UDC 517.9

OSCILLATION CRITERIA FOR FIRST-ORDER LINEAR DIFFERENCE EQUATIONS WITH SEVERAL DELAY ARGUMENTS*

КРИТЕРІЇ ОСЦИЛЯЦІЇ РОЗВ'ЯЗКІВ ЛІНІЙНИХ РІЗНИЦЕВИХ РІВНЯНЬ ПЕРШОГО ПОРЯДКУ З ДЕКІЛЬКОМА ЗАПІЗНЕННЯМИ В АРГУМЕНТАХ

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The difference equation with delayed arguments

$$\Delta u(k) + \sum_{i=1}^{m} p_i(k)\, u(\tau_i(k)) = 0$$

is considered, where $\Delta u(k) = u(k+1) - u(k)$, $p_i : N \to R$, $\tau_i : N \to N$, $\lim_{k\to+\infty} \tau_i(k) = +\infty$, $i = 1, \ldots, m$. *In the paper sufficient conditions are established for all proper solutions of the above equation to be oscillatory.*

Розглянуто різницеве рівняння з запізненнями в аргументах

$$\Delta u(k) + \sum_{i=1}^{m} p_i(k)\, u(\tau_i(k)) = 0,$$

де $\Delta u(k) = u(k+1) - u(k)$, $p_i : N \to R$, $\tau_i : N \to N$, $\lim_{k\to+\infty} \tau_i(k) = +\infty$, $i = 1, \ldots, m$. *Знайдено достатні умови для того, щоб всі правильні розв'язки рівняння були осцилюючими.*

1. Introduction. The aim of this work is to study the difference equation

$$\Delta u(k) + \sum_{i=1}^{m} p_i(k)\, u(\tau_i(k)) = 0, \tag{1.1}$$

where $\Delta u(k) = u(k+1) - u(k)$ and for $1 \le i \le m$,

$$p_i : N \to R^+, \quad \tau_i : N \to N, \tag{1.2}$$

$$\tau_i(k) \le k - 1 \quad \text{for} \quad k \in N \quad \text{and} \quad \lim_{k\to+\infty} \tau_i(k) = +\infty. \tag{1.3}$$

* The work was supported by the Sh. Rustaveli National Science Foindation (Grant No. 31/09).

Define

$$\tau_*(\cdot) = \min\{\tau_i(\cdot) : i = 1, \ldots, m\}.$$

Definition 1.1. *Let* $N_n = \{n, n+1, \ldots\}$ *and* $n_0 = \min\{\tau_*(k) : k \in N_n\}$. *We will coll a function* $u : N_{n_0} \to R$ *a proper solution of the equation (1.1) if it satisfies (1.1) on* N_n *and*

$$\sup\{|u(i)| : i \geq k\} > 0 \quad \textit{for any} \quad k \in N_{n_0}.$$

Definition 1.2. *We say that a solution* $u : N_{n_0} \to R$ *of (1.1) is oscillatory if for any* $k \in N_{n_0}$ *there exist* $n_1, n_2 \in N_k$ *such that* $u(n_1) \cdot u(n_2) \leq 0$. *Otherwise the solution is called nonoscillatory.*

The oscillation theory of delay differential equations has been extensively developed [1–8]. The oscillation theory of discrete analogues of delay difference equations has also attracted a growing attention in recent years. In particular, the problem of establishing sufficient conditions for the oscillation of all solutions of the equation

$$\Delta u(k) + p(k)\, u(\tau(k)) = 0, \quad k \in N,$$

has been the subject of many recent investigations (see, for example, [9–13]).

2. Some auxiliary lemmas. Let $k_0 \in N$. We denote by $\mathbf{U}_{k_0}$ the set of all solutions of (1.1) such that $u(k) > 0$ for $k \geq k_0$.

Lemma 2.1. *Let* $k_0 \in N$, $\mathbf{U}_{k_0} \neq \varnothing$, $\tau_i(k) \leq k-1$, $i = 1, \ldots, m$, τ_i *are noncreasing functions and for each* $i = 1, \ldots, m$,

$$\liminf_{k \to +\infty} \sum_{j=\tau_i(k)}^{k-1} p_i(j) > 0. \tag{2.1}$$

Then

$$\limsup_{k \to +\infty} \frac{u(\tau_i(k))}{u(k+1)} \leq \frac{4}{c_i^2}, \quad i = 1, \ldots, m. \tag{2.2}$$

Proof. By (2.1) it is clear that, for any $i = 1, \ldots, m$ and $\varepsilon \in (0, c_i)$ there exists $k_i \in N$ such that

$$\sum_{j=\tau_i(k)}^{k-1} p_i(j) > c_i - \varepsilon \quad \text{for} \quad k \geq k_i \quad i = 1, \ldots, m. \tag{2.3}$$

Let $u : [k_0, +\infty) \to (0, +\infty)$ be a positive solution of equation (1.1). According to (1.3), without loss of generality we can assume that

$$u(\tau_i(t)) > 0 \quad \text{for} \quad k \geq k_0, \quad i = 1, \ldots, m.$$

Thus, from (1.1) we have

$$\Delta u(k) = -\sum_{i=1}^{m} p_i(k)\, u(\tau_i(k)) \le 0 \quad \text{for} \quad k \ge k_0,$$

and $u(k)$ is an nonincreasing function. Let $k \in N_{k_0}$ and $\varepsilon \in (0, c_i)$. Then by (2.3) either

$$p_i(k) \ge \frac{c_i - \varepsilon}{2} \tag{2.4}$$

or, if $p_i(k) < \dfrac{c_i - \varepsilon}{2}$ then there exists $k^* > k$ such that

$$\sum_{j=k}^{k^*-1} p_i(j) < \frac{c_i - \varepsilon}{2} \quad \text{and} \quad \sum_{j=k}^{k^*} p_i(j) \ge \frac{c_i - \varepsilon}{2}. \tag{2.5}$$

Let (2.4) be fulfilled. Then from (1.1) we obtain

$$u(k) - u(k+1) = \sum_{j=1}^{m} p_j(k)\, u(\tau_j(k)) \ge p_i(k)\, u(\tau_i(k)) \ge \frac{c_i - \varepsilon}{2}\, u(\tau_i(k)) \tag{2.6}$$

and by (2.3),

$$u(\tau_i(k)) - u(k) \ge \sum_{j=\tau_i(k)}^{k-1} p_i(j)\, u(\tau_i(j)) \ge u(\tau_i(k-1)) \sum_{j=\tau_i(k)}^{k-1} p_i(j) \ge$$

$$\ge (c_i - \varepsilon)\, u(\tau_i(k-1)). \tag{2.7}$$

Combining the inequalities (2.6) and (2.7) we get

$$u(k) \ge \frac{(c_i - \varepsilon)^2}{2}\, u(\tau_i(k-1)). \tag{2.8}$$

Assume now that (2.5) holds. It is clear that

$$\sum_{j=\tau_i(k^*)}^{k-1} p_i(j) = \sum_{j=\tau_i(k^*)}^{k^*-1} p_i(j) - \sum_{j=k}^{k^*-1} p_i(j) \ge (c_i - \varepsilon) - \frac{c_i - \varepsilon}{2} = \frac{c_i - \varepsilon}{2}. \tag{2.9}$$

Summing up (1.1) from k to k^* and using the fact that the function u is nonicreasing and the function τ_i is nondecreasing by (2.5) we have

$$u(k) - u(k^*+1) = \sum_{j=k}^{k^*} \sum_{\ell=1}^{m} p_\ell(j)\, u(\tau_\ell(j)) \ge u(\tau_i(k^*)) \sum_{j=k}^{k^*} p_i(j) \ge \frac{c_i - \varepsilon}{2}\, u(\tau_i(k^*)). \tag{2.10}$$

Analogously we can find that

$$u(\tau_i(k^*)) \geq \frac{c_i - \varepsilon}{2}\, u(\tau_i(k-1)). \tag{2.11}$$

Consequently, according to (2.8), (2.10) and (2.11), for any $k \in N_{k_0}$,

$$\frac{u(\tau_i(k-1))}{u(k)} \leq \frac{4}{(c_i - \varepsilon)^2},$$

i.e.,

$$\limsup_{k\to+\infty} \frac{u(\tau_i(k))}{u(k+1)} \leq \frac{4}{(c_i - \varepsilon)^2}$$

which, for arbitrary small values of ε, implies (2.2).

Lemma 2.1 is proved.

Lemma 2.2. *Let $k_0 \in N$, $\mathbf{U}_{k_0} \neq \varnothing$, $u \in \mathbf{U}_{k_0}$, $\tau_i(k) \leq k-1$, $i = 1, \ldots, m$, τ_i be noncreasing functions and the condition (2.1) be satisfied. Then*

$$\lim_{k\to+\infty} u(k) \exp\left(\sum_{j=1}^{k-1}\sum_{i=1}^{m} \lambda_i p_i(j)\right) = +\infty \quad \textit{for any} \quad \lambda_i > \frac{4}{c_i^2}. \tag{2.12}$$

Proof. Since all the conditions of Lemma 2.1 are fulfilled, for any $\gamma_i > \dfrac{4}{c_i^2}$ there exists $k_1 \geq k_0$ such that for each $i \in \{1, \ldots, m\}$,

$$\frac{u(\tau_i(k))}{u(k+1)} \leq \gamma_i \quad \text{for} \quad k \geq k_1. \tag{2.13}$$

For any $k^* \geq k_1$,

$$\sum_{k=k_1}^{k^*} \frac{\Delta u(k)}{u(k+1)} = \sum_{k=k_1}^{k^*} \left(1 - \frac{u(k)}{u(k+1)}\right) = k^* - k_1 + 1 - \sum_{k=k_1}^{k^*} \exp\left(\ln \frac{u(k)}{u(k+1)}\right) \leq$$

$$\leq k^* - k_1 + 1 - \sum_{k=k_1}^{k^*} \left(1 + \ln \frac{u(k)}{u(k+1)}\right) = -\sum_{k=k_1}^{k^*} (\ln u(k) - \ln u(k+1)) =$$

$$= \ln u(k^*+1) - \ln u(k_1) = \ln \frac{u(k^*+1)}{u(k_1)}.$$

From (1.1), we have

$$\sum_{k=k_1}^{k^*} \frac{\Delta u(k)}{u(k+1)} = -\sum_{k=k_1}^{k^*} \sum_{i=1}^{m} p_i(k) \frac{u(\tau_i(k))}{u(k+1)}.$$

Combining (2.13) with the last two relations, we obtain

$$-\sum_{k=k_1}^{k^*}\sum_{i=1}^{m}\gamma_i\,p_i(k)\le\ln\frac{u(k^*+1)}{u(k_1)}$$

and, consequently,

$$u(k^*+1)\ge u(k_1)\exp\left(-\sum_{k=k_1}^{k^*}\sum_{i=1}^{m}\gamma_i\,p_i(k)\right).$$

By (2.1) it is obvious that

$$\sum_{j=1}^{+\infty}p_i(j)=+\infty.$$

Therefore if $\lambda_i>\dfrac{4}{c_i^2}$, the last inequality yields

$$\lim_{k^*\to+\infty}u(k^*+1)\exp\left(\sum_{j=k_1}^{k^*}\sum_{i=1}^{m}\lambda_i\,p_i(j)\right)=+\infty,$$

i.e., (2.12) holds.

Lemma 2.2 is proved.

Now consider the difference inequality

$$\Delta u(k)+\sum_{i=1}^{m}q_i(k)\,u(\sigma_i(k))\le 0, \tag{2.14}$$

where

$$q_i:N\to R_+,\quad \sigma_i:N\to N\quad\text{and}\quad\lim_{k\to+\infty}\sigma_i(k)=+\infty,\quad i=1,\ldots,m. \tag{2.15}$$

Lemma 2.3. *Assume that (2.1) is satisfied and for sufficiently large* k,

$$\begin{gathered}\sigma_i(k)\le\tau_i(k)\le k-1,\quad p_i(k)\le q_i(k)\quad\textit{for}\quad k\in N,\\ \lim_{k\to+\infty}\sigma_i(k)=+\infty,\quad i=1,\ldots,m,\end{gathered} \tag{2.16}$$

and $u:N_{k_0}\to(0,+\infty)$ *is a positive solution of (2.14) for a certain* $k_0\in N$*. Then, there exist* $k_1>k_0$ *such that* $\mathbf{U}_{k_1}\ne\varnothing$ *and* $u^*:N_{k_0}\to R_+$ *is a solution of (1.1) which satisfies the condition*

$$0<u^*(k)\le u(k)\quad\textit{for}\quad k\ge k_1. \tag{2.17}$$

Proof. Let $u : N_{k_0} \to R_+$ a positive solution of (2.14). By (2.16) and (2.1), it is clear that there exists $k_1 \geq k_0$ such that

$$u(\sigma_i(k)) > 0 \quad \text{and} \quad \sum_{i=1}^{m} \sum_{j=\tau_i(k)}^{k-1} p_i(j) > 0 \quad \text{for} \quad k \geq k_1. \tag{2.18}$$

Summing up (2.14) from k to n and making $n \to +\infty$ we have

$$u(k) > \sum_{j=k}^{+\infty} \sum_{i=1}^{m} q_i(j)\, u(\sigma_i(j)) \quad \text{for} \quad k \geq k_1. \tag{2.19}$$

Assuming that $k^* = \min\{\tau_*(k) : k \in N_{k_1}\}$ where $\tau_*(k) = \min\{\tau_i(k) : 1 \leq i \leq n\}$ and consider the sequence of functions $u_i : N_{k^*} \to R$, $i = 1, 2, \ldots$, defined as follows:

$$u_1(k) = u(k) \quad \text{for} \quad k \in N_{k^*}, \tag{2.20}$$

$$u_j(k) = \begin{cases} \displaystyle\sum_{i=k}^{+\infty} \sum_{\ell=1}^{m} p_\ell(i)\, u_{j-1}(\tau_\ell(i)) & \text{for} \quad k \in N_{k_1}, \\ u(k) \quad \text{for} \quad k \in [k^*, k_1), & j = 2, 3, \ldots. \end{cases} \tag{2.21}$$

By induction we will prove that

$$u_j(k) \leq u_{j-1}(k) \quad \text{for} \quad k \in N_{k_1}, \quad j = 2, 3, \ldots. \tag{2.22}$$

Indeed, by (2.16) and (2.20) using the fact that the function u is nonincreasing, we have

$$u_2(k) = \sum_{i=k}^{+\infty} \sum_{\ell=1}^{m} p_\ell(i)\, u_1(\tau_\ell(i)) = \sum_{i=k}^{+\infty} \sum_{\ell=1}^{m} p_\ell(i)\, u(\tau_\ell(i)) \leq$$

$$\leq \sum_{i=k}^{+\infty} \sum_{\ell=1}^{m} q_\ell(i)\, u(\sigma_\ell(i)) \leq u(k) = u_1(k)$$

and supposing that $u_{j-1}(k) \leq u_{j-2}(k)$ for $k \in N_{k_1}$, we have

$$u_j(k) = \sum_{i=k}^{+\infty} \sum_{\ell=1}^{m} p_\ell(i)\, u_{j-1}(\tau_\ell(i)) \leq \sum_{i=k}^{+\infty} \sum_{\ell=1}^{m} p_\ell(i)\, u_{j-2}(\tau_\ell(i)) = u_{j-1}(k).$$

Thus (2.22) holds. Without loss of generality by (2.1) assume that

$$\sum_{j=\tau(k)}^{k-1} p_i(j) > 0 \quad \text{for} \quad k \geq k_1. \tag{2.23}$$

Define $\lim_{j\to+\infty} u_j(k) = u^*(k)$ (according to (2.22), this limit exists). Therefore, from (2.21), we get

$$u^*(k) = \sum_{i=k}^{+\infty}\sum_{\ell=1}^{m} p_\ell(i)\, u^*(\tau_\ell(i)) \quad \text{for} \quad k \in N_{k_1}. \tag{2.24}$$

Now we will show that $u^*(k) > 0$ for $k > k_1$. Assume, for the sake of contradiction, that there exists $k_2 > k_1$, such that $u^*(k) = 0$ for $k \ge k_2$ and $u^*(k) > 0$ for $k \in [k^*, k_2)$. Denote by N^* the set of natural numbers k for which $\tau_i(k) \ge k_2$, $i = 1, \ldots, n$, and $\overline{k} = \min N^*$. By (2.16), (2.23) and (2.24) we have $\overline{k} \ge k_2$. Therefore, $\alpha_\ell = \min\{u^*(\tau_\ell(i)) : \tau_\ell(\overline{k}) \le i \le \overline{k}-1\} > 0$ and according to (2.1) and (2.23), we obtain

$$u^*(k_2) = \sum_{i=k_2}^{+\infty}\sum_{\ell=1}^{m} p_\ell(i)\, u^*(\tau_\ell(i)) \ge \sum_{\ell=1}^{m}\alpha_\ell \sum_{i=\tau_2(\overline{k})}^{\overline{k}-1} p_\ell(i) > 0,$$

which in view of $u^*(k_2) = 0$, leads a contradiction. Therefore, $u^*(k) > 0$ for $k \ge k_1$. Hence, equation (1.1) has a positive solution u^* satisfying the condition (2.17).

Lemma 2.3 is proved.

Lemma 2.4. *Assume that* $k_0 \in N$, $\mathbf{U}_{k_0} \ne \varnothing$, $\tau_i(k) \le k-1$, $i = 1, \ldots, m$, *and the condition (2.1) is fulfilled. Then, for any* $\lambda_i > \dfrac{4}{c_i^2}$, *condition (2.12) holds.*

Proof. Since $u : N_{k_0} \to (0, +\infty)$ is a solution of (1.1), it is clear that u is a solution of the inequality

$$\Delta u(k) + \sum_{i=1}^{m} p_i(k)\, u(\sigma_i(k)) \le 0 \quad \text{for} \quad k \ge k_1,$$

where $\sigma_i(k) = \max\{\tau_i(j) : 1 \le j \le k,\ j \in N\}$ and $k_1 > k_0$ is a sufficiently large number.

We will show that

$$\liminf_{k\to+\infty} \sum_{j=\sigma_i(k)}^{k-1} p_i(j) = c_i \quad i = 1, \ldots, m. \tag{2.25}$$

Assume that (2.25) in not satisfied. Then there exist $i_0 \in \{1, \ldots, m\}$ and a sequence $\{k_\ell\}_{\ell=1}^{+\infty}$ of natural numbers such that $\sigma_{i_0}(k_\ell) \ne \tau_{i_0}(k_\ell)$, $\ell = 1, 2, \ldots,$ and

$$\liminf_{\ell\to+\infty} \sum_{j=\sigma_{i_0}(k_\ell)}^{k_\ell-1} p_{i_0}(j) = c^*_{i_0} < c_{i_0}. \tag{2.26}$$

From the definition of the function σ_i and in view of $\sigma_{i_0}(k_\ell) \ne \tau_{i_0}(k_\ell)$, for any k_ℓ there exists $k'_\ell < k_\ell$ such that $\sigma_{i_0}(k) = \sigma_{i_0}(k_\ell)$ for $k'_\ell \le k \le k_\ell$, $\lim_{\ell\to+\infty} k'_\ell = +\infty$, and $\sigma_{i_0}(k'_\ell) = \tau_{i_0}(k'_\ell)$. Thus

$$\sum_{j=\tau_{i_0}(k'_\ell)}^{k'_\ell-1} p_{i_0}(j) = \sum_{j=\sigma_{i_0}(k'_\ell)}^{k'_\ell-1} p_{i_0}(j) = \sum_{j=\sigma_{i_0}(k_\ell)}^{k'_\ell-1} p_{i_0}(j) \le \sum_{j=\sigma_{i_0}(k_\ell)}^{k_\ell-1} p_{i_0}(j)$$

and by (2.26) we have

$$\liminf_{\ell\to+\infty}\sum_{j=\tau_{i_0}(k'_\ell)}^{k'_\ell-1} p_{i_0}(j) \le \liminf_{\ell\to+\infty}\sum_{j=\sigma_{i_0}(k_\ell)}^{k_\ell-1} p_{i_0}(j) = c^*_{i_0} < c_{i_0}.$$

In view of (2.1) the last inequality leads to a contradiction and consequently (2.25) holds. Now by Lemma 2.3, we conclude that the equation (1.1) has a solution $u^*(k)$ such that

$$0 < u^*(k) \le u(k) \quad \text{for} \quad k \in N_{k_1}, \tag{2.27}$$

where $k_1 > k_0$ is sufficiently large. Hence, taking into account that the functions σ_i are nondecreasing, in view of Lemma 2.3, we obtain

$$\lim_{k\to+\infty} u^*(k)\exp\left(\sum_{j=1}^{k-1}\sum_{i=1}^{m}\lambda_i\, p_i(j)\right) = +\infty \quad \text{for any} \quad \lambda_i > \frac{4}{c_i^2}.$$

Therefore, by (2.27), we get

$$\lim_{k\to+\infty} u(k)\exp\left(\sum_{j=1}^{k-1}\sum_{i=1}^{m}\lambda_i\, p_i(j)\right) = +\infty \quad \text{for any} \quad \lambda_i > \frac{4}{c_i^2}, \quad i = 1,\ldots,m.$$

Lemma 2.4 is proved.

Lemma 2.5. *Let* $\varphi,\psi : N \to (0,+\infty)$, ψ *be nondecreasing and*

$$\lim_{k\to+\infty}\varphi(k) = +\infty, \tag{2.28}$$

$$\liminf_{k\to+\infty}\psi(k)\,\widetilde{\varphi}(k) = 0, \tag{2.29}$$

where $\widetilde{\varphi}(k) = \inf\{\varphi(s) : s \ge k, s \in N\}$. *Then there exists an increasing sequence of natural numbers* $\{k_i\}_{i=1}^{+\infty}$ *such that*

$$\lim_{i\to+\infty} k_i = +\infty, \quad \widetilde{\varphi}(k_i) = \varphi(k_i), \quad \psi(k)\,\widetilde{\varphi}(k) \ge \psi(k_i)\,\widetilde{\varphi}(k_i),$$

$$k = 1,2,\ldots,k_i, \quad i = 1,2,\ldots.$$

We refer the reader to [13] for a proof of Lemma 2.5. For a continuous case, analogous of Lemma 2.5, see [14] (Lemma 7.1).

Lemma 2.6. *Let* $\tau_i : N \to N$, $i = 1,\ldots,m$, *and (1.3) be fulfilled. Then there exists a nondecreasing function* $\sigma : N \to N$ *such that*

$$\begin{aligned} &\text{(i) } \lim_{k\to+\infty}\sigma(k) = +\infty,\\ &\text{(ii) } \sigma(k) \le \min\{\tau_i(k) : i = 1,\ldots,m\},\\ &\text{(iii) } \sigma(N_k) \supset \cup_{i=1}^{m}\tau_i(N_k) \quad \textit{for any} \quad k \in N. \end{aligned} \tag{2.30}$$

Proof. Consider the sequence

$$A = \{a_1, a_2, \ldots, a_m, \ldots, a_{2m}, \ldots\} = \{\tau_1(1), \ldots, \tau_m(1), \tau_1(2), \ldots, \tau_m(2), \ldots\}$$

and denote by τ the function $\tau : N \to A$ thus defined. By (1.3) it is obvious that

$$\lim_{k\to+\infty} \tau(k) = +\infty \quad \text{and} \quad \tau(N_k) \supset \tau_i(N_k), \tag{2.31}$$

$$k = 1, \ldots, m, \quad \text{for any} \quad k \in N.$$

Introduce the following sets:

$$s \in A_1 \Leftrightarrow s \in N, \quad \tau(s) = \inf\{\tau(k) : k \in N\},$$

$$s \in A_j \Leftrightarrow s \in N, \quad \tau(s) = \inf\{\tau(k) : k \in N \setminus \cup_{i=1}^{j-1} A_i\},$$

$$j = 2, 3, \ldots,$$

and denote $\xi_j = \max A_j$, $j = 1, 2, \ldots$, $\xi_1^0 = \xi_1$, $\xi_j^0 = \max\{\xi_j, \xi_{j-1}^0 + 1\}$, $j = 2, 3, \ldots$. We will construct the function σ as follows:

$$\sigma(k) = \tau(\xi_1) \quad \text{for} \quad 1 \le k \le \xi_1^0,$$

$$\sigma(k) = \tau(\xi_j) \quad \text{for} \quad \xi_{j-1}^0 < k \le \xi_j^0, \quad j = 2, 3, \ldots.$$

The function σ is obviously nondecreasing and satisfies the conditions (i) and (ii). We also have $\sigma(N_k) \supset \tau(N_k)$ for any $k \in N$. Therefore, in view of (2.31) it is obvious that the condition (iii) is also satisfied.

Lemma 2.6 is proved.

Remark 2.1. Let $\tau_i : N \to N$, $i = 1, \ldots, m$, $p : N \to R_+$, (1.3) be fulfilled and

$$\limsup_{k\to+\infty} \sum_{s=\tau_i(k)}^{k-1} p(s) < +\infty, \quad i = 1, \ldots, m.$$

Then

$$\limsup_{k\to+\infty} \sum_{s=\sigma(k)}^{k-1} p(s) < +\infty,$$

where the function σ is given by Lemma 2.6.

Lemma 2.7. *Let $k_0 \in N$, $U_{k_0} \ne \varnothing$, (1.2), (1.3) be fulfilled. Then for any $u \in U_{k_0}$ we have*

$$\limsup_{k=+\infty} u(k) \exp\left(\sum_{j=1}^{k-1} \sum_{i=1}^{m} p_i(j)\right) < +\infty. \tag{2.32}$$

Proof. Since $U_{k_0} \neq \varnothing$ (see Definition 2.1), (1.1) has a positive solution $u : N_{k_0} \to (0, +\infty)$. From the equality

$$\sum_{j=k_1}^{k} \frac{\Delta u(j)}{u(j)} = \sum_{j=k_1}^{k} \left(\frac{\Delta u(j+1)}{u(j)} - 1 \right) = \sum_{j=k_1}^{k} \left(\exp \left(\ln \frac{\Delta u(j+1)}{u(j)} - 1 \right) \right),$$

$$\text{for} \quad k_1 \geq k_0,$$

since $e^x \geq 1 + x$, we obtain

$$\sum_{j=k_1}^{k} \frac{\Delta u(j)}{u(j)} \geq \sum_{j=k_1}^{k} \ln \frac{\Delta u(j)}{u(j)} = \ln \frac{\Delta u(k+1)}{u(k_1)}. \tag{2.33}$$

Taking into account that the function u is nonincreasing, from (1.1) we have

$$\Delta u(k) = -\sum_{j=1}^{m} p_j(k)\, u(\tau_j(k)) \leq -u(k) \sum_{j=1}^{m} p_j(k) \quad \text{for} \quad k \geq k_1,$$

where k_1, is a sufficiently large number.

Consequently,

$$\sum_{j=k_1}^{k} \frac{\Delta u(j)}{u(j)} \leq -\sum_{j=k_1}^{k} \sum_{i=1}^{m} p_i(j). \tag{2.34}$$

Combining the inequalities (2.33) and (2.34), we obtain

$$u(k+1) \leq u(k_1) \exp \left(-\sum_{j=k_1}^{k} \sum_{i=1}^{m} p_i(j) \right),$$

that is (2.32) holds.

Lemma 2.7 is proved.

Lemma 2.8 (Abel transformation). *Let* $\{a_i\}_{i=1}^{+\infty}$ *and* $\{b_i\}_{i=1}^{+\infty}$ *be sequences of nonegative numbers and* $\sum_{i=1}^{+\infty} a_i < +\infty$. *Then*

$$\sum_{i=1}^{k} a_i\, b_i = A_1 b_1 - A_{k+1}\, b_{k+1} - \sum_{i=1}^{k} A_{i+1}(b_i - b_{i+1}),$$

where $A_i = \sum_{j=i}^{+\infty} a_j$.

3. Necessary conditions for existence of positive solutions.

Theorem 3.1. *Let* $k_0 \in N$, $\mathbf{U}_{k_0} \neq \varnothing$, *(1.2), (1.3) and (2.1) be fulfilled and*

$$\limsup_{k=+\infty} \sum_{j=\tau_j(k)}^{k-1} p^*(j) < +\infty. \tag{3.1}$$

Then there exists $\lambda \in \left[1, \frac{4}{c_0^2}\right]$ *such that*

$$\limsup_{\varepsilon \to 0+} \left(\liminf_{k \to +\infty} \exp\left((\lambda + \varepsilon) \sum_{i=1}^{k-1} p^*(s) \right) \times \right.$$

$$\left. \times \sum_{j=k}^{+\infty} \sum_{i=1}^{m} p_i(j) \exp\left(-(\lambda + \varepsilon) \sum_{s=1}^{\tau_i(j)-1} p^*(s) \right) \right) \leq 1, \tag{3.2}$$

where $c_0 = \min\{c_i : i = 1, \ldots, m\}$, $p^*(k) = \sum_{i=1}^{m} p_i(k)$.

Proof. Since $\mathbf{U}_{k_0} \neq \varnothing$ (see Definition 2.1), (1.1) has a positive solution $u : N_{k_0} \to (0, +\infty)$. According to Lemma 2.7, (2.32) holds. On the other hand, since all the conditions of Lemma 2.4 are satisfies, we conclude that condition (2.12) holds.

Denote by λ the set of all λ for with

$$\lim_{k = +\infty} u(k) \exp\left(\lambda \sum_{j=1}^{k-1} \sum_{i=1}^{m} p_i(j) \right) = +\infty \tag{3.3}$$

is fulfilled and denote $\lambda_0 = \inf \lambda$. In view of (3.3) and (2.32) it is obvious that $\lambda_0 \in \left[1, \frac{4}{c_0^2}\right]$. It is obvious that for any $\varepsilon > 0$

$$\lim_{k \to +\infty} u(k) \exp\left((\lambda_0 + \varepsilon) \sum_{j=1}^{k-1} p^*(j) \right) = +\infty \tag{3.4}$$

and

$$\liminf_{k \to +\infty} u(k) \exp\left((\lambda_0 - \varepsilon) \sum_{j=1}^{k-1} p^*(j) \right) = 0. \tag{3.5}$$

According to Lemma 2.6, there exists a nondecreasing function σ such that (2.30) is fulfilled. Hence, by virtue of (3.4), (3.5) and (2.30), it is clear that for any $\varepsilon > 0$, the functions

$$\varphi(k) = u(\sigma(k)) \exp\left((\lambda_0 + \varepsilon) \sum_{j=1}^{\sigma(k)-1} p^*(j) \right) \tag{3.6}$$

and

$$\psi(k) = \exp\left(-2\varepsilon \sum_{j=1}^{\sigma(k)-1} p^*(j) \right) \tag{3.7}$$

satisfy the condition of Lemma 2.5, for sufficiently large k. Hence, there exists an increasing sequence $\{k_j\}_{j=1}^{+\infty}$ of natural numbers,

$$\psi(k_j)\,\widetilde{\varphi}(k_j) \le \psi(k)\,\widetilde{\varphi}(k) \quad \text{for} \quad k^* \le k \le k_j, \tag{3.8}$$

$$\widetilde{\varphi}(k_j) = \varphi(k_j), \quad j = 1, 2, \ldots, \tag{3.9}$$

where k^* is a sufficiently large number. By (2.30) and (3.4) it is clear that

$$\widetilde{\varphi}(k) = \inf\left\{ u(\sigma(k)) \exp\left((\lambda_0 + \varepsilon) \sum_{j=1}^{\sigma(s)-1} p^*(j) \right) : s \ge k, s \in N \right\} \le$$

$$\le \inf\left\{ u(\tau_i(s)) \exp\left((\lambda_0 + \varepsilon) \sum_{j=1}^{\tau_i(s)-1} p^*(j) \right) : s \ge k, s \in N \right\}, \quad i = 1, \ldots, m. \tag{3.10}$$

By (3.10), from (1.1) we get

$$u(\sigma(k_\ell)) \ge \sum_{j=\sigma(k_\ell)}^{+\infty} \sum_{i=1}^{m} p_i(j)\, u(\tau_i(j)) = \sum_{i=1}^{m} \sum_{j=\sigma(k_\ell)}^{+\infty} p_i(j) \times$$

$$\times \exp\left(-(\lambda_0 + \varepsilon) \sum_{s=1}^{\tau_i(j)-1} p^*(s) \right) u(\tau_i(j)) \times$$

$$\times \exp\left((\lambda_0 + \varepsilon) \sum_{s=1}^{\tau_i(j)-1} p^*(s) \right) \ge$$

$$\ge \sum_{i=1}^{m} \sum_{j=\sigma k_\ell}^{+\infty} p_i(j) \exp\left(-(\lambda_0 + \varepsilon) \sum_{s=1}^{\tau_i(j)-1} \right) \widetilde{\varphi}(j), \quad \ell = 1, 2, \ldots,$$

where $p^*(s) = \sum_{i=1}^{m} p_i(s)$, that is,

$$u(\sigma(k_\ell)) \ge \sum_{i=1}^{m} \left\{ \sum_{j=\sigma(k_\ell)}^{k_\ell - 1} p_i(j) \exp\left(-(\lambda_0 + \varepsilon) \sum_{s=1}^{\tau_i(j)-1} p^*(s) \right) \widetilde{\varphi}(j) + \right.$$

$$\left. + \sum_{j=k_\ell}^{+\infty} p_i(j) \exp\left(-(\lambda_0 + \varepsilon) \sum_{s=1}^{\tau_i(j)-1} p^*(s) \right) \widetilde{\varphi}(j) \right\}.$$

Thus by (3.8), and using the fact that the function $\widetilde{\varphi}$ is noncreasing, the last inequality yields

$$u(\sigma(k_\ell)) \ge \sum_{i=1}^{m} \left\{ \sum_{j=\sigma(k_\ell)}^{k_\ell - 1} p_i(j) \widetilde{\varphi} \exp\left(-2\varepsilon \sum_{s=1}^{j-1} p^*(s) \right) \times \right.$$

$$\times \exp\left(-(\lambda_0+\varepsilon)\sum_{s=1}^{j-1}p^*(s)\right)+\exp\left(-(\lambda_0+\varepsilon)\sum_{s=1}^{\tau_i(j)-1}p^*(s)\right)+$$

$$+\sum_{j=k_\ell}^{+\infty}p_i(j)\exp\left(-(\lambda_0+\varepsilon)\sum_{s=1}^{\tau_i(j)-1}p^*(s)\right)\widetilde{\varphi}(j)\Bigg\}\geq$$

$$\geq\sum_{i=1}^{m}\Bigg\{\widetilde{\varphi}(j)\exp\left(-2\varepsilon\sum_{s=1}^{k_\ell-1}p^*(s)\right)\times$$

$$\times\sum_{j=\sigma(k_\ell)}^{k_\ell-1}p_i(j)\exp\left(2\varepsilon\sum_{s=1}^{j-1}p^*(s)\right)\exp\left(-(\lambda_0+\varepsilon)\sum_{s=1}^{\tau_i(j)-1}p^*(s)\right)+$$

$$+\widetilde{\varphi}(k_\ell)\sum_{j=k_\ell}^{+\infty}p_i(j)\exp\left(-(\lambda_0+\varepsilon)\sum_{s=1}^{\tau_i(j)-1}p^*(s)\right)\Bigg\},\quad \ell=1,2,\ldots. \tag{3.11}$$

Put

$$I_i(k_\ell,\varepsilon)=\sum_{j=\sigma(k_\ell)}^{k_\ell-1}p_i(j)\exp\left(2\varepsilon\sum_{s=1}^{j-1}p^*(s)\right)\exp\left(-(\lambda_0+\varepsilon)\sum_{s=1}^{\tau_i(j)-1}p^*(s)\right).$$

Using the Lemma 2.8, we have

$$I_i(k_\ell,\varepsilon)=\exp\left(2\varepsilon\sum_{s=1}^{\sigma(k_\ell)-1}p^*(s)\right)\sum_{j=\sigma(k_\ell)}^{+\infty}p_i(j)\exp\left(-(\lambda_0+\varepsilon)\sum_{s=1}^{\tau_i(j)-1}p^*(s)\right)-$$

$$-\exp\left(2\varepsilon\sum_{s=1}^{k_\ell-1}p^*(s)\right)\sum_{j=k_\ell+1}^{+\infty}p_i(j)\exp\left(-(\lambda_0+\varepsilon)\sum_{s=1}^{\tau_i(j)-1}p^*(s)\right)+$$

$$+\sum_{j=\sigma(k)}^{k}\left(\exp\left(2\varepsilon\sum_{s=1}^{j}p^*(s)\right)-\exp\left(2\varepsilon\sum_{s=1}^{j-1}p^*(s)\right)\right)\times$$

$$\times\sum_{k=j+1}^{+\infty}p_i(k)\exp\left(-(\lambda_0+\varepsilon)\sum_{s=1}^{\tau_i(k)-1}p^*(s)\right).$$

Since

$$\exp\left(2\varepsilon\sum_{s=1}^{j}p^*(s)\right)-\exp\left(2\varepsilon\sum_{s=1}^{j-1}p^*(s)\right)\geq 0$$

from the last equality we obtain

$$I_i(k_\ell,\varepsilon) = \exp\left(2\varepsilon \sum_{s=1}^{\sigma(k_\ell)-1} p^*(s)\right) \sum_{j=\sigma(k_\ell)}^{+\infty} p_i(j)\times$$

$$\times \exp\left(-(\lambda_0+\varepsilon) \sum_{s=1}^{\tau_i(j)-1} p^*(s)\right) - \exp\left(2\varepsilon \sum_{s=1}^{k_\ell-1} p^*(s)\right)\times$$

$$\times \sum_{j=k_\ell+1}^{+\infty} p_i(j) \exp\left(-(\lambda_0+\varepsilon) \sum_{s=1}^{\tau_i(j)-1} p^*(s)\right). \tag{3.12}$$

According to (3.11) and (3.12) we get

$$u(\sigma(k_\ell)) \geq \widetilde{\varphi}(k_\ell) \sum_{s=1}^{m} \exp\left(-2\varepsilon \sum_{s=1}^{k_\ell-1} p^*(s)\right)\times$$

$$\times \exp\left(2\varepsilon \sum_{s=1}^{\sigma(k_\ell)-1} p^*(s)\right) \sum_{j=\sigma(k_\ell)}^{+\infty} p_i(j) \exp\left(-(\lambda_0+\varepsilon) \sum_{s=1}^{\tau_i(j)-1} p^*(s)\right).$$

Therefore, by (3.6) and (3.9) the last inequality implies

$$\exp\left((\lambda_0+\varepsilon) \sum_{s=1}^{\sigma(k_\ell)-1} p^*(s)\right) \sum_{i=1}^{m} \sum_{j=\sigma(k_\ell)}^{+\infty} p_i(j)\times$$

$$\times \exp\left(-(\lambda_0+\varepsilon) \sum_{s=1}^{\tau_i(j)-1} p^*(s)\right) \leq \exp\left(2\varepsilon \sum_{\sigma(k_\ell)}^{k_\ell-1} p^*(s)\right). \tag{3.13}$$

By (3.1) and Remark 2.1, there exists $M > 0$ such that $\sum_{s=\sigma(k_\ell)}^{k_\ell-1} p^*(s) \leq M$, $\ell = 1, 2, \ldots$. From (3.13) we have

$$\limsup_{\ell\to+\infty} \exp\left((\lambda_0+\varepsilon) \sum_{s=1}^{\sigma(k_\ell)-1} p^*(s)\right) \sum_{i=1}^{m} \sum_{j=\sigma(k_\ell)}^{+\infty} p_i(j) \exp\left(-(\lambda_0+\varepsilon) \sum_{s=1}^{\tau_i(j)-1} p^*(s)\right) \leq e^{2\varepsilon M},$$

a.e.

$$\liminf_{k\to+\infty} \exp\left((\lambda_0+\varepsilon) \sum_{s=1}^{k-1} p^*(s)\right) \sum_{i=1}^{m} \sum_{j=k}^{+\infty} p_i(j) \exp\left(-(\lambda_0+\varepsilon) \sum_{s=1}^{\tau_i(j)-1} p^*(s)\right) \leq e^{2\varepsilon M},$$

which implies (3.2).

Theorem 3.1 is proved.

4. Sufficient conditions for oscillation. In this section, using Theorem 3.1, sufficient conditions will be established for oscillation of all solutions of the equation (1.1) which generalizes the results given in [12].

Theorem 4.1. *Assume that the conditions (1.2), (1.3), (2.1), (3.1) are satisfied and, for any* $\lambda \in \left[1, \frac{4}{c_0^2}\right]$,

$$\limsup_{\varepsilon \to 0+} \left(\liminf_{k \to +\infty} \exp\left((\lambda+\varepsilon) \sum_{s=1}^{\sigma(k-1} p^*(s) \right) \sum_{j=k}^{+\infty} \sum_{i=i}^{m} p_i(j) \times \right.$$

$$\left. \times \exp\left(-(\lambda+\varepsilon) \sum_{s=1}^{\tau_i(j)-1} p^*(s) \right) \right) > 1. \tag{4.1}$$

Then all proper solutions of equation (1.1) oscillate, where

$$c_0 = \min\{c_i : i = 1, \ldots, m\} \quad \textit{and} \quad p^*(s) = \sum_{i=i}^{m} p_i(s). \tag{4.2}$$

Proof. Suppose the contrary. Let $u : N_{k_0} \to (0, +\infty)$ with $k_0 \in N$ be a positive proper solution of the equation (1.1), i.e., $\mathbf{U}_{k_0} \neq \varnothing$. Taking into account Theorem 3.1 we will conclude that there exists $\lambda_0 \in \left[1, \frac{4}{c_0^2}\right]$ such that the inequality (3.2) holds for $\lambda = \lambda_0$. But this contradicts the condition (4.1). The obtained contradiction proves the theorem.

Theorem 4.2. *Let the conditions (1.2), (1.3), (2.1), (3.1) be satisfied and*

$$\sum_{k=1}^{+\infty} \left(\frac{1}{m} \sum_{i=1}^{m} p_i(k) - \left(\prod_{i=1}^{m} p_i(k) \right)^{\frac{1}{m}} \right) < +\infty, \tag{4.3}$$

and, for any $\lambda \in \left[1, \frac{4}{c_0^2}\right]$,

$$\limsup_{\varepsilon \to 0+} \left(\liminf_{k \to +\infty} \exp\left((\lambda+\varepsilon) \sum_{i=1}^{k-1} p^*(i) \right) \sum_{j=k}^{+\infty} p^*(j) \right.$$

$$\left. \times \exp\left(-\frac{\lambda+\varepsilon}{m} \sum_{i=i}^{m} \sum_{s=1}^{\tau_i(j)-1} p^*(s) \right) \right) > 1. \tag{4.4}$$

Then all proper solutions of equation (1.1) oscillate, where c_0 *and* p^* *are given by (4.2).*

Proof. To prove the theorem, it suffices to show that by (4.3), (4.4) implies (4.1). By (3.1) there exists $k^* \in N$ and $M > 0$ such that

$$\sum_{i=1}^{m} \sum_{s=\tau_j(k)}^{k-1} p^*(s) \le M \quad \text{for} \quad k \ge k^*. \tag{4.5}$$

Denote

$$\rho(\varepsilon,k) = \exp\left((\lambda+\varepsilon)\sum_{j=1}^{k-1} p^*(j)\right)\sum_{j=k}^{+\infty}\sum_{i=1}^{m} p_i(j)\exp\left(-(\lambda+\varepsilon)\sum_{s=1}^{\tau_i(j)-1} p^*(s)\right).$$

Using the arithmetic mean-geometric mean inequality, for $k \geq k^*$ we get

$$\rho(\varepsilon,k) = \exp\left((\lambda+\varepsilon)\sum_{j=1}^{k-1} p^*(j)\right) m\sum_{j=k}^{+\infty}\frac{1}{m}\sum_{i=1}^{m} p_i(j)\exp\left(-(\lambda+\varepsilon)\sum_{s=1}^{\tau_i(j)-1} p^*(s)\right) \geq$$

$$\geq \exp\left((\lambda+\varepsilon)\sum_{j=1}^{k-1} p^*(j)\right) m\sum_{j=k}^{+\infty}\left(\prod_{i=1}^{m} p_i(j)\right)^{\frac{1}{m}} \times$$

$$\times \exp\left(-\frac{\lambda+\varepsilon}{m}\sum_{i=1}^{m}\sum_{s=1}^{\tau_i(j)-1} p^*(s)\right) = \exp\left((\lambda+\varepsilon)\sum_{j=1}^{k-1} p^*(j)\right) \times$$

$$\times\, m\sum_{j=k}^{+\infty}\left[\frac{1}{m}p^*(j) - \left(\frac{1}{m}p^*(j) - \left(\prod_{i=1}^{m} p_i(j)\right)^{\frac{1}{m}}\right)\right] \times$$

$$\times \exp\left(-\frac{\lambda+\varepsilon}{m}\sum_{i=1}^{m}\sum_{s=1}^{\tau_i(j)-1} p^*(s)\right).$$

By (4.5) and the last inequality, we have

$$\rho(\varepsilon,k) \geq \exp\left((\lambda+\varepsilon)\sum_{j=1}^{k-1} p^*(j)\right)\sum_{j=k}^{+\infty} p^*(j)\exp\left(-\frac{\lambda+\varepsilon}{m}\sum_{i=1}^{m}\sum_{s=1}^{\tau_i(j)-1} p^*(s)\right) -$$

$$-\, m\exp\left((\lambda+\varepsilon)\sum_{j=1}^{k-1} p^*(j)\right)\sum_{j=k}^{+\infty}\left(\frac{1}{m}p^*(j) - \left(\prod_{i=1}^{m} p_i(j)\right)^{\frac{1}{m}}\right) \times$$

$$\times \exp\left(-(\lambda+\varepsilon)\sum_{s=1}^{j-1} p^*(s)\right)\exp\left(\frac{\lambda+\varepsilon}{m}\sum_{i=1}^{m}\sum_{s=\tau_i(j)}^{j-1} p^*(j)\right) \geq$$

$$\geq \exp\left((\lambda+\varepsilon)\sum_{j=1}^{k-1} p^*(s)\right)\sum_{j=k}^{+\infty} p^*(s)\,\exp\left(-\frac{\lambda+\varepsilon}{m}\sum_{i=1}^{m}\sum_{s=1}^{\tau_i(j)-1} p^*(j)\right) -$$

$$-\, m\exp\left(\frac{1}{m}(\lambda+\varepsilon)M\right)\sum_{j=k}^{+\infty}\left(\frac{1}{m}p^*(j) - \left(\prod_{i=1}^{m} p_i(j)\right)^{\frac{1}{m}}\right). \qquad (4.6)$$

On the other hand by (4.3) it is obvious that

$$\lim_{k\to+\infty}\sum_{j=k}^{+\infty}\left(\frac{1}{m}\sum_{i=1}^{m}p_i(j)-\left(\prod_{i=1}^{m}p_i(j)\right)^{\frac{1}{m}}\right)=0.$$

Therefore, according to (4.4), from (4.6), we get

$$\limsup_{\varepsilon\to0+}\left(\liminf_{k\to+\infty}\rho(\varepsilon,k)\right)>1,$$

i.e., (4.1) is fulfilled, which proves the validity of the theorem.

In a manner similar to the above, we can prove the following theorem.

Theorem 4.3. *Let the conditions (1.2), (1.3), (2.1), (3.1) be fulfilled and*

$$\sum_{s=1}^{+\infty}|p_i(s)-p_j(s)|<+\infty,\quad i,j=1,\ldots,m, \tag{4.7}$$

and, for any $\lambda\in\left[1,\dfrac{4}{c_0^2}\right]$,

$$\limsup_{\varepsilon\to0}\left(\liminf_{k\to+\infty}\exp\left((\lambda+\varepsilon)\sum_{i=1}^{k-1}p^*(i)\right)\sum_{j=k}^{+\infty}p^*(j)\exp\left(-(\lambda+\varepsilon)\sum_{s=1}^{\tau_i(j)-1}p^*(s)\right)\right)>1.$$

Then all proper solutions of equation (1.1) oscillate, where c_0 *and* p^* *are given by (4.2).*

Theorem 4.4. *Let the conditions (1.2), (1.3), (2.1), (3.1) and (4.3) be fulfilled. Then the condition*

$$\liminf_{k\to+\infty}\sum_{i=1}^{m}\left(\sum_{s=\tau_i(k)}^{k-1}p^*(s)\right)>\frac{m}{e} \tag{4.8}$$

is sufficient for all proper solutions of equation (1.1) to be oscillatory, where p^* *is given by (4.2).*

Proof. To prove the theorem, it suffices to show that (4.8) implies (4.4). Indeed, by (4.8) there exists k^* and $\varepsilon_0>0$ such that

$$\sum_{i=1}^{m}\left(\sum_{s=\tau_i(k)}^{k-1}p^*(s)\right)>\frac{m+\varepsilon_0}{e}\quad\text{for}\quad k\geq k^*. \tag{4.9}$$

Thus

$$\rho_1(\varepsilon,k) = \exp\left((\lambda+\varepsilon)\sum_{i=1}^{k-1}p^*(i)\right)\sum_{j=k}^{+\infty}p^*(j)\exp\left(-\frac{\lambda+\varepsilon}{m}\sum_{i=1}^{m}\sum_{s=1}^{\tau_i(j)-1}p^*(s)\right) =$$

$$= \exp\left((\lambda+\varepsilon)\sum_{i=1}^{k-1}p^*(i)\right)\sum_{j=k}^{+\infty}p^*(j)\exp\left(-(\lambda+\varepsilon)\sum_{s=1}^{j-1}p^*(s)\right)\times$$

$$\times\exp\left(\frac{\lambda+\varepsilon}{m}\sum_{i=1}^{m}\sum_{s=\tau_i(j)}^{j-1}p^*(s)\right)$$

and by (4.9) and the last equality, we get

$$\rho_1(\varepsilon,k) \ge \exp\left((\lambda+\varepsilon)\frac{m+\varepsilon_0}{m}\right)\exp\left((\lambda+\varepsilon)\sum_{i=1}^{k-1}p^*(i)\right)\times$$

$$\times\sum_{j=k}^{+\infty}p^*(j)\exp\left(-(\lambda+\varepsilon)\sum_{s=1}^{j-1}p^*(s)\right) \quad \text{for} \quad k \ge k^*. \tag{4.10}$$

Defining

$$\sum_{s=1}^{j-1}p^*(s) = a_{j-1}$$

and we will show that

$$\liminf_{k\to+\infty}\exp\left((\lambda+\varepsilon)a_{k-1}\right)\sum_{s=k}^{+\infty}p^*(j)\exp\left(-(\lambda+\varepsilon)a_{j-1}\right) \ge \frac{1}{\lambda+\varepsilon}. \tag{4.11}$$

Indeed, by (2.1) it is obvious that

$$\sum_{j=1}^{+\infty}p_i(j) = +\infty, \quad i = 1,\ldots,m,$$

that is, $\lim_{j\to+\infty}a_j = +\infty$. Therefore,

$$\exp((\lambda+\varepsilon)a_{k-1})\sum_{j=k}^{+\infty}p^*(j)\exp(-(\lambda+\varepsilon)a_{j-1}) =$$

$$= \exp((\lambda+\varepsilon)a_{k-1})\sum_{j=k}^{+\infty}(a_j - a_{j-1})\exp(-(\lambda+\varepsilon)a_{j-1}) =$$

$$= \exp((\lambda+\varepsilon)a_{k-1})\sum_{j=k}^{+\infty}\exp(-(\lambda+\varepsilon)a_{j-1})\int_{a_{j-1}}^{a_j}ds \ge$$

$$\geq \exp((\lambda+\varepsilon)a_{k-1}) \sum_{j=k}^{+\infty} \int_{a_{j-1}}^{a_j} \exp(-(\lambda+\varepsilon)s)ds =$$

$$= \exp((\lambda+\varepsilon)a_{k-1}) \sum_{j=k}^{+\infty} \exp(-(\lambda+\varepsilon)s)ds = \frac{1}{\lambda+\varepsilon}.$$

Hence, by (4.11) and (4.10) we obtain

$$\limsup_{\varepsilon\to 0+} \left(\liminf_{k\to+\infty} \rho_1(\varepsilon,k) \right) \geq \exp\left((\lambda+\varepsilon) \frac{m+\varepsilon_0}{m} \right) \frac{1}{\lambda+\varepsilon} \geq \frac{m+\varepsilon_0}{m} > 1,$$

that is, condition (4.4) holds, which proves the validity of the theorem.

Using Theorem 4.3, similarly to Theorem 4.4 one can prove the following theorem.

Theorem 4.5. *Let the conditions (1.2), (1.3), (2.1), (3.1) and (4.7) be fulfilled. Then the condition*

$$\liminf_{k\to+\infty} \sum_{i=1}^{m} \left(\sum_{s=\tau_i(k)}^{k-1} p_i(s) \right) > \frac{1}{e} \tag{4.12}$$

is sufficient for all proper solutions of equation (1.1) to be oscillatory.

Example. Let $m \in N$, $\lambda \in (0,+\infty)$, $\alpha_i \in (0,1)$, $i = 1,\ldots,m$. Consider the equation (1.1), where

$$\tau_i(k) = [\alpha_i k], \quad p_i(k) = \frac{k^{-\lambda} - (k+1)^{-\lambda}}{[\alpha_1 k]^{-\lambda} + \ldots + [\alpha_m k]^{-\lambda}}, \quad i = 1,\ldots,m, \tag{4.13}$$

where $[a]$ denotes the integer part of a. It is obvious that

$$k^{1+\lambda}\left(k^{-\lambda} - (k+1)^{-\lambda}\right) \to \lambda \quad \text{for} \quad k \to +\infty. \tag{4.14}$$

According to (4.13) and (4.14), we get

$$\liminf_{k\to+\infty} \sum_{i=1}^{m} \left(\sum_{s=\tau_i(k)}^{k-1} p_i(k) \right) = \liminf_{k\to+\infty} \left(\sum_{j=[\alpha_i k]}^{k-1} \frac{j(j^{-\lambda} - (j+1)^{-\lambda})}{[\alpha_1 j]^{-\lambda} + \ldots + [\alpha_m j]^{-\lambda}} \frac{1}{j} \right) =$$

$$= \frac{\lambda}{\alpha_1^{-\lambda} + \ldots + \alpha_m^{-\lambda}} \ln \frac{1}{\alpha_1 \ldots \alpha_m}. \tag{4.15}$$

Using the arithmetic mean-geometric inequality we obtain

$$\frac{\lambda}{\alpha_1^{-\lambda} + \ldots + \alpha_m^{-\lambda}} \ln \frac{1}{\alpha_1 \ldots \alpha_m} \leq \frac{\frac{\lambda}{m}}{(\alpha_1 \ldots \alpha_m)^{-\frac{\lambda}{m}}} \ln \frac{1}{\alpha_1 \ldots \alpha_m} =$$

$$= \frac{\lambda}{m} (\alpha_1 \ldots \alpha_m)^{\frac{\lambda}{m}} \ln \frac{1}{\alpha_1 \ldots \alpha_m}. \tag{4.16}$$

Since

$$\max\left\{\frac{\lambda}{m}(\alpha_1\ldots\alpha_m)^{\frac{\lambda}{m}}\ \ln\frac{1}{\alpha_1\ldots\alpha_m}\ :\ \lambda\in(0,+\infty)\right\}=\frac{1}{e}$$

and if $\alpha_i=\alpha_1,\ i=1,\ldots,m$, then for $\lambda=-\dfrac{1}{\ln\alpha_1}$ we have

$$\liminf_{k\to+\infty}\sum_{i=1}^{m}\left(\sum_{s=\tau_i(k)}^{k-1}p_i(s)\right)=\frac{1}{e}. \tag{4.17}$$

By (4.15)–(4.17), for any $\varepsilon>0$ there exists $\Delta>0$ such that, if $\sum_{j=1}^{m}|\alpha_1-\alpha_j|<\Delta$, then

$$\frac{1-\varepsilon}{e}\le\liminf_{k\to+\infty}\sum_{i=1}^{m}\left(\sum_{j=\tau_i(k)}^{k-1}p_i(j)\right)\le\frac{1}{e}.$$

According to (4.13), it easy to see that the function $u(k)=k^{-\lambda}$ is a positive solution of (1.1). This example shows that (4.12) is an optimal condition.

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Received 08.08.13