbf{x}) = \omega(\mathbf{x}; Y_0(\mathbf{x}), \underline{Y}_\alpha(\mathbf{x}'), a_{\alpha\beta}(\mathbf{x}'))$. If the thermodynamic potential is a functional of the form

$$\Omega = \int d^3x \, \omega(\mathbf{x}) \; ,$$

$$\omega(\mathbf{x}) = \omega(\mathbf{x}; Y_0(\mathbf{x}), Y_\alpha(\mathbf{x}'), a_{\alpha\beta}(\mathbf{x}')) \; , \; Y_\alpha = \underline{Y}_\beta \, a_{\beta\alpha} \; ,$$

then in terms of the new set of variables we have

$$\dot{s}_\alpha = -\varepsilon_{\alpha\beta\gamma} \left(\frac{Y_\beta}{Y_0} \frac{\delta\Omega}{\delta Y_\gamma} + \frac{1}{Y_0} a_{\mu\beta} \left(\frac{\delta\Omega}{\delta a_{\mu\gamma}} \right)_Y \right) . \qquad (3.12)$$

In the local limit, when the density of the thermodynamic potential ω is represented as

$$\omega(\mathbf{x}) = \omega(Y_0(\mathbf{x}), \underline{Y}_\alpha(\mathbf{x}), a(\mathbf{x}), \underline{\omega}_k(\mathbf{x})) \; ,$$

from (3.11) we obtain

$$\dot{s}_\alpha = -\frac{1}{Y_0} \, \varepsilon_{\alpha\beta\gamma} \, a_{\mu\beta} \, \frac{\partial\omega}{\partial a_{\mu\gamma}} - \nabla_k j_{\alpha k} \; , \qquad (3.13)$$

$$j_{\alpha k} = \frac{1}{Y_0} \frac{\partial\omega}{\partial \underline{\omega}_{\beta k}} \, a_{\beta\alpha} \; .$$

The first term on the right side of Eq.(3.13) takes into account the anisotropy.

We obtain the equation of motion for the orthogonal rotation matrix. Substituting expression (3.10) in (3.7) and ignoring the influence of the dissipative processes, we find

$$\dot{b}_{\alpha\beta}(\mathbf{x}, \hat{w}) = i b_{\alpha\gamma}(\mathbf{x}, \hat{w}) \, \varepsilon_{\gamma\beta\lambda} \operatorname{Tr} \hat{w}[H, \hat{\chi}_\lambda(\mathbf{x}, \hat{w})] \; .$$

Noting further that in the main approximation for the statistical operator $w(Y, a)$ the stationary condition (2.3) holds, we can rewrite the last equation in the form

$$\dot{b}_{\alpha\beta}(\mathbf{x}, \hat{w}) = i b_{\alpha\gamma}(\mathbf{x}, \hat{w}) \, \varepsilon_{\gamma\beta\lambda} \, h_\sigma \operatorname{Tr} \hat{w}[\hat{s}_\sigma, \hat{\chi}_\lambda(\mathbf{x}, \hat{w})] =$$
$$= b_{\alpha\gamma}(\mathbf{x}, \hat{w}) \, \varepsilon_{\gamma\beta\lambda} \, h_\lambda \; ,$$

where the relation (3.5) is taken into account. We note that the asymptotic relation (3.8) contains the rotation matrix $b(\mathbf{x}, \hat{\rho})$, which is defined by the relations (3.1). On the other hand, Eqs. (2.23) for the flux densities of the additive motion integrals contain the Cartan forms which are the functions of the rotation matrix $a(\mathbf{x})$ that enters into the locally equilibrium Gibbs distribution and, in general, is not identical to the matrix $b(\mathbf{x}, \hat{w})$. Therefore, to close the equations of motion we must establish a connection between these two orthogonal matrices. Using the relations (3.1), (3.2), and (2.8), we have

$$1 b(\mathbf{x}, w) \, \Delta(\mathbf{x}, w) = 1 b(\mathbf{x}, w) \, \tilde{a}(\mathbf{x})\Delta(\mathbf{x}, \underline{w}) = 0 \; ,$$

$$\mathbf{m}b(\mathbf{x}, w)\Delta_2(\mathbf{x}, w) = \mathbf{m}b(\mathbf{x}, w)\,\tilde{a}(\mathbf{x})\Delta_2(\mathbf{x}, \underline{w}) = 0 \ .$$

We see, therefore, that the rotation matrix $b(\mathbf{x}, w)\,\tilde{a}(\mathbf{x}) = b(\mathbf{x}, \underline{w})$ is a function of the arguments $\underline{Y}$, a, $\underline{\omega}_k$. The variables $\underline{Y}$ and $\underline{\omega}_k$ change slowly in space and the time and the dependence on the matrix a is weak because of the small anisotropy. Therefore, the equation of motion for the rotation matrix $a(\mathbf{x})$ in the main approximation with respect to the spatial gradients and small anisotropy can be represented in the form

$$\dot{a}_{\alpha\beta}(\mathbf{x}) \approx b_{\alpha\gamma}^{-1}(\mathbf{x}, \underline{w})\dot{b}_{\gamma\beta}(\mathbf{x}, w) = a_{\alpha\gamma}(\mathbf{x})\varepsilon_{\gamma\beta\lambda}\,h_\lambda \ .$$
(3.14)

Thus, we obtain a closed system of equations for the magnets considered by us, without regard for the dissipative processes:

$$\dot{s}_\alpha = -\frac{1}{Y_0}\,\varepsilon_{\alpha\beta\gamma}\left(Y_\beta\,\frac{\delta\Omega}{\delta Y_\gamma} + a_{\mu\beta}\,\frac{\delta\Omega}{\delta a_{\mu\gamma}}\right),$$
$$\dot{a}_{\alpha\beta} = -a_{\alpha\gamma}\,\varepsilon_{\gamma\beta\lambda}\,\frac{Y_\lambda}{Y_0} \ .$$
(3.15)

The equation for the energy density is

$$\dot{\varepsilon} = -\nabla_k\,\frac{1}{Y_0^2}\,\frac{\partial\omega}{\partial s_\alpha}\,\frac{\partial\omega}{\partial\omega_{\alpha k}} \ .$$
(3.16)

By virtue of (2.15), Eqs.(3.15) can be written in the form

$$\dot{s}_\alpha = \varepsilon_{\alpha\beta\gamma}\left(\frac{\delta H}{\delta s_\beta}\,s_\gamma + \frac{\delta H}{\delta a_{\mu\beta}}\,a_{\mu\gamma}\right), \quad \dot{a}_{\alpha\beta} = \varepsilon_{\beta\rho\gamma}\,a_{\alpha\gamma}\,\frac{\delta H}{\delta s_\rho} \ .$$
(3.17)

Since the energy density ε is a function of the quantities σ, a, $\underline{s}$, and $\underline{\omega}_k$ [see Eq. (2.21)] and since the quantities σ, $\underline{s}$, and $\underline{\omega}_k$ for the weakly inhomogeneous and weakly anisotropic states vary slowly in space and time, it is useful to change to the variables ε, a, $\underline{s}$, and $\underline{\omega}_k$ in Eqs. (3.15) and (3.16). By virtue of (1.13) and (2.15), we obtain from (3.15) and (3.16) a closed system of equations

$$\dot{\varepsilon} = -\nabla_k\,\frac{\partial\varepsilon}{\partial s_\alpha}\,\frac{\partial\varepsilon}{\partial\underline{\omega}_{\alpha k}} \ , \quad \dot{a}_{\alpha\beta} = \varepsilon_{\alpha\rho\nu}\,a_{\rho\beta}\,\frac{\partial\varepsilon}{\partial\underline{s}_\nu} \ ,$$

$$\dot{s}_\alpha = -\nabla_k\,\frac{\partial\varepsilon}{\partial\underline{\omega}_{\alpha k}} + \varepsilon_{\alpha\xi\mu}\left(\underline{s}_\xi\,\frac{\partial\varepsilon}{\partial\underline{s}_\mu} + \underline{\omega}_{\xi k}\,\frac{\partial\varepsilon}{\partial\underline{\omega}_{\mu k}} + a_{\xi\rho}\,\frac{\partial\varepsilon}{\partial a_{\mu\rho}}\right).$$
(3.18)

As a result of the equation of motion for the rotation matrix, we find the equation of motion for the Cartan form $\underline{\omega}_{\alpha k}$

$$\dot{\underline{\omega}}_{\alpha k} = -\nabla_k\,\frac{\partial\varepsilon}{\partial\underline{s}_\alpha} + \varepsilon_{\alpha\beta\gamma}\,\underline{\omega}_{\beta k}\,\frac{\partial\varepsilon}{\partial\underline{s}_\gamma} \ . \quad (3.19)$$

From these equations and from the thermodynamic relation (2.21) follows the adiabaticity of the processes in the approximation considered by us, $\dot{\sigma} = 0$.

4. Ferrimagnet

In this section we consider the thermodynamics and hydrodynamics of the weakly anisotropic, two-sublattice ferrimagnet with noncompensated sublattices. We see that there is an essential simplifying circumstance which allows to consider the two-sublattice ferrimagnet (for brevity simply «ferrimagnet») as a special case of the magnet with total symmetry breaking relatively to spin rotations with a special kind of dependence of the thermodynamic quantities on the rotation matrix.

For the ferrimagnet the source $\hat{G}$ in the statistical operator (2.1) is defined by the formula

$$\hat{G} = \int d^3x\,\mathbf{l}(\mathbf{x}, t)\,\hat{\Delta}(\mathbf{x}) = \hat{G}(t) \ , \quad \hat{\Delta}^+(\mathbf{x}) = \hat{\Delta}(\mathbf{x}) \ ,$$

$$|\mathbf{l}(\mathbf{x})| = 1 \ .$$
(4.1)

The real vector l_α ($l_\alpha^* = l_\alpha$) has a sense of the antiferromagnetism vector. Assuming that the equilibrium state is the spiral ordering state [see Eq. (2.3)], we find the form of the function $l_\alpha(\mathbf{x}, t)$

$$\mathbf{l}(\mathbf{x}, t) = \xi a(\varphi^0)a(\varphi(\mathbf{x}, t)) \equiv \xi a(\mathbf{x}, t) \ ,$$
$$\varphi_\alpha(\mathbf{x}, t) = n_\alpha(\mathbf{p}\mathbf{x} - ht) \ , \quad |\xi| = 1 \ ,$$
(4.2)

where ξ_α is a constant real unit vector, and φ_α^0 is a uniform rotation. Thus, the statistical equilibrium state of the ferrimagnet is characterized by the thermodynamic forces Y_0 and Y_α, by the rotation angles φ_α^0, and by the spiral vector p_k. The difference from the case of the total symmetry violation [see Eq. (2.4)] is that in Eq. (4.2) ξ_α is a constant real vector. Therefore, rotations around the vector ξ_α do not change the antiferromagnetism vector l_α and, hence, ferrimagnet is characterized by the two independent rotation angles φ_α^0.

In accordance with (2.1), (2.7), and (4.1), the locally equilibrium distribution of the ferrimagnet is defined by the formula

$$w_\nu(Y(\mathbf{x}'), l(\mathbf{x}')) =$$

$$= \exp\{\Omega_\nu - \int d^3x'\, Y_a(\mathbf{x}')\hat{\zeta}_a(\mathbf{x}') - \nu\hat{G}_l\} \equiv U_a^+ \underline{w}_\nu\, U_a\;,$$

(4.3)

$$\hat{G}_l = U_a^+ G U_a = \int d^3x'\, \xi a(\mathbf{x}')\hat{\Delta}(\mathbf{x}') \equiv \int d^3x'\, \mathbf{l}(\mathbf{x}')\hat{\Delta}(\mathbf{x}')\;,$$

where

$$\underline{w}_\nu = \exp\{\Omega_\nu - \int d^3x\, [\,Y_0(\mathbf{x})\hat{\underline{\varepsilon}}(\mathbf{x}) + \underline{Y}_\alpha(\mathbf{x})\hat{s}_\alpha(\mathbf{x})] - \nu\hat{G}\},$$

$$\hat{G} = \int d^3x\, \xi\hat{\Delta}(\mathbf{x})\;.$$

Here the thermodynamic forces $Y_a(\mathbf{x})$ and the vector $l_\alpha(\mathbf{x})$ are the arbitrary functions of coordinates. In the state of the total equilibrium $Y_a(\mathbf{x}) = Y_a$ and the structure of the antiferromagnetism vector l_α is defined by formula (4.2). From (4.3) and the normalization condition $\mathrm{Tr}\, w = 1$ it follows that the locally equilibrium thermodynamic potential Ω is a functional of the quantities $Y_a(\mathbf{x})$, $l_\alpha(\mathbf{x})$:

$$\Omega = \Omega(Y(\mathbf{x}'), l(\mathbf{x}')) = \int d^3x\, \omega(\mathbf{x};\, Y(\mathbf{x}'), l(\mathbf{x}'))\;.$$

(4.4)

Note the next important peculiarity. It follows from (4.3) that the antiferromagnetism vector l_α is related to the rotation matrix $a_{\alpha\beta}$ by the relation $l_\alpha = \xi_\beta\, a_{\beta\alpha}$. This allows us to consider the ferrimagnet as a particular case of the magnet with total symmetry breaking relative to spin rotations, for which the dependence of the thermodynamic potential Ω on the rotation matrix $a_{\alpha\beta}$ occurs only through the combination $\xi_\beta\, a_{\beta\alpha} \equiv l_\alpha$. Therefore, we can use the results obtained earlier with allowance for the indicated peculiarity without repeating the calculations of the second section.

We write the main thermodynamic identity for the locally equilibrium states. Variational derivatives of the thermodynamic potential $\Omega = \Omega(Y, l)$

with respect to the thermodynamic forces Y_a are defined by the equations

$$\left(\frac{\delta\Omega}{\delta Y_0}\right)_l = \varepsilon\;,\qquad \left(\frac{\delta\Omega}{\delta Y_\alpha}\right)_l = s_\alpha\;.$$

(4.5)

We find the variational derivative $(\delta\Omega/\delta l)_Y$. Since in the case of the ferrimagnet

$$\left(\frac{\delta\Omega}{\delta a_{\rho\nu}}\right)_Y = \xi_\rho\left(\frac{\delta\Omega}{\delta l_\nu}\right)_Y\;,$$

then scalary multiplying both parts of the last relation on the vector ξ_ρ ($\xi_\rho^2 = 1$), we have

$$\left(\frac{\delta\Omega}{\delta l_\nu}\right)_Y = \left(\frac{\delta\Omega}{\delta a_{\rho\nu}}\right)_Y \xi_\rho\;.$$

(4.6)

The derivative $(\delta\Omega/\delta a)_{\underline{Y}}$ is defined by formula (2.13), and the derivative $(\delta\Omega/\delta a)_Y$ is related to the derivative $(\delta\Omega/\delta a)_{\underline{Y}}$ by the relation

$$\left(\frac{\delta\Omega}{\delta a_{\alpha\beta}}\right)_Y = \left(\frac{\delta\Omega}{\delta a_{\alpha\beta}}\right)_{\underline{Y}} - \left(\frac{\delta\Omega}{\delta Y_\beta}\right)_a a_{\alpha\mu}\, Y_\mu\;.$$

(4.7)

Using (4.6), (4.7), and (2.13) for the derivative $(\delta\Omega/\delta l)_Y$, we obtain

$$\left(\frac{\delta\Omega}{\delta l_\alpha(\mathbf{x})}\right)_Y = \frac{i}{2}\,\varepsilon_{\alpha\gamma\mu}\, l_\mu(\mathbf{x})Y_0(\mathbf{x})\, \mathrm{Tr}\, w[\hat{s}_\gamma(\mathbf{x}), H] -$$

$$- \left(\frac{\delta\Omega}{\delta Y_\alpha(\mathbf{x})}\right)_l l_\mu(\mathbf{x})Y_\mu(\mathbf{x})\;.$$

(4.8)

Thus, the thermodynamic identity for the locally equilibrium states of the weakly anisotropic ferrimagnet has the form

$$\delta\Omega =$$

$$= \int d^3x\left(\varepsilon(\mathbf{x})\,\delta Y_0(\mathbf{x}) + s_\alpha(\mathbf{x})\delta Y_\alpha(\mathbf{x}) + \left(\frac{\delta\Omega}{\delta l_\alpha}\right)_Y \delta l_\alpha(\mathbf{x})\right).$$

(4.9)

Taking into account the expression for the entropy density $\sigma = -\omega + Y_a\,\zeta_a$, we see that to an accuracy of ∇Y_0 the following equations are valid:

$$\frac{\delta H}{\delta\sigma} = \frac{1}{Y_0}\;,\qquad \frac{\delta H}{\delta s_\alpha} = -\frac{Y_\alpha}{Y_0}\;,\qquad \frac{\delta H}{\delta l_\alpha} = \frac{1}{Y_0}\frac{\delta\Omega}{\delta l_\alpha}\;.\quad (4.10)$$

Let us consider the local limit of the relations which we obtained when the density of the thermodynamic potential ω depends on the variables Y, l, and $\nabla_k l$ (or, in the last case, on the quantity $v_{\alpha k} = - \varepsilon_{\alpha \beta \gamma} l_\beta \nabla_k l_\gamma$):

$$\omega(\mathbf{x}) = \omega(Y(\mathbf{x}), l(\mathbf{x}), v_k(\mathbf{x})) . \qquad (4.11)$$

The connection between the variables Y, l, and v_k, and the variables $\underline{Y}$, a, and $\underline{\omega}_k$ is given by

$$Y_\alpha = \underline{Y}_\beta \, a_{\beta \alpha} , \quad l_\alpha = \xi_\beta \, a_{\beta \alpha} ,$$
$$v_{\alpha k} = (\delta_{\rho v} - \xi_\rho \, \xi_v) \, a_{\rho \alpha} \, \underline{\omega}_{v k} . \qquad (4.12)$$

The quantity $\underline{j}_k$ was obtained in the second section of Eq. (2.18) for the spin flux density. Taking it into account and also Eqs. (4.11) and (4.12) for the spin flux density in the case of the ferrimagnet, we find

$$\underline{j}_{\alpha k} = \frac{1}{Y_0} \frac{\partial \omega}{\partial v_{\alpha k}} \, \delta^\perp_{\alpha \beta} \, \tilde{a}_{\beta \gamma} , \quad \delta^\perp_{\alpha \beta} \equiv \delta_{\alpha \beta} - l_\alpha \, l_\beta$$

or

$$j_{\alpha k} = \frac{1}{Y_0} \frac{\partial \omega}{\partial v_{\alpha k}} . \qquad (4.13)$$

Here we took into account that since variations of the quantities l_α and $v_{\alpha k}$ are not independent (see Ref. 9), $l_\alpha \, \partial \omega / \partial v_{\alpha k} = 0$. Besides, it is easy to obtain the relation that connects the derivatives $(\partial \omega / \partial l)_{Y,v}$ and $(\partial \omega / \partial a)_{\underline{Y}, \underline{\omega}}$:

$$\left(\frac{\partial \omega}{\partial l_v} \right)_{Y,v} = \xi_\mu \left(\frac{\partial \omega}{\partial a_{\mu v}} \right)_{\underline{Y}, \underline{\omega}} - \left(\frac{\partial \omega}{\partial Y_v} \right)_{l,v} l_\rho \, Y_\rho .$$

Using Eq. (2.19), we thus have

$$\left(\frac{\partial \omega}{\partial l_v} \right)_{Y,v} = i \, \frac{Y_0}{2} \, \varepsilon_{\alpha \beta v} \, l_\beta \, \mathrm{Tr} \, w [\hat{S}_\alpha , \hat{\varepsilon}(\mathbf{x})] -$$

$$- \left(\frac{\partial \omega}{\partial Y_v} \right)_{l,v} l_\rho \, Y_\rho . \qquad (4.14)$$

The second law of thermodynamics in the local limit can thus be written in the form

$$d\omega = \varepsilon d \, Y_0 + s d \, Y + \frac{\partial \omega}{\partial l} \, dl + Y_0 \, j_k \, dv_k , \qquad (4.15)$$

where the derivative $(\partial \omega / \partial l)_{Y,v}$ is defined by Eq.(4.14). Using the entropy density σ, we can rewrite the last relation as follows:

$$d\varepsilon = T \, d\sigma + h ds + \frac{\partial \varepsilon}{\partial l} \, dl + j_k \, dv_k . \qquad (4.16)$$

Here we have taken into account that

$$\left(\frac{\partial \varepsilon}{\partial l} \right)_{\sigma, s, v_k} = \frac{1}{Y_0} \left(\frac{\partial \omega}{\partial l} \right)_{Y_0, Y, v_k} .$$

To find the energy flux density in the locally equilibrium state we use, by analogy with the case of the total symmetry violation, the relation (2.22). Within accuracy of ∇Y_0 we then obtain

$$q_k = h_\alpha \, j_{\alpha k} . \qquad (4.17)$$

Therefore, in accordance with (4.16),

$$\zeta_{a k} = \frac{\partial \varepsilon}{\partial v_{\alpha k}} \frac{\partial \zeta_a}{\partial s_\alpha} .$$

Let us consider the hydrodynamic stage of evolution of a weakly anisotropic ferrimagnet at times $t \gg \tau_0$ (τ_0 is the relaxation time). To construct hydrodynamics we will use, by analogy with the case of a magnet with total symmetry breaking, the reduced description hypothesis. The reduced description parameters are the densities of the additive motion integrals $\zeta_a(\mathbf{x})$ and the antiferromagnetism vector $g_\alpha(\mathbf{x}, \hat{\rho})$ considered as a functional of the nonequilibrium statistical operator $\hat{\rho}^*$. In such a scheme the equation of motion for the antiferromagnetism vector is obtained on the basis of the operator $\hat{g}_\alpha(\mathbf{x}, \hat{\rho})$ of the antiferromagnetism vector. As in the case of the magnet with total symmetry breaking, we have used the operator $\hat{\chi}(\mathbf{x}, \hat{\rho})$ [see Eq. (3.4)] to derive the equation of motion for the rotation matrix. However, we can at once obtain the equation of motion for the vector l_α by using the connection between the antiferromagnetism vector l_α and the rotation matrix $a_{\alpha \beta}$, $l_\alpha = \xi_\beta \, a_{\beta \alpha}$, indicated above. After

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* As for the magnet with total symmetry breaking in the case of the antiferromagnet, it is necessary to differentiate between the antiferromagnetism vector l_α that enters into the Gibbs disctribution and the antiferromagnetism vector $g_\alpha(\mathbf{x}, \hat{\rho})$ considered as a functional of the nonequalibrium statistical operator $\hat{\rho}$.

convolution of both sides of Eq. (3.14) with the constant vector ξ_ρ we have

$$\dot{l}_\alpha = \varepsilon_{\alpha\beta\gamma} h_\beta l_\gamma$$

or $\qquad\qquad\qquad\qquad\qquad\qquad (4.18)$

$$\dot{l}_\alpha = -\varepsilon_{\alpha\beta\gamma} \frac{Y_\beta}{Y_0} l_\gamma .$$

Further, the equation for the spin density s_α in the general case was found earlier and is given by Eq. (3.12). Assuming that the density of the thermodynamic potential ω depends on the variables Y_0, Y_α, and l_α,

$$\omega(\mathbf{x}) = \omega(\mathbf{x}; Y_0(\mathbf{x}), Y_\alpha(\mathbf{x}'), l_\alpha(\mathbf{x}')) , \quad l_\alpha = \xi_\beta a_{\beta\alpha} ,$$

[we recall that in the derivation of Eq. (3.12) the potential Ω is considered under the local dependence on the inverse temperature Y_0], then, switching from Eq. (3.12) to the new variables, we obtain

$$\dot{s}_\alpha = -\frac{1}{Y_0} \varepsilon_{\alpha\beta\gamma} \left(Y_\beta \frac{\delta\Omega}{\delta Y_\gamma} + l_\beta \frac{\delta\Omega}{\delta l_\gamma} \right) . \qquad (4.19)$$

Thus, Eqs. (4.18) and (4.19) are the dissipation-free dynamic equations for the weakly anisotropic ferrimagnet. The equation for the energy density is

$$\dot{\varepsilon} = -\nabla_k \frac{1}{Y_0^2} \frac{\partial\omega}{\partial s_\alpha} \frac{\partial\omega}{\partial v_{\alpha k}} \qquad (4.20)$$

[here we have used the expression (4.17) for the energy flux density q_k].

Using (4.10), we can write Eqs. (4.18) and (4.19) in the form

$$\dot{s}_\alpha = \varepsilon_{\alpha\beta\gamma} \left(\frac{\delta H}{\delta s_\beta} s_\gamma + \frac{\delta H}{\delta l_\beta} l_\gamma \right), \quad \dot{l}_\alpha = \varepsilon_{\alpha\beta\gamma} \frac{\delta H}{\delta s_\beta} l_\gamma .$$
$$(4.21)$$

Equations (4.21) coincide with the corresponding equations obtained in the framework of the Hamiltonian approach in Ref. 9.

Since the energy density ε is a function of the quantities σ, s, l, and v_k [see Eq. (4.16)], for verification of the adiabaticity condition we switch in Eqs. (4.18) and (4.19) to the variables ε, s, l, and v_k. By virtue of (4.16), from (4.18) and (4.19) we obtain in the local limit the system of equations

$$\dot{s}_\alpha = \varepsilon_{\alpha\beta\gamma} \left(\frac{\partial\varepsilon}{\partial s_\beta} s_\gamma + \frac{\partial\varepsilon}{\partial l_\beta} l_\gamma + \frac{\partial\varepsilon}{\partial v_{\beta k}} v_{\gamma k} \right) - \nabla_k \frac{\partial\varepsilon}{\partial v_{\alpha k}} ,$$
$$(4.22)$$

$$\dot{l}_\alpha = \varepsilon_{\alpha\beta\gamma} \frac{\partial\varepsilon}{\partial s_\beta} l_\gamma , \quad \dot{\varepsilon} = -\nabla_k \frac{\partial\varepsilon}{\partial s_\alpha} \frac{\partial\varepsilon}{\partial v_{\alpha k}} .$$

From the equation for the antiferromagnetism vector follows the equation for the quantity $v_{\alpha k}$:

$$\dot{v}_{\alpha k} = \varepsilon_{\alpha\beta\gamma} \frac{\partial\varepsilon}{\partial s_\beta} v_{\gamma k} - (\delta_{\alpha\beta} - l_\alpha l_\beta) \nabla_k \frac{\partial\varepsilon}{\partial s_\beta} .$$

Using (4.16) and (4.22), we have $\dot{\sigma} = 0$, which proves the adiabaticity of the processes in the approximation which we are considering.

5. Spectrum of spin waves

To find the spectrum of spin waves for the multisublattice magnet with total symmetry breaking we linearize the system of equations (3.18) and choose as parameters, which describe the deviation from equilibrium, the quantities $\delta s_\alpha(\mathbf{x},t) = \underline{s}_\alpha(\mathbf{x},t) - \underline{s}_\alpha^{(0)}$ and $\delta a_{\alpha\beta}(\mathbf{x}, t) = \varepsilon_{\alpha\gamma\rho} \delta\varphi_\gamma(\mathbf{x}, t) \times a_{\rho\beta}^{(0)}(\mathbf{x}, t)$, where $\underline{s}^{(0)}$ and $a^{(0)}$ are the equilibrium values. The matrix of rotation $a_{\alpha\beta}^{(0)}(\mathbf{x}, t)$ satisfies the equation $\dot{a}_{\alpha\beta}^{(0)} = \varepsilon_{\alpha\rho\gamma} (\partial\varepsilon/\partial\underline{s}_\gamma) a_{\rho\beta}^{(0)}$. The variation of the right Cartan form is

$$\delta\underline{\omega}_{\alpha k}(a) = \underline{\omega}_{\alpha k}(ba) - \underline{\omega}_{\alpha k}(a) = \nabla_k \delta\varphi_\alpha - \varepsilon_{\alpha\beta\gamma} \underline{\omega}_{\beta k} \delta\varphi_\gamma ,$$
$$b_{\alpha\beta} = \delta_{\alpha\beta} - \varepsilon_{\alpha\beta\gamma} \delta\varphi_\gamma .$$

Assuming $\underline{\delta s}$, $\delta\varphi \propto \exp i(\mathbf{kx} - \omega t)$, we obtain the system of equations

$$(i\omega I - hN - if + f'N - T)_{\alpha\beta} \delta\varphi_\beta = \varepsilon_{\alpha\beta} \delta\underline{s}_\beta ,$$
$$(5.1)$$

$$(-i\omega I + i\tilde{f} + hN - M\varepsilon - N\tilde{f}' - \tilde{T})_{\alpha\beta} \delta\underline{s}_\beta =$$

$$= (D - ND''N - Mf'N + G'N +$$

$$+ iD'N + iN\tilde{D}' + iMf - iG - iQ +$$

$$+ i\tilde{Q} + NQ' - \tilde{Q}'N + H + MT)_{\alpha\beta} \delta\varphi_\beta ,$$

where

$$\varepsilon_{\alpha\beta} = \frac{\partial^2\varepsilon}{\partial\underline{s}_\alpha \partial\underline{s}_\beta} , \quad f_{\alpha\beta} = k_i \frac{\partial^2\varepsilon}{\partial\underline{s}_\alpha \partial\underline{\omega}_{\beta i}} ,$$

$$f'_{\alpha\beta} = p_i \frac{\partial^2 \varepsilon}{\partial \underline{s}_\alpha \, \partial \underline{\omega}_{\beta i}} \ , \quad D_{\alpha\beta} = k_i \, k_l \frac{\partial^2 \varepsilon}{\partial \underline{\omega}_{\alpha i} \, \partial \underline{\omega}_{\beta l}} \ ,$$

$$D'_{\alpha\beta} = k_i \, p_l \frac{\partial^2 \varepsilon}{\partial \underline{\omega}_{\alpha i} \, \partial \underline{\omega}_{\beta l}} \ , \quad D''_{\alpha\beta} = p_i \, p_l \frac{\partial^2 \varepsilon}{\partial \underline{\omega}_{\alpha i} \, \partial \underline{\omega}_{\beta l}} \ ,$$

$$N_{\alpha\beta} = \varepsilon_{\alpha\gamma\beta} \, n_\gamma \ , \quad M_{\alpha\beta} = \varepsilon_{\alpha\gamma\beta} \, \underline{s}_\gamma \ ,$$

$$G_{\alpha\beta} = \varepsilon_{\alpha\gamma\beta} \, k_i \frac{\partial \varepsilon}{\partial \underline{\omega}_{\gamma i}} \ , \quad G'_{\alpha\beta} = \varepsilon_{\alpha\gamma\beta} \, p_i \frac{\partial \varepsilon}{\partial \underline{\omega}_{\gamma i}} \ ,$$

$$T_{\alpha\beta} = \frac{\partial^2 \varepsilon}{\partial \underline{s}_\alpha \, \partial \varphi_\beta} \ , \quad Q_{\alpha\beta} = k_i \frac{\partial^2 \varepsilon}{\partial \underline{\omega}_{\alpha i} \, \partial \varphi_\beta} \ ,$$

$$Q'_{\alpha\beta} = p_i \frac{\partial^2 \varepsilon}{\partial \underline{\omega}_{\alpha i} \, \partial \varphi_\beta} \ , \quad H_{\alpha\beta} = \frac{\partial^2 \varepsilon}{\partial \varphi_\alpha \, \partial \varphi_\beta} \ .$$

The terms connected with the matrices T, Q, Q', and H take into account the anisotropy. Eliminating $\delta \underline{s}$ from Eqs. (5.1) and equating to zero the determinant of the equation for $\delta\varphi$, we find the next dispersion relation for determination of the spectrum of spin waves:

$$\det \left(\omega^2 a + \omega(ib_1 + b_2) + ic_1 + c_2 \right) \equiv \det A = 0 \ , \tag{5.2}$$

$$a = -\varepsilon^{-1} \ , \quad b_1 = M - aRN - N\tilde{R}a + aT - \tilde{T}a \ ,$$

$$R = f' - Ih \ , \quad b_2 = -af - \tilde{f}a \ , \tag{5.3}$$

$$c_1 = -G + (D' + \tilde{f}aR)N + N(\tilde{D}' + \tilde{R}af) +$$
$$+ \tilde{T}af - \tilde{f}aT - Q + \tilde{Q},$$

$$c_2 = D - ND''N + \tilde{f}af - N\tilde{R}aRN + \tilde{T}aT + NQ' -$$
$$- \tilde{Q}'N + H + (G' - hM)N \ .$$

It is easy to see that the matrices a and b_2 are symmetrical and that the matrices b_1 and c_1 are antisymmetrical. In the matrix c_2 all terms are explicitly symmetrical, except the term $(G' - hM)N$. For the study of the symmetry of this matrix we note that equilibrium values $\underline{s}_\alpha$, p_k, and $\underline{h}_\alpha$, as follows from Eqs. (3.18), are related by the relation

$$N_{\alpha\beta}\left(\underline{s}_\beta \, \underline{h} - p_k \frac{\partial \varepsilon}{\partial \underline{\omega}_{\beta k}} \right) = \frac{\partial \varepsilon}{\partial \varphi_\alpha} \ . \tag{5.4}$$

In the absence of anisotropy the right side of Eq. (5.4) is equal to zero. Using Eq. (5.4), it is not difficult to see that $((G' - hM)N)_{\alpha\beta} = ((G' - hM)N)_{\beta\alpha}$. Hence, in the absence of anisot-

ropy the matrix A is a Hermitian that leads to real values of the spin wave frequencies [8]. In the presence of anisotropy we have

$$((G' - hM)N)_{\alpha\beta} - ((G' - hM)N)_{\beta\alpha} =$$
$$= (n_\alpha \, N_{\beta\gamma} - n_\beta \, N_{\alpha\gamma}) \frac{\partial \varepsilon}{\partial \varphi_\gamma} \ .$$

For the Hermitian character of the matrix A it follows that the relation

$$N_{\alpha\gamma} \frac{\partial \varepsilon}{\partial \varphi_\gamma} \equiv \varepsilon_{\alpha\beta\gamma} \, n_\beta \frac{\partial \varepsilon}{\partial \varphi_\gamma} = 0 \ , \tag{5.5}$$

which means that the anisotropy should be such that the rotations around the direction $n_\alpha = \underline{h}_\alpha / \underline{h}$ do not change the energy functional ε. If the energy functional ε does not satisfy the relation (5.5), then the matrix A is not a Hermitian and complex frequencies, in general, arise in the spectrum. This means that the corresponding state is unstable. Therefore, when the matrix A is not a Hermitian, the exchange and anisotropy constants should satisfy certain inequalities for the spectrum of spin waves to be real. Further, we assume that Eq. (5.5) is satisfied.

We rewrite the dispersion relation (5.2) in the form

$$\sum_{n=0}^{6} A_n(\mathbf{k})\omega^n = 0 \ , \tag{5.6}$$

where the coefficients A_n ($n = 0, ..., 6$) in terms of the convolution

$$|abc| = \frac{1}{6} \varepsilon_{\alpha\beta\gamma} \, \varepsilon_{uvt} \, a_{\alpha u} \, b_{\beta v} \, c_{\gamma t}$$

are defined by the formulas

$$A_0 = |c_2 \, c_2 \, c_2| - 3|c_1 \, c_2 \, c_1| \ ,$$

$$A_1 = -6|b_1 \, c_1 \, c_2| - 3|b_2 \, c_1 \, c_1| - 3|b_2 \, c_2 \, c_2| \ ,$$

$$A_2 = -3|ac_1 \, c_1| + 3|ac_2 \, c_2| + 3|b_2 \, b_2 \, c_2| -$$
$$- 6|b_1 \, b_2 \, c_1| - 3|b_1 \, b_1 \, c_2| \ ,$$

$$A_3 = |b_2 \, b_2 \, b_2| + 6|a \, b_2 \, c_2| - 6|a \, b_1 \, c_1| - 3|b_1 \, b_1 \, b_2| \ ,$$

$$A_4 = 3|aac_2| - 3|ab_1 \, b_1| + 3|ab_2 \, b_2| \ ,$$

$$A_5 = 3|aab_2| \ ,$$

$$A_6 = |aaa| \ .$$

We carry out the analysis of the possible spectra of spin waves in the limit of small wave vectors $\mathbf{k}$. Note that in the absence of anisotropy and at $\omega = 0$, $k = 0$ we have det $A(0, 0) = $ det c_2 and by virtue of the evident form of the matrix c_2 (5.3), det $A(0, 0) = 0$. This means that in the isotropic case the system has at least two Goldstone modes [since $A_{2l+1}(k = 0) = 0$]. In the presence of anisotropy the situation changes: det $A(0, 0) = $ det $c_2 \neq 0$ which means that all modes of the anisotropic magnet, in general, are activation modes. However, the order of the activation frequencies with respect to anisotropy may be different, and it changes from first to third. Since we are considering a small anisotropy, we take into account the anisotropy in the linear approximation. In the given approximation the modes, whose activation frequencies are quadratic and cubic in anisotropy, become activationless. Let us consider some special cases of equilibrium values of the quantities $\underline{s}$, $\underline{h}$, and p_k.

1. $\underline{s} = 0$, $\underline{h} = 0$, $p_k = 0$.

The dispersion relation (5.6) has the form

$$A_6 \, \omega^6 + (A_4' + A_4'' k^2)\omega^4 + A_2'' k^2 \omega^2 + A_0''' k^4 = 0 \ .$$

$$(5.7)$$

Here we have evidently given the dependence of the modulus $|\mathbf{k}|$ in the coefficients A_n in Eq. (5.6). Solution of Eq. (5.7) yields two pairs of Goldstone and one pair of activation modes

$$\omega_{1,2}^2 = F_{1,2} \, k^2 \ , \quad \omega_3^2 = \omega_0^2 + F_3 \, k^2 \ ,$$

where

$$\omega_0^2 = -\frac{A_4'}{A_6} \ ,$$

$$F_{1,2} = \frac{1}{2A_4'} \{-A_2'' \pm \sqrt{(A_2'')^2 - 4A_0''' A_4'} \ \} \ ,$$

$$F_3 = \frac{A_6 \, A_2'' - A_4' \, A_4''}{A_6 \, A_4'} \ .$$

We present here for comparison the spectra of spin waves for the isotropic magnet [13,14] in the case under consideration

$$\omega_i^2 = \lambda_i^2 k^2 \ , \quad i = 1, 2, 3 \ .$$

2. $\underline{s} \neq 0$, $\underline{h} = 0$, $p_k = 0$.

The dispersion relation has the form

$$A_6 \, \omega^6 + (A_4' + A_4'' k^2)\omega^4 + (A_2' + A_2'' k^2)\omega^2 + A_0''' k^4 = 0 \ .$$

$$(5.8)$$

At small k we have one pair of Goldstone and two pairs of activation modes

$$\omega_{1,2}^2 = \omega_\pm^2 + R_\pm \, k^2 \ , \quad \omega_3^2 = R_3 \, k^4 \ .$$

Here

$$\omega_\pm^2 = \frac{1}{2A_6} \{-A_4' \pm \sqrt{(A_4')^2 - 4A_2' A_6} \} \ ,$$

$$R_\pm = \mp \, \frac{A_2'' + A_4'' \, \omega_\pm^2}{A_6(\omega_+^2 - \omega_-^2)} \ , \quad R_3 = -\frac{A_0'''}{A_2'} \ .$$

Similarly for the isotropic magnet [8] we have

$$\omega_1^2 = \nu_1 \, k^4 \ , \quad \omega_2^2 = \nu_2 \, k^2 \ , \quad \omega_3^2 = \omega_0^2 + \nu_3 \, k^2 \ .$$

In connection with the appearance of the activation frequencies in the isotropic magnet, it should be noted that the quantities $\underline{s}_\alpha$ and $\underline{\omega}_{\alpha k}$ are invariant relative to the right uniform rotations with the matrix b:

$$s \to s' = bs \ , \quad a \to a' = a\tilde{b} \ .$$

The energy density in the isotropic case $\varepsilon = \varepsilon(\underline{s}, \underline{\omega}_k)$ is therefore also invariant relative to the right rotations. However, it need not necessarily be invariant relative to the left rotations

$$s \to s' = sb \ , \quad a \to a' = \tilde{b}a \ ,$$

when, for example [8], the quantities $\underline{s}_\alpha$ and $\underline{\omega}_{\alpha k}$ enter ε in convolution with some «foreign» vector, which characterizies the given magnet

$$\varepsilon = \varepsilon(l_\alpha \, \underline{s}_\alpha \ , \ l_\alpha \, \underline{\omega}_{\alpha k})$$

(l_α is the unit vector of anisotropy, which is connected with the left rotations). This is the reason for the occurrence of the activation branches in the spectrum. Such situation is characteristic of the considered exchange multisublattice magnets, whose state is described, jointly with the spin density, by the additional dynamic variable which is the matrix of rotation; if the state of the magnet is characterized only by the spin density, then the activation frequencies are absent. We emphasize that in the equations of motion for the additive motion integrals the expansion in terms of spatial gradients begins with the linear terms on gradients

and in the equation of motion for the rotation matrix the expansion begins with zero-order terms on gradients, which corresponds to the precession motion with the corresponding activation frequencies.

3. $\underline{s} \neq 0$, $\underline{h} \neq 0$, $p_k = 0$.

Equation (5.6) has the form

$$A_6\, \omega^6 + (A_4' + A_4''k^2)\omega^4 + (A_2' + A_2''k^2)\omega^2 +$$

$$+ A_0' + A_0''k^2 = 0 \; . \qquad (5.9)$$

In this case all branches are activation branches

$$\omega_i^2 = \omega_{0i}^2 + c_i^2 k^2 \; , \quad i = 1,\, 2,\, 3 \; ,$$

where the activation frequencies ω_{0i}^2 are determined from the cubic equation, which is obtained from Eq. (5.9) at $k = 0$. For the isotropic magnet in this case [8] we have one pair of Goldstone branches and two pairs of activation branches

$$\omega_{1,2}^2 = \omega_{\pm}^2 + \mu_{\pm}\, k^2 \; , \quad \omega_3^2 = \mu_3\, k^2 \; .$$

4. $\underline{s} \neq 0$, $\underline{h} \neq 0$, $p_k \neq 0$.

This is the most general case. Analysis of the dispersion relation shows that there are six activation branches whose spatial anisotropy is caused by the presence of the spiral structure

$$\omega_i = \omega_{0i} + c_i(\mathbf{pk}) + d_i'(\mathbf{pk})^2 + c_i''k^2 \; .$$

For the isotropic magnet in the given case [8] we have

$$\omega_{1,2} = \alpha(\mathbf{pk}) \pm \sqrt{\beta k^2 + \gamma(\mathbf{pk})^2} \; ,$$

$$\omega_i = \omega_{0i} + \lambda_i'(\mathbf{pk}) + \mu_i'(\mathbf{pk})^2 + \mu_i''k^2 \quad (i = 3, \, \ldots, \, 6) \; .$$

6. Conclusions

Thus, on the basis of generalization the quasiaverages method for the weakly anisotropic, locally equilibrium states and with use of the reduced description method with the elements of the matrix of rotation as additional dynamic variables, we have built thermodynamics and have found the equations of low-frequency dynamics of the multisublattice magnets with strong exchange interaction. In some special cases the results are in agreement with the results of the phenomenological approach based on employment of the Hamiltonian formalism [8,9]. Except for the multisublattice magnets, the concept of total spontaneous symmetry breaking relative to spin rotations has been used in the Hamiltonian approach for the description of the low-frequency dynamics of the superfluid B-phase of ^{3}He (Ref. 19) and of the quantum spin crystals [20].

1. L. D. Landau and E. M. Lifshitz, *Phys. Zc. Sowjet.* **8**, 153 (1935).
2. A. I. Akhiezer, V. G. Bar'yakhtar, and S. V. Peletminsky, *Spin Waves*, North-Holland, Amsterdam (1960).
3. A. A. Isayev and S. V. Peletminsky, *Teor. Mat. Fiz.* **102**, 470 (1995).
4. H. Goldstein, *Classical Mechanics*, Addison-Wesley Press (1950).
5. L. A. Takhtadzhyan and L. D. Faddeev, *Hamiltonian Approach in Theory of Solitons*, Nauka, Moscow (1986) (in Russian).
6. I. E. Dzyaloshinskii and G. E. Volovick, *Ann. Phys.* **125**, 67 (1980).
7. S. J. B. Einchcomb and A. J. McKane, *Phys. Rev.* **E51**, 2974 (1995).
8. M. Yu. Kovalevsky, S. V. Peletminsky, and A. L. Shishkin, *Ukr. Fiz. Zh.* **36**, 245 (1991).
9. A. A. Isayev, M. Yu. Kovalevsky, and S. V. Peletminsky, *Teor. Mat. Fiz.* **95**, 58 (1993).
10. N. N. Bogolyubov, *Physica* **26**, 1 (1960).
11. J. Goldstone, *J. Nuovo cim.* **19**, 154 (1961).
12. D. Forster, *Hydrodynamic Fluctuations, Broken Symmetry and Correlation Functions*, Benjamin, Reading, MA (1975).
13. D. V. Volkov and A. A. Gheltukhin, *Izv. Akad. Nauk SSSR, Ser. Fiz.* **14**, 1487 (1980).
14. B. I. Halperin and W. M. Saslow, *Phys. Rev.* **B16**, 2154 (1977).
15. A. F. Andreev and V. I. Marchenko, *Usp. Fiz. Nauk* **130**, 39 (1980).
16. B. I. Halperin and P. C. Hohenberg, *Phys. Rev.* **188**, 898 (1969).
17. A. I. Akhiezer and S. V. Peletminsky, *Methods of Statistical Physics*, Pergamon, Oxford (1981).
18. V. G. Bar'yakhtar, V. G. Belykh, and T. K. Soboleva, *Teor. Mat. Fiz.* **77**, 311 (1988).
19. M. Yu. Kovalevsky and A. A. Rozhkov, *Fiz. Nizk. Temp.* **21**, 1138 (1995).
20. A. A. Isayev and M. Yu. Kovalevsky, *Fiz. Nizk. Temp.* **20**, 1125 (1994).