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PERIODIC SOLUTIONS OF SECOND-ORDER NEUTRAL FUNCTIONAL DIFFERENTIAL SYSTEMS WITH TIME-VARYING OPERATOR AND DELAYS*

ПЕРІОДИЧНІ РОЗВ'ЯЗКИ НЕЙТРАЛЬНИХ ФУНКЦІОНАЛЬНО-ДИФЕРЕНЦІАЛЬНИХ СИСТЕМ ДРУГОГО ПОРЯДКУ З ЗАЛЕЖНИМ ВІД ЧАСУ ОПЕРАТОРОМ ТА ЗАПІЗНЕННЯМИ

Zhengxin Wang

College Sci., Nanjing Univ. Posts and Telecommunications
Nanjing 210003, China
and Research Center for Complex Systems and Network Sci. and Southeast Univ.
Nanjing 210096, China
e-mail: zwang@njupt.edu.cn

Jinde Cao

Research Center for Complex Systems and Network Sci. and Southeast Univ.
Nanjing 210096, China
and King Abdulaziz Univ.
Jeddah 21589, Saudi Arabia
e-mail: jdcao@seu.edu.cn

Shiping Lu

College Math. and Statistics, Nanjing Univ. Information Sci. and Technology
Nanjing 210044, China
and Anhui Normal Univ.
Wuhu 241000, China
e-mail: lushiping88@sohu.com

This paper aims to investigate the existence of periodic solutions for second-order neutral functional differential systems with time-varying operator and delays. The interesting issue is that the coefficient matrix of the neutral difference operator is not a constant matrix, which is different from the other literature. Some properties of the time-varying neutral difference operator and the results on the existence of periodic solutions of second-order neutral functional differential systems are obtained. Moreover, a numerical simulation is given to illustrate the theoretical results.

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Досліджується існування періодичних розв'язків функціонально-диференціальних систем другого порядку з залежним від часу оператором та запізненнями. При цьому матриця коефіцієнтів нейтрального різницевого оператора не є сталою на відміну від випадків, описаних у літературі. Отримано деякі властивості нейтрального змінного за часом різницевого оператора та результати щодо існування періодичних розв'язків нейтральних функціонально-диференціальних систем другого порядку. Наведено числовий приклад для ілюстрації теоретичних результатів.

1. Introduction. In recent years, the problem on the existence of periodic solutions to functional differential systems was extensively studied, see [1, 2, 4, 5, 7–12] and their reference lists. In [2], Các discussed the existence of 2π-periodic solutions of the systems

$$x'' + \frac{d}{dt}[\operatorname{grad} F(x(t))] + g(t, x(t)) = e(t),$$

under various asymptotic behaviors of g by applying a theorem in [12]. By using Mawhin's generalized continuation theorem, Ge [4] proved three theorems for the existence of harmonic solutions to the systems

$$\ddot{x} + \frac{d}{dt}\operatorname{grad} F(x) + \operatorname{grad} G(x) = p(t),$$

where $F \in C^2(R^n, R)$, $G \in C^1(R^n, R)$, $p \in C(R, R^n)$ and $p(t+T) \equiv p(t)$. Besides, Kiguradze and Mukhigulashvili [7] studied the existence and uniqueness of an ω-periodic solution of two-dimensional nonautonomous differential systems.

Recently, Agarwal and Chen [1] studied the periodic solutions of the first-order differential system

$$x' = G(t, x(t)), \quad x(0) = x(2\pi).$$

By applying the inverse function theorem, they obtained the existence and uniqueness results for periodic solutions. Mallet-Paret and Nussbaum [11] studied the stability of periodic solutions of state-dependent delay-differential equations of the form

$$\dot{z}(t) = g(z(t), z(t - r_1), \ldots, z(t - r_n)),$$

where $z : R \to R^m$, $r_i = r_i(z(t))$, $i = 1, 2, \ldots, n$.

On the other hand, neutral functional differential systems were also studied. In virtue of an extension of Mawhin's continuation theorem which was established by the author, Lu [8] studied the existence of periodic solutions to a second-order p-Laplacian neutral functional differential systems in the form

$$\frac{d}{dt}\varphi_p[(x(t) + Cx(t-\tau))'] = f(t, x(t), x(t-\mu(t)), x'(t)).$$

Lu and Ge [9] studied the following second-order neutral differential systems:

$$\frac{d^2}{dt^2}(x(t) + Cx(t-r)) + \frac{d}{dt}\operatorname{grad} F(x(t)) + \operatorname{grad} G(x(t-\tau(t))) = p(t),$$

which the coefficient matrix of the neutral difference operator being the constant matrix C. Under this and other assumptions, they presented some results on the existence of periodic solutions.

Very recently, Henríquez, Pierri, and Prokopczyk in [5] studied the existence of periodic solutions of an abstract neutral functional differential equations with finite and infinite delays of the form

$$\frac{d}{dt}(x(t) - Bx(t)) = Ax(t) + L(x_t) + f(t), \quad t \in R,$$

where $x(t) \in X$ and X is a Banach space. Lu, Xu and Xia [10] studied new properties of the D-operator which is described as

$$D(x_t) = x(t) - Bx(t-\tau),$$

and the existence of periodic solutions for the neutral functional differential equation

$$\frac{dDx_t}{dt} = f(t, x_t).$$

It is easy to see that the coefficient matrices C of the difference operator D, $[Dx](t) = x(t) + Cx(t-r)$, are all constant matrices in above studies including the zero matrix. In some cases, however, the coefficient matrix C is not a constant matrix which is related to the time-variable t. The studies of the existence of periodic solutions to functional differential systems with time-varying coefficient matrix $C(t)$ are rare. Therefore, it is worthwhile to study how to obtain the existence of periodic solution from the existing results. Since the difference operator contains a time-varying matrix $C(t)$, this paper is different from the literature.

Motivated by the above studies, this paper discusses the existence of periodic solutions to the following second-order neutral functional differential systems with time-varying coefficient matrix and deviating arguments:

$$\frac{d^2}{dt^2}(x(t) - C(t)x(t-r)) + \frac{d}{dt}\operatorname{grad} F(x(t)) + \operatorname{grad} G(x(t-\gamma(t))) = e(t), \tag{1.1}$$

where $F \in C^2(R^n, R)$, $G \in C^1(R^n, R)$, $e \in C(R, R^n)$ with $e(t+T) \equiv e(t)$ and $\int_0^T e(t)dt = \mathbf{0}$, where $\mathbf{0}$ is the n-dimensional zero vector, $\gamma \in C^1(R, R)$ and $\gamma(t+T) \equiv \gamma(t)$ for a constant $T > 0$, r is a constant. $C \in C^2(R, R^{n\times n})$ is a symmetric matrix for all $t \in R$. Because the coefficient matrix in the difference operator is not a constant matrix, the methods of estimating a priori bounds of periodic solutions in [2, 4, 8–10] can not be used directly. For this reason, in this paper, we analyze some properties of the linear difference operator $D : [Dx](t) = x(t) - C(t)x(t-r)$ firstly, and obtain some results on the existence of D^{-1} and on properties of D^{-1}. Then the existence results of periodic solutions to Eq. (1.1) are obtained by applying Mawhin's continuation theorem.

The main contributions of this paper include: (1) The difference operator contains a time-varying matrix, which is rare and different from the constant matrix case. The paper obtains new properties of time-varying difference operator. (2) The paper investigates periodic solutions of the neutral functional differential systems under the time-varying difference operator. The methods to estimate the *priori bounds* are different from those in the literature.

The rest of the paper is organized as follows. Section 2 introduces preliminary results. The point is that properties of time-varying difference operator are obtained. Section 3 investigates existence of periodic solutions of neutral functional differential system (1.1). Section 4 gives an example and two simulations to verify the theoretical results.

2. Preliminaries. In this section, we will make some preparations. At first, we recall Mawhin's continuation theorem [3].

Let X and Y be Banach spaces and $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero, here $D(L)$ denotes the domain of L. This means that $\operatorname{Im} L$ is closed in Y and $\dim \ker L = \dim(Y/\operatorname{Im} L) < +\infty$. Consider the supplementary subspaces X_1, Y_1 of X, Y respectively, such that $X = \ker L \bigoplus X_1$ and $Y = \operatorname{Im} L \bigoplus Y_1$, and let $P : X \to \ker L$ and $Q : Y \to Y_1$ be natural projectors. Clearly, $\ker L \cap (D(L) \cap X_1) = \{0\}$, thus the restriction $L_P := L|_{D(L)\cap X_1}$ is invertible. Denote by K_P the inverse of L_P.

Now, let Ω be an open bounded subset of X with $D(L) \cap \Omega \neq \varnothing$. A map $N : \overline{\Omega} \to Y$ is said to be L-compact in $\overline{\Omega}$, if $QN(\overline{\Omega})$ is bounded and the operator $K_P(I - Q)N : \overline{\Omega} \to X$ is compact.

Lemma 2.1 [3]. *Suppose that X and Y are two Banach spaces, and $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero. Furthermore, $\Omega \subset X$ is an open bounded set and $N : \overline{\Omega} \to Y$ is L-compact on $\overline{\Omega}$. If:*

(1) $Lx \neq \lambda Nx$, for all $x \in \partial\Omega \cap D(L)$, $\lambda \in (0, 1)$;

(2) $QNx \neq 0$, for all $x \in \partial\Omega \cap \ker L$;

(3) $\deg(JQN, \Omega \cap \ker L, 0) \neq 0$, where $J : \operatorname{Im} Q \to \ker L$ is an isomorphism.

Then the equation $Lx = Nx$ has a solution in $\overline{\Omega} \cap D(L)$.

We denote by $(\cdot, \cdot)$ the inner product in R^n, moreover, $|x| = \sqrt{(x, x)}$ is the Euclidean norm for $x \in R^n$. In order to apply Lemma 2.1, we take

$$X = \{x \in C^1(R, R^n) : x(t + T) \equiv x(t)\},$$

with the norm $\|x\| = \max_{t\in[0,T]} \sqrt{|x(t)|^2 + |x'(t)|^2}$, for all $x \in X$, and

$$Y = \{x \in C(R, R^n) : x(t + T) \equiv x(t)\},$$

with the norm $\|y\|_0 = \max_{t\in[0,T]} |y(t)|$, for all $y \in Y$. Then X and Y are both Banach spaces. Denote

$$I_n = \{1, 2, \ldots, n\}, \quad [x, y] = \int_0^T (x(t), y(t))dt, \quad \text{for all} \quad x, y \in Y.$$

A vector-valued function $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^\top$ is continuous if $x_i(t)$ are continuous functions for all $t \in R, i \in I_n$. Similarly, a matrix-valued function $A(t) = (a_{ij}(t))_{n\times n}$ is continuous if $a_{ij}(t)$ are continuous for all $t \in R$, $i, j \in I_n$, and the definitions of their derivatives and integrals are similar. Let

$$x'(t) = (x'_1(t), x'_2(t), \ldots, x'_n(t))^\top, \quad A'(t) = (a'_{ij}(t))_{n\times n},$$

$$\int_0^T x(t)dt = \left(\int_0^T x_1(t)dt, \int_0^T x_2(t)dt, \ldots, \int_0^T x_n(t)dt\right)^\top,$$

and

$$\int_0^T A(t)dt = \left(\int_0^T a_{ij}(t)dt\right)_{n\times n}.$$

Let $Z = \{A \in C(R, R^{n\times n}) : A(t+T) \equiv A(t)\}$ represent the set of all continuous matrix-valued functions, with the norm $\|A\|_m = \max_{t\in[0,T]} \|A(t)\|_F$, where

$$\|A(t)\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij}(t))^2}.$$

Then Z is a Banach space. Suppose that $C \in C^2(R, R^{n\times n})$ and $C(t) \equiv C(t+T)$. We denote

$$\widehat{C} = \max_{t\in[0,T]}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij}(t))^2}, \quad \widehat{C}_1 = \max_{t\in[0,T]}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}(a'_{ij}(t))^2}, \quad \widehat{C}_2 = \max_{t\in[0,T]}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}(a''_{ij}(t))^2}.$$

Then $\widehat{C}, \widehat{C}_1, \widehat{C}_2$ are all finite constants.

The following lemma holds by calculation, moreover, it also can be found in books on matrix analysis such as [6].

Lemma 2.2. *Let $a, b \in Y$ and $A, B \in Z$. Then the following relations hold:*
(1) $\|A+B\|_F \le \|A\|_F + \|B\|_F$,
(2) $\|AB\|_F \le \|A\|_F\|B\|_F$,
(3) $|Aa| \le \|A\|_F|a|$,
(4) $\|\epsilon A\|_F \le |\epsilon|\|A\|_F$, *where $\epsilon \in R$ is a constant,*
(5) $|a+b| \le |a| + |b|$.

These results, as well as conclusion (3), which shows compatibility of the Frobenius norm and the Euclidean norm, are very crucial to obtain *a priori bounds* for periodic solutions to Eq. (1.1).

Lemma 2.3 [9]. *Let $0 \le \alpha \le T$ be a constant, $s \in C(R,R)$ with $s(t+T) \equiv s(t)$ and* $\max_{t\in[0,T]} |s(t)| \le \alpha$, *then for all $x \in X$ we have*

$$\int_0^T |x(t) - x(t-s(t))|^2 dt \le 2\alpha^2 \int_0^T |x'(t)|^2 dt.$$

Defining an operator D as follows:

$$D : Y \to Y, \quad [Dx](t) = x(t) - C(t)x(t-r). \tag{2.1}$$

It is easy to see that D is a continuous linear operator with $\|D\| \le 1 + \widehat{C}$.

Lemma 2.4. *If $\widehat{C} < 1$, then D has its continuous inverse D^{-1} with the following properties:*
(1) $\|D^{-1}\| \le \dfrac{1}{1-\widehat{C}}$,

(2) $[D^{-1}f](t) = f(t) + \sum_{j=1}^{\infty}\prod_{i=1}^{j} C(t-(i-1)r)f(t-jr)$, *for all* $f \in Y$,

(3) $|[D^{-1}f](t)| \leq \dfrac{\|f\|_0}{1-\widehat{C}}$, *for all* $f \in Y$,

(4) $\displaystyle\int_0^T |[D^{-1}f](t)|dt \leq \frac{1}{1-\widehat{C}}\int_0^T |f(t)|dt$, *for all* $f \in Y$,

(5) $[Df'](t) = [Df]'(t) + C'(t)f(t-r)$, *for all* $f \in X$.

Proof. Define a linear operator

$$E : Y \to Y, \quad [Ex](t) = C(t)x(t-r).$$

Then $|[Ex](t)| \leq \|C(t)\|_F|x(t-r)| \leq \widehat{C}\|x\|_0$, so $\|E\| \leq \widehat{C}$. Moreover, for all $f \in Y$,

$$[E^j f](t) = \prod_{i=1}^{j} C(t-(i-1)r)f(t-jr),$$

which yields

$$\left|\sum_{j=1}^{\infty}[E^j f](t)\right| \leq \left|\sum_{j=1}^{\infty}\prod_{i=1}^{j} C(t-(i-1)r)f(t-jr)\right| \leq$$

$$\leq \sum_{j=1}^{\infty}\prod_{i=1}^{j} \|C(t-(i-1)r)\|_F|f(t-jr)| \leq$$

$$\leq \sum_{j=1}^{\infty}\widehat{C}^j\|f\|_0 \leq \frac{\widehat{C}}{1-\widehat{C}}\|f\|_0.$$

Since $D = id - E$ and $\|E\| \leq \widehat{C} < 1$ it follows that

$$D^{-1} : Y \to Y, \quad D^{-1} = (id - E)^{-1} = id + \sum_{j=1}^{\infty} E^j,$$

and $\|D^{-1}\| \leq \dfrac{1}{1-\|E\|} \leq \dfrac{1}{1-\widehat{C}}$. For all $f \in Y$,

$$[D^{-1}f](t) = f(t) + \sum_{j=1}^{\infty}[E^j f](t) = f(t) + \sum_{j=1}^{\infty}\prod_{i=1}^{j} C(t-(i-1)r)f(t-jr),$$

furthermore,

$$|[D^{-1}f](t)| = \left|f(t) + \sum_{j=1}^{\infty}[E^j f](t)\right| \leq \frac{\|f\|_0}{1-\widehat{C}}.$$

On the other hand, for all $f \in Y$,

$$\int_0^T |[D^{-1}f](t)|dt = \int_0^T \left| f(t) + \sum_{j=1}^{\infty} \prod_{i=1}^{j} C(t-(i-1)r) f(t-jr) \right| dt \leq$$

$$\leq \int_0^T |f(t)|dt + \int_0^T \sum_{j=1}^{\infty} \left\| \prod_{i=1}^{j} C(t-(i-1)r) \right\|_F |f(t-jr)|dt \leq$$

$$\leq \int_0^T |f(t)|dt + \sum_{j=1}^{\infty} \widehat{C}^j \int_0^T |f(t)|dt \leq \frac{1}{1-\widehat{C}} \int_0^T |f(t)|dt.$$

Therefore, the conclusions (1) to (4) hold, and the conclusion (5) can be obtained by a direct calculation.

Lemma 2.4 is proved.

Now, we define

$$L : D(L) \subset X \to Y, \quad Lx = \frac{d^2(Dx)}{dt^2}, \tag{2.2}$$

where $D(L) = \{x : x \in C^2(R, R^n), x(t+T) \equiv x(t)\}$ and

$$N : X \to Y, \quad [Nx](t) = -\frac{d}{dt} \operatorname{grad} F(x(t)) - \operatorname{grad} G(x(t-\gamma(t))) + e(t). \tag{2.3}$$

Let $x \in \ker L$, one has $\dfrac{d^2(Dx)}{dt^2} = 0$, that is, $x(t) - C(t)x(t-r) = \omega_1 t + \omega_2$, where $\omega_1, \omega_2 \in R^n$. Since $x(t) - C(t)x(t-r)$ is T-periodic, we have $\omega_1 = 0$. Let $\psi_i(t)$ be a solution to $x(t) - C(t)x(t-r) = e_i$, where $e_i = (\underbrace{0, 0, \ldots, 1}_{i}, \ldots, 0)^\top \in R^n$. Let $\Psi(t) = (\psi_1(t), \psi_2(t), \ldots, \psi_n(t)) \in R^{n\times n}$, $K = (k_1, \ldots, k_n)^\top \in R^n$. Then for all $\omega = \sum_{i=1}^n k_i e_i \in R^n$, $\Psi(t)K$ is a solution to $x(t) - C(t)x(t-r) = \omega$. Therefore, $\ker L = \{\Psi(t)K : K \in R^n\}$. Furthermore, $\ker L = \operatorname{codim} \operatorname{Im} L = R^n$, $\operatorname{Im} L = \left\{ y \in Y : \int_0^T y(t)\, dt \right\}$. Hence, L is a Fredholm operator with index zero. Define operators P, Q as follows, respectively,

$$P : X \to \ker L, \quad Px = \Psi(t) \frac{\int_0^T \Psi(t)x(t)dt}{\int_0^T \|\Psi(t)\|_F^2 dt}, \quad Q : Y \to \operatorname{Im} Q, \quad Qy = \frac{1}{T} \int_0^T y(t)dt.$$

Then $\operatorname{Im} P = \ker L$, $\ker Q = \operatorname{Im} L$. Let K_P represent the inverse of $L_{\ker P \cap D(L)}$, so $K_P : \operatorname{Im} L \to D(L) \cap \ker P$. Since $\operatorname{Im} L \subset Y$ and $D(L) \cap \ker P \subset X$, K_P is an embedding operator. Furthermore, K_P is a completely operator in $\operatorname{Im} L$, which together with (2.3), makes it is easy to see that N is L-compact on $\overline{\Omega}$, where Ω is an arbitrary open bounded subset of X.

For the sake of convenience, we list the following assumptions.

(H_1) There is a positive constant η such that $(H(v)x, x) \geq \eta|x|^2$, where $H(x) = \dfrac{\partial^2 F(x)}{\partial x^2}$, for all $v, x \in R^n$.

(H_2) There is a positive constant M such that

$$x_i \frac{\partial G}{\partial x_i} > 0 \quad \text{or} \quad x_i \frac{\partial G}{\partial x_i} < 0, \quad \text{for all} \quad i \in I_n \quad \text{with} \quad |x_i| > M.$$

(H_3) There is a positive constant L such that $|\operatorname{grad} G(x) - \operatorname{grad} G(y)| \leq L|x - y|$, for all $x, y \in R^n$.

(H_4) There is an integer $m \geq 0$ such that $|\gamma(t) - mT| \leq \alpha$ and $\gamma'(t) < 1$ for all $t \in [0, T]$.

Since the matrices of $H(x)$ and $C(t)$ are symetric, it follows that $C'(t)$, $H(x)C(t)C(t)H(x)$ and $H(x)C'(t)C'(t)H(x)$ are also symmetric for all $x \in R^n$, $t \in R$. Therefore the matrices $C(t)$, $C'(t)$, $H(x)C(t)C(t)H(x)$ and $H(x)C'(t)C'(t)H(x)$ have n real eigenvalues respectively. Furthermore, the maximums of n eigenvalues of $C(t)$ and $C'(t)$ are finite respectively, which are denoted by λ_0 and λ_1, and $\lambda_0 \leq \widehat{C}$, $\lambda_1 \leq \widehat{C}_1$ from [6]. Throughout this paper, we denote the eigenvalues of $H(x)C(t)C(t)H(x)$ and $H(x)C'(t)C'(t)H(x)$ by $\mu_1, \mu_2, \ldots, \mu_n$ and $\mu'_1, \mu'_2, \ldots, \mu'_n$ respectively, moreover, we assume that $\mu_M := \max_{i \in I_n}\{\max_{x \in R^n, t \in R} |\mu_i|\} < +\infty$ and $\mu'_M := := \max_{i \in I_n}\{\max\limits_{x \in R^n, t \in R} |\mu'_i|\} < +\infty$.

3. Main results. Based on properties of the difference operator D and Mawhin's continuation theorem, some results on the existence of periodic solutions of the neutral functional differential systems (1.1) are obtained.

Theorem 3.1. *Under the assumptions* (H_1)–(H_4), *Eq. (1.1) has at least one* T*-periodic solution if* $\widehat{C} < \dfrac{1}{n+1}$ *and* $\eta > \sqrt{n\mu'_M}T + \sqrt{\mu_M} + \sqrt{2}L\alpha + \sqrt{2n\lambda_1}\alpha LT + \sqrt{2\lambda_0}\alpha L + n\sqrt{\lambda_1}L_1T^2 + +\sqrt{n\lambda_0}L_1T$.

Proof. Let $\Omega_1 = \{x \in D(L) \cap X : Lx = \lambda Nx, \lambda \in (0, 1)\}$. If $x(\cdot) \in \Omega_1$, then from (2.2) and (2.3), we have

$$\frac{d^2}{dt^2}(x(t) - C(t)x(t-r)) + \lambda \frac{d}{dt} \operatorname{grad} F(x(t)) + \lambda \operatorname{grad} G(x(t - \gamma(t))) = \lambda e(t). \tag{3.1}$$

Integrating both sides of Eq. (3.1) over $[0, T]$, we obtain $\displaystyle\int_0^T \operatorname{grad} G(x(t - \gamma(t)))dt = 0$, that is, for all $i \in I_n$,

$$\int_0^T \frac{\partial G(x(t - \gamma(t)))}{\partial x_i}\, dt = 0. \tag{3.2}$$

Now, we claim that there exists a constant $t_{0i} \in R$ (which depends on the subscript i) such that

$$|x_i(t_{0i})| \leq M, \quad \text{for all} \quad i \in I_n. \tag{3.3}$$

In fact, if $x_i(t-\gamma(t)) > M\ \forall t \in [0,T]$, $i \in I_n$, then from (H_2) we know

$$\int\limits_0^T \frac{\partial G(x(t-\gamma(t)))}{\partial x_i}\, dt \neq 0,$$

which is in contradiction to formula (3.2). So there exists a constant $\xi_i \in [0,T]$ (which is related to the subscript i) such that

$$x_i(\xi_i-\gamma(\xi_i)) \le M \quad \forall i \in I_n. \tag{3.4}$$

For the same reason, there exists a constant $\eta_i \in [0,T]$ (which is related to the subscript i) such that

$$x_i(\eta_i-\gamma(\eta_i)) \ge -M \quad \forall i \in I_n. \tag{3.5}$$

Case 1. If $x_i(\xi_i-\gamma(\xi_i)) < -M$, then according to (3.5) and the intermediate value theorem for a continuous function we know there exists a constant t_{1i} between $\xi_i-\gamma(\xi_i)$ and $\eta_i-\gamma(\eta_i)$ such that $x_i(t_{1i}) = -M$, $i \in I_n$. Take $t_{0i} = t_{1i}$.

Case 2. If $x_i(\xi_i-\gamma(\xi_i)) \ge -M$, then according to (3.4) we obtain $|x_i(\xi_i-\gamma(\xi_i))| \le M$, $i \in I_n$. Take $t_{0i} = \xi_i-\gamma(\xi_i)$.

Combining Cases 1 and 2, it is easy to see that (3.3) holds. Let $t_{2i} = k_iT + t_{0i}$, where k_i is an integer and $t_{2i} \in [0,T]$, $i \in I_n$, so $|x_i(t_{2i})| = |x_i(t_{0i})| \le M\ \forall i \in I_n$. Therefore,

$$|x_i(t)| \le |x_i(t_{2i})| + \int\limits_{t_{2i}}^{t} |x_i'(t)dt| \le M + \int\limits_0^T |x_i'(t)|\, dt \le M + \int\limits_0^T |x'(t)|\, dt \ \ \forall t \in [0,T], \ \ i \in I_n. \tag{3.6}$$

Moreover,

$$\|x\|_0 \le \sqrt{n}M + \sqrt{n}\int\limits_0^T |x'(t)|\, dt. \tag{3.7}$$

It follows from Eq. (3.1) that

$$\left[\frac{d^2}{dt^2}\left(x(t)-C(t)x(t-r)\right) + \lambda\frac{d}{dt}\,\mathrm{grad}F(x(t)) + \right.$$
$$\left. + \lambda\,\mathrm{grad}\,G(x(t-\gamma(t))), \frac{d}{dt}\left(x(t)-C(t)x(t-r)\right)\right] =$$
$$= \left[\lambda e(t), \frac{d}{dt}\left(x(t)-C(t)x(t-r)\right)\right]. \tag{3.8}$$

As $[x''(t), x'(t)] = 0$, $[\operatorname{grad} G(x(t)), x'(t)] = 0$, it follows from (3.8) that

$$\begin{aligned}
\eta \int_0^T |x'(t)|^2 dt \le |[H(x(t))x'(t), x'(t)]| \le |[H(x(t))x'(t), C'(t)x(t-r) + C(t)x'(t-r)]| + \\
+ |[\operatorname{grad} G(x(t-\gamma(t))), x'(t) - C'(t)x(t-r) - C(t)x'(t-r)]| + \\
+ |[e(t), x'(t) - C'(t)x(t-r) - C(t)x'(t-r)]| \le \\
\le |[H(x(t))x'(t), C'(t)x(t-r)]| + |[H(x(t))x'(t), C(t)x'(t-r)]| + \\
+ |[\operatorname{grad} G(x(t-\gamma(t))) - \operatorname{grad} G(x(t)), x'(t)]| + \\
+ |[\operatorname{grad} G(x(t-\gamma(t))) - \operatorname{grad} G(x(t)), C'(t)x(t-r)]| + \\
+ |[\operatorname{grad} G(x(t)), C'(t)x(t-r)]| + \\
+ |[\operatorname{grad} G(x(t-\gamma(t))) - \operatorname{grad} G(x(t)), C(t)x'(t-r)]| + \\
+ |[\operatorname{grad} G(x(t)), C(t)x'(t-r)]| + \\
+ |[e(t), x'(t)]| + |[e(t), C'(t)x(t-r)]| + |[e(t), C(t)x'(t-r)]|. \qquad (3.9)
\end{aligned}$$

We have

$$\begin{aligned}
|[H(x(t))x'(t), C'(t)x(t-r)]| &= |[C'^{\top}(t)H(x(t))x'(t), x(t-r)]| = \\
&= |[C'(t)H(x(t))x'(t), x(t-r)]| \le \\
&\le |[C'(t)H(x(t))x'(t), C'(t)H(x(t))x'(t)]|^{\frac{1}{2}} |[x(t), x(t)]|^{\frac{1}{2}} \le \\
&\le \sqrt{\mu'_M} \left(\int_0^T |x'(t)|^2 \, dt \right)^{\frac{1}{2}} \left(\int_0^T |x(t)|^2 \, dt \right)^{\frac{1}{2}}, \qquad (3.10)
\end{aligned}$$

and

$$\begin{aligned}
|[H(x(t))x'(t), C(t)x'(t-r)]| &= |[C(t)H(x(t))x'(t), x'(t-r)]| \le \\
&\le |[C(t)H(x(t))x'(t), C(t)H(x(t))x'(t)]|^{\frac{1}{2}} |[x'(t), x'(t)]|^{\frac{1}{2}} \le \\
&\le \sqrt{\mu_M} \int_0^T |x'(t)|^2 \, dt. \qquad (3.11)
\end{aligned}$$

Let $\gamma(t) - mT = s(t)$, then by Lemma 2.3 we can obtain the following inequalities:

$$|[\operatorname{grad} G(x(t-\gamma(t))) - \operatorname{grad} G(x(t)), x'(t)]| \le L \int_0^T |x(t-s(t)) - x(t)||x'(t)|\,dt \le$$
$$\le L \left(\int_0^T |x(t-s(t)) - x(t)|^2 dt \right)^{\frac{1}{2}} \times$$
$$\times \left(\int_0^T |x'(t)|^2 dt \right)^{\frac{1}{2}} \le \sqrt{2} L\alpha \int_0^T |x'(t)|^2\,dt, \tag{3.12}$$

$$|[\operatorname{grad} G(x(t-\gamma(t))) - \operatorname{grad} G(x(t)), C'(t)x(t-r)]| \le \sqrt{\lambda_1} \left(\int_0^T |x(t)|^2 dt \right)^{\frac{1}{2}} \times$$
$$\times \left(\int_0^T |\operatorname{grad} G(x(t-\gamma(t))) - \operatorname{grad} G(x(t))|^2 dt \right)^{\frac{1}{2}} \le$$
$$\le \sqrt{2\lambda_1} L\alpha \left(\int_0^T |x'(t)|^2 dt \right)^{\frac{1}{2}} \left(\int_0^T |x(t)|^2 dt \right)^{\frac{1}{2}}, \tag{3.13}$$

and

$$|[\operatorname{grad} G(x(t-\gamma(t))) - \operatorname{grad} G(x(t)), C(t)x'(t-r)]| \le \sqrt{2\lambda_0} L\alpha \int_0^T |x'(t)|^2\,dt. \tag{3.14}$$

From assumption (H_3),

$$|\operatorname{grad} G(x)| \le L|x| + |\operatorname{grad} G(0)| := L|x| + g_0 \le \begin{cases} L + g_0 := g_1, & \text{for} \quad |x| \le 1, \\ (L + g_0)|x| := L_1|x|, & \text{for} \quad |x| > 1. \end{cases}$$

Let $\triangle_1 = \{t \in [0,T] : |x(t)| \leq 1\}$, $\triangle_2 = \{t \in [0,T] : |x(t)| > 1\}$, then

$$|[\operatorname{grad} G(x(t)), C'(t)x(t-r)]| \leq |[\operatorname{grad} G(x(t)), \operatorname{grad} G(x(t))]|^{\frac{1}{2}} |[C'(t)x(t-r), C'(t)x(t-r)]|^{\frac{1}{2}} \leq$$

$$\leq \sqrt{\lambda_1} \left(\int_0^T |x(t)|^2 dt \right)^{\frac{1}{2}} \left(\left(\int_{\triangle_1} + \int_{\triangle_2} \right) |\operatorname{grad} G(x(t))|^2 dt \right)^{\frac{1}{2}} \leq$$

$$\leq \sqrt{\lambda_1} \left(\int_0^T |x(t)|^2 dt \right)^{\frac{1}{2}} \left(g_1^2 T + L_1^2 \int_0^T |x(t)|^2 dt \right)^{\frac{1}{2}} \leq$$

$$\leq g_1 \sqrt{\lambda_1 T} \left(\int_0^T |x(t)|^2 dt \right)^{\frac{1}{2}} + L_1 \sqrt{\lambda_1} \int_0^T |x(t)|^2 \, dt, \tag{3.15}$$

and

$$|[\operatorname{grad} G(x(t)), C(t)x'(t-r)]| \leq \sqrt{\lambda_0} \left(\int_0^T |x'(t)|^2 dt \right)^{\frac{1}{2}} \left(\left(\int_{\triangle_1} + \int_{\triangle_2} \right) |\operatorname{grad} G(x(t))|^2 \, dt \right)^{\frac{1}{2}} \leq$$

$$\leq \sqrt{\lambda_0} \left(\int_0^T |x'(t)|^2 \, dt \right)^{\frac{1}{2}} \left(g_1^2 T + L_1^2 \int_0^T |x(t)|^2 \, dt \right)^{\frac{1}{2}} \leq$$

$$\leq g_1 \sqrt{\lambda_0 T} \left(\int_0^T |x'(t)^2 \, dt \right)^{\frac{1}{2}} +$$

$$+ L_1 \sqrt{\lambda_0} \left(\int_0^T |x'(t)|^2 dt \right)^{\frac{1}{2}} \left(\int_0^T |x(t)|^2 dt \right)^{\frac{1}{2}}. \tag{3.16}$$

Furthermore,

$$|[e(t), x'(t)]| + |[e(t), C'(t)x(t-r)]| + |[e(t), C(t)x'(t-r)]| \leq$$

$$\leq \widetilde{e} \left(\int_0^T |x'(t)^2 \, dt \right)^{\frac{1}{2}} + \widetilde{e} \sqrt{\lambda_1} \left(\int_0^T |x(t)|^2 \, dt \right)^{\frac{1}{2}} + \widetilde{e} \sqrt{\lambda_0} \left(\int_0^T |x'(t)|^2 \, dt \right)^{\frac{1}{2}}, \tag{3.17}$$

where $\widetilde{e} = \left(\int_0^T |e(t)^2\,dt\right)^{\frac{1}{2}}$. Substituting (3.7) and (3.10) – (3.17) into (3.9) we obtain

$$\eta\int\limits_0^T |x'(t)|^2 dt \le \left(T\sqrt{n\mu'_M}+\sqrt{\mu_M}+L\alpha\sqrt{2}+\alpha LT\sqrt{2n\lambda_1}+\alpha L\sqrt{2\lambda_0}+nL_1T^2\sqrt{\lambda_1}+L_1T\sqrt{n\lambda_0}\right)\times$$

$$\times\int\limits_0^T |x'(t)|^2 dt + \left(M\sqrt{n\mu'_M T} + \alpha LM\sqrt{2nT\lambda_1} + 2nML_1T^{\frac{3}{2}}\sqrt{\lambda_1} + g_1T^{\frac{3}{2}}\sqrt{n\lambda_1} + \right.$$

$$\left. + g_1\sqrt{\lambda_0 T} + L_1M\sqrt{nT\lambda_0} + \widetilde{e} + \widetilde{e}T\sqrt{n\lambda_1} + \widetilde{e}\sqrt{\lambda_0}\right)\left(\int\limits_0^T |x'(t)|^2\,dt\right)^{\frac{1}{2}} +$$

$$+ nTM^2L_1\sqrt{\lambda_1} + g_1MT\sqrt{n\lambda_1} + \widetilde{e}M\sqrt{n\lambda_1 T}. \tag{3.18}$$

By $\eta > T\sqrt{n\mu'_M}+\sqrt{\mu_M}+L\alpha\sqrt{2}+\alpha LT\sqrt{2n\lambda_1}+\alpha L\sqrt{2\lambda_0}+nL_1T^2\sqrt{\lambda_1}+L_1T\sqrt{n\lambda_0}$ and (3.18), $\int_0^T |x'(t)|^2\,dt$ is bounded. So there exists a positive constant M_1 such that

$$\int\limits_0^T |x'(t)|^2\,dt \le M_1, \tag{3.19}$$

which, together with (3.7), gives

$$\|x\|_0 \le \sqrt{n}M + \sqrt{nTM_1}. \tag{3.20}$$

From the definition of the operator D we know that, for $x \in D(L)$,

$$[Dx''](t) = [Dx]''(t) + 2C'(t)x'(t-r) + C''(t)x(t-r),$$

which, together with Eq. (3.1), yields

$$x''(t) + D^{-1}\left[\lambda\frac{d}{dt}\operatorname{grad}F(x(t)) + \lambda\operatorname{grad}G(x(t-\gamma(t)))\right] =$$

$$= D^{-1}[\lambda e(t) + 2C'(t)x'(t-r) + C''(t)x(t-r)],$$

so by Lemma 2.4 we obtain

$$\int_0^T |x''(t)|^2\,dt \le \frac{1}{1-\widehat{C}}\left(\int_0^T \|H(x(t))\|_F|x'(t)|\,dt + \int_0^T |\mathrm{grad}\,G(x(t-\gamma(t)))|\,dt + \int_0^T |e(t)|\,dt + \right.$$

$$\left. + 2\int_0^T \|C'(t)\|_F|x'(t-r)|dt + \int_0^T \|C''(t)\|_F|x(t-r)|dt\right) \le$$

$$\le \frac{1}{1-\widehat{C}}\,[H_0\sqrt{TM_1} + G_0T + \widetilde{e}\sqrt{T} + 2\widehat{C}_1\sqrt{TM_1}\widehat{C}_2(\sqrt{n}M + \sqrt{nTM_1})T] := M_2, \tag{3.21}$$

where H_0 and G_0 are the maximums of $\|H(x(t))\|_F$ and $|\mathrm{grad}\,G(x(t-\gamma(t)))|$ on $\{x : \|x\|_0 \le \le \sqrt{n}M + \sqrt{nTM_1}\}$. Therefore,

$$|x'(t)| = \left(\sum_{i=1}^n |x_i'(t)|^2\right)^{\frac{1}{2}} \le \left(T\sum_{i=1}^n \int_0^T |x_i''(t)|^2 dt\right)^{\frac{1}{2}} \le \sqrt{nTM_2} \quad \forall t \in [0,T],$$

that is, $\|x'\|_0 \le \sqrt{nTM_2}$, which together with (3.20), shows that $\|x\| \le \sqrt{n}M + \sqrt{nTM_1} + +\sqrt{nTM_2}$.

Let $\Omega_2 = \{x \in X : QNx = 0, x \in \ker L\}$. If $x \in \Omega_2$, then $x = \Psi(t)K, K \in R^n$ satisfying

$$-\frac{1}{T}\int_0^T \mathrm{grad}\,G(x(t-\gamma(t)))dt \equiv -\frac{1}{T}\int_0^T \frac{1}{1-\gamma'(\mu(t))}\,\mathrm{grad}\,G(\Psi(t)K)\,dt = \mathbf{0},$$

where $\mu(s)$ is the inverse of $t-\gamma(t)$.

In view of $\psi_i(t) = [D^{-1}e_i](t) = e_i + \sum_{j=1}^\infty \prod_{k=1}^j C(t-(k-1)r)e_i$, we have

$$|k_i\psi_{ii}(t)| = \left|k_i + \sum_{j=1}^\infty k_i\left(\prod_{k=1}^j C(t-(k-1)r)e_i\right)_i\right| \ge$$

$$\ge |k_i| - |k_i|\sum_{j=1}^\infty\prod_{k=1}^j \widehat{C} = |k_i| - |k_i|\frac{\widehat{C}}{1-\widehat{C}} := (1-\rho)|k_i|,$$

and

$$|\psi_{il}(t)| = \left|\sum_{j=1}^\infty\left(\prod_{k=1}^j C(t-(k-1)r)e_i\right)_l\right| \le \sum_{j=1}^\infty\prod_{k=1}^j \widehat{C} = \frac{\widehat{C}}{1-\widehat{C}} := \rho,$$

where $l \ne i$, for all $t \in [0,T]$. Then $|k_i| - \rho\sum_{j=1}^n |k_j| \le M$. Otherwise, if $|k_i| - \rho\sum_{j=1}^n |k_j| > M$, then

$$|(\Psi(t)K)_i| = \left|\sum_{j=1}^n k_j\psi_{ji}(t)\right| = \left|k_i\psi_{ii}(t) - \sum_{j\ne i}^n k_j\psi_{ji}(t)\right| \ge$$

$$\geq |k_i| - \rho|k_i| - \rho\sum_{j\neq i}^{n}|k_j| = |k_i| - \rho\sum_{j=1}^{n}|k_j| > M,$$

for all $t \in [0,T]$, and

$$\int_0^T \frac{1}{1-\gamma'(\mu(t))}\,\mathrm{grad}\,G(\Psi(t)K)dt \neq \mathbf{0},$$

by assumption (H_2). This is a contradiction. Therefore, $|k_i| - \rho\sum_{j=1}^{n}|k_j| \leq M$, which together with $\sum_{j=1}^{n}|k_j| \leq \sqrt{n}|K|$, gives $|k_i| - \sqrt{n}\rho|K| \leq M$, that is, $|k_i| \leq \sqrt{n}\rho|K| + M$. Furthermore,

$$|K| = \left(\sum_{j=1}^{n}|k_j|^2\right)^{\frac{1}{2}} \leq \sqrt{n}(\sqrt{n}\rho|K| + M) = n\rho|K| + \sqrt{n}M,$$

therefore, $|K| \leq \dfrac{\sqrt{n}M}{1-n\rho}$. On the other hand,

$$\Psi(t) = \big([D^{-1}e_1](t), [D^{-1}e_2](t), \ldots, [D^{-1}e_n](t)\big) =$$

$$= \left(e_1 + \sum_{j=1}^{\infty}\prod_{k=1}^{j}C(t-(k-1)r)e_1, \ldots, e_n + \sum_{j=1}^{\infty}\prod_{k=1}^{j}C(t-(k-1)r)e_n\right) =$$

$$= I + \sum_{j=1}^{\infty}\prod_{k=1}^{j}C(t-(k-1)r)I,$$

where I is the identity matrix, moreover,

$$\|\Psi\|_m = \max_{t\in[0,T]}\|\Psi(t)\|_F \leq \|I\|_F + \sum_{j=1}^{\infty}\prod_{k=1}^{j}\max_{t\in[0,T]}\|C(t-(k-1)r)\|_F\|I\|_F \leq$$

$$\leq \sqrt{n} + \sum_{j=1}^{\infty}\prod_{k=1}^{j}\widehat{C}\sqrt{n} = \sqrt{n}\frac{1}{1-\widehat{C}} = \frac{\sqrt{n}\rho}{\widehat{C}}.$$

Then $\|x\|_0 = \max_{t\in[0,T]}|\Psi(t)K| \leq \|\Psi\|_m|K| \leq \dfrac{\sqrt{n}\rho}{\widehat{C}}\dfrac{\sqrt{n}M}{1-n\rho} = \dfrac{n\rho M}{\widehat{C}(1-n\rho)} := M_3$.

Now, if we let $\Omega = \{x : x \in X, \|x\| \leq \sqrt{nTM_1} + \sqrt{nTM_2} + M_3 + \sqrt{n}M\}$, then $\Omega_1 \cup \Omega_2 \subset \Omega$. So the condition (1) and condition (2) of Lemma 2.1 are satisfied.

If $x \in \partial\Omega \cap \ker L$, then $x(t) = \Psi(t)K$, $K \in R^n$ with $\|x\|_0 > \sqrt{n}M$, hence there exists an $i \in I_n$ such that $|(\Psi(t)K)_i| > M$, $t \in [0,T]$. Otherwise, $|(\Psi(t)K)_i| \leq M$, $t \in [0,T]$, for all $i \in I_n$, then $\|x\|_0 = \max_{t\in[0,T]}\left(\sum_{i=1}^{n}|(\Psi(t)K)_i|^2\right)^{\frac{1}{2}} \leq \sqrt{n}M$. It is a contradiction. It follows from $|(\Psi(t)K)_i| > M$, $t \in [0,T]$ that $QNx = -\dfrac{1}{T}\displaystyle\int_0^T \frac{1}{1-\gamma'(\mu(t))}\,\mathrm{grad}\,G(\Psi(t)K)dt \neq \mathbf{0}$ and $\mathrm{sgn}\left(x_i\dfrac{\partial G(x)}{\partial x_i}\right) = (-1)^{s_i}, s_i \in \{0,1\}, i \in I_n$.

Let

$$H(\mu, x) = \mu Sx + (1-\mu)JQNx,$$

where $S = \operatorname{diag}\{(-1)^{s_1}, (-1)^{s_2}, \ldots, (-1)^{s_n}\}$ and $J(x) = x$ for all $x \in R^n$. Therefore, for all $(\mu, x) \in [0,1] \times \partial\Omega \cap \ker L$, $H(\mu, x) \neq 0$ and

$$\deg\{JQN, \Omega \cap \ker L, 0\} = \deg\{H(0,\cdot), \Omega \cap \ker L, 0\} = \deg\{H(1,\cdot), \Omega \cap \ker L, 0\} =$$

$$= \deg\{id, \Omega \cap \ker L, 0\} \neq 0.$$

Now, we know that conditions in Lemma 2.1 are all satisfied, therefore $Lx = Nx$ has at least one solution in X.

Theorem 3.1 is proved.

If we replace the assumption (H_3) with the following assumption:

(H_5) There is a positive constant ℓ such that $|\operatorname{grad} G(x)| \leq \ell|x| + w$, for all $x \in R^n$.

Then the following theorem holds.

Theorem 3.2. *Under the assumptions* (H_1), (H_2) *and* (H_5), *Eq. (1.1) has at least one* T-*periodic solution if* $\widehat{C} < \dfrac{1}{n+1}$ *and* $\eta > \sqrt{n\mu'_M}T + \sqrt{\mu_M} + \ell_1 T + \ell_1 n T^2\sqrt{\lambda_1} + \ell_1 T\sqrt{\lambda_0}$.

Proof. From assumption (H_5) we obtain

$$|\operatorname{grad} G(x)| \leq \ell|x| + w \leq \begin{cases} \ell + w := w_1, & \text{for} \quad |x| \leq 1, \\ (\ell + w)|x| := \ell_1|x|, & \text{for} \quad |x| > 1. \end{cases} \tag{3.22}$$

It follows from (3.8) that

$$\begin{aligned}
\eta\int_0^T |x'(t)|^2 dt &\leq |[H(x(t))x'(t), x'(t)]| \leq |[H(x(t))x'(t), C'(t)x(t-r) + C(t)x'(t-r)]| + \\
&+ |[\operatorname{grad} G(x(t-\gamma(t))), x'(t) - C'(t)x(t-r) - C(t)x'(t-r)]| + \\
&+ |[e(t), x'(t) - C'(t)x(t-r) - C(t)x'(t-r)]| \leq \\
&\leq |[H(x(t))x'(t), C'(t)x(t-r)]| + |[H(x(t))x'(t), C(t)x'(t-r)]| + \\
&+ |[\operatorname{grad} G(x(t-\gamma(t))), x'(t)]| + |[\operatorname{grad} G\,(x(t-\gamma(t))), C'(t)x(t-r)]| + \\
&+ |[\operatorname{grad} G(x(t-\gamma(t))), C(t)x'(t-r)]| + |[e(t), x'(t)]| + \\
&+ |[e(t), C'(t)x(t-r)]| + |[e(t), C(t)x'(t-r)]|.
\end{aligned} \tag{3.23}$$

Let $\nabla_1 = \{t \in [0,T] : |x(t-\gamma(t))| \leq 1\}$, $\nabla_2 = \{t \in [0,T] : |x(t-\gamma(t))| > 1\}$. Then it follows from (3.7) and (3.22) that

$$|[\operatorname{grad} G(x(t-\gamma(t))), x'(t)]| \leq \left(\int_0^T |x'(t)|^2 dt\right)^{\frac{1}{2}} \left(\left(\int_{\nabla_1} + \int_{\nabla_2}\right) \operatorname{grad} G(x(t-\gamma(t)))|^2 dt\right)^{\frac{1}{2}} \leq$$

$$\leq \left(\int_0^T |x'(t)|^2 dt\right)^{\frac{1}{2}} \left(w_1^2 T + \ell_1^2 \|x\|_0^2 T\right)^{\frac{1}{2}} \leq$$

$$\leq \left(\int_0^T |x'(t)|^2 dt\right)^{\frac{1}{2}} \left(w_1\sqrt{T} + \ell_1 M\sqrt{nT} + \ell_1 T \left(\int_0^T |x'(t)|^2 dt\right)^{\frac{1}{2}}\right) \leq$$

$$\leq \left(w_1\sqrt{T} + \ell_1 M\sqrt{nT}\right) \left(\int_0^T |x'(t)|^2 dt\right)^{\frac{1}{2}} + \ell_1 T \int_0^T |x'(t)|^2\, dt. \tag{3.24}$$

We can also obtain the following two inequalities:

$$|[\text{grad}\, G(x(t-\gamma(t))), C'(t)x(t-r)]| \leq \sqrt{\lambda_1}(w_1 TM\sqrt{n} + n\ell_1 TM^2) +$$
$$+ \sqrt{\lambda_1}\left(w_1 T^{\frac{3}{2}}\sqrt{n} + 2n\ell_1 MT^{\frac{3}{2}}\right)\left(\int_0^T |x'(t)|^2\, dt\right)^{\frac{1}{2}} +$$
$$+ \ell_1 nT^2\sqrt{\lambda_1}\int_0^T |x'(t)|^2\, dt, \tag{3.25}$$

and

$$|[\text{grad}\, G(x(t-\gamma(t))), C(t)x'(t-r)]| \leq \sqrt{\lambda_0}\left(w_1\sqrt{T} + \ell_1 M\sqrt{nT}\right)\left(\int_0^T |x'(t)|^2 dt\right)^{\frac{1}{2}} +$$
$$+ \ell_1 T\sqrt{\lambda_0}\int_0^T |x'(t)|^2 dt. \tag{3.26}$$

Substituting (3.7), (3.10), (3.11), (3.17), (3.24), (3.25) and (3.26) into (3.23) we obtain

$$\eta\int_0^T |x'(t)|^2 dt \leq (T\sqrt{n\mu'_M} + \sqrt{\mu_M} + \ell_1 T + \ell_1 nT^2\sqrt{\lambda_1} + \ell_1 T\sqrt{\lambda_0})\int_0^T |x'(t)|^2 dt +$$
$$+ (M\sqrt{n\mu'_M T} + w_1\sqrt{T} + \ell_1 M\sqrt{nT} + w_1 T^{\frac{3}{2}}\sqrt{n\lambda_1} + 2n\ell_1 MT^{\frac{3}{2}}\sqrt{\lambda_1} +$$
$$+ w_1\sqrt{T\lambda_0} + \ell_1 M\sqrt{nT\lambda_0})\left(\int_0^T |x'(t)|^2 dt\right)^{\frac{1}{2}} +$$
$$+ \sqrt{\lambda_1}(w_1 TM\sqrt{n} + n\ell_1 TM^2) + \widetilde{e}M\sqrt{n\lambda_1 T}. \tag{3.27}$$

By $\eta > T\sqrt{n\mu'_M} + \sqrt{\mu_M} + \ell_1 T + \ell_1 n T^2\sqrt{\lambda_1} + \ell_1 T\sqrt{\lambda_0}$ and (3.27) we know there exists a positive constant M_3 such that

$$\int_0^T |x'(t)|^2 dt \leq M_3. \tag{3.28}$$

The remainder proof is similar to Theorem 3.1.

Theorem 3.2 is proved.

Remark 3.1. It is easy to see from assumption (H_2) that the sign of $x_i \dfrac{\partial G}{\partial x_i}$ is different for every $i \in I_n$.

Remark 3.2. As the coefficient matrix $C(t)$ is not a constant matrix C, the results in this paper can not be obtained from [1, 2, 4, 5, 8–11].

4. A numerical example. In this section, we consider a example and give two simulations to illustrate the theoretical results of Section 3.

Example 4.1. Consider the following systems:

$$\left(x(t) - \begin{pmatrix} \frac{1}{64}\sin t & \frac{1}{64}\cos t \\ \frac{1}{64}\cos t & \frac{1}{64}\sin t \end{pmatrix} x(t-2) \right)'' + \frac{d}{dt}\,\mathrm{grad}\, F(x(t)) +$$

$$+ \,\mathrm{grad}\, G(x(t-\gamma(t))) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}, \tag{4.1}$$

where $x(t) = (x_1(t), x_2(t))^\top$, $F(x) = x_1^2 + x_2^2$, $G = \dfrac{1}{32}(x_1^2 + x_2^2)$ and $\gamma(t) = \dfrac{1}{16}\cos t$. By calculation, we obtain $\lambda_0 = \lambda_1 = 0.0221$, $\alpha = L = L_1 = \dfrac{1}{16}$, $T = 2\pi$, $\eta = 2$, $\mu_M = \mu'_M = 0.002$. It is easy to verify that (H_1)–(H_4) hold. Furthermore, $\widehat{C} = \dfrac{\sqrt{2}}{64} < \dfrac{1}{3}$ and $\eta = 2 > \sqrt{n\mu'_M}T + \sqrt{\mu_M} + \sqrt{2}L\alpha + \sqrt{2n\lambda_1}\alpha L T + \sqrt{2\lambda_0}\alpha L + n\sqrt{\lambda_1}L_1 T^2 + \sqrt{n\lambda_0}L_1 T = 1.5196$. Therefore, it follows from Theorem 3.1 that Eq. (4.1) has at least one 2π-periodic solutions. Simulation results are shown in Figures 4.1 and 4.2. As showed in Figures 4.1 and 4.2, there exists a periodic solution, and solutions $(x_1(t), x_2(t))$ which start from 4 initial values tend to it.

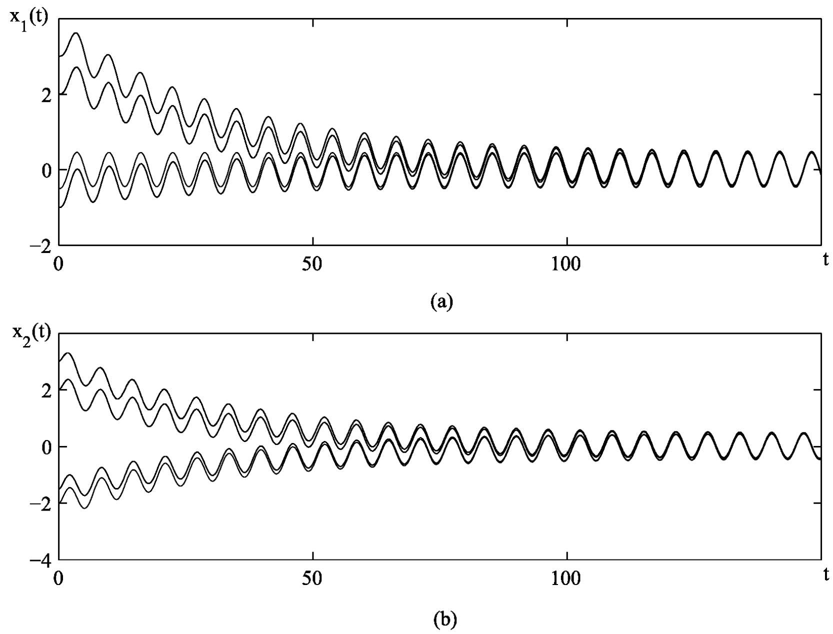

Fig. 4.1. Evolution of (a) the $x_1(t)$ and (b) the $x_2(t)$ with 4 initial values.

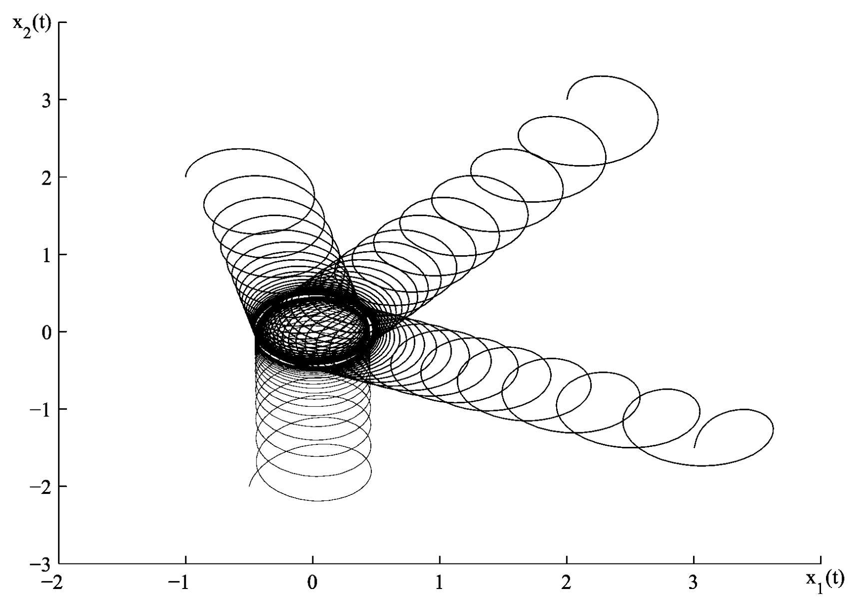

Fig. 4.2. Phase plane of systems (4.1) with 4 initial points.

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