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GLOBAL EXISTENCE RESULTS FOR FUNCTIONAL DIFFERENTIAL INCLUSIONS WITH DELAY

ІСНУВАННЯ ГЛОБАЛЬНИХ РОЗВ'ЯЗКІВ ФУНКЦІОНАЛЬНО-ДИФЕРЕНЦІАЛЬНИХ ВКЛЮЧЕНЬ ІЗ ЗАПІЗНЕННЯМ

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Our aim in this work is to study the existence of solutions of a functional differential inclusion with finite delay. We use the Bohnenblust – Karlin fixed point theorem for the existence of solutions.

Вивчено питання існування розв'язків функціонально-диференціальних включень зі скінченним запізненням. Для доведення існування розв'язків було використано теорему Бохнебласта – Карліна про нерухому точку.

1. Introduction. In this work we are going to prove the existence of solutions of a class of semilinear functional evolution inclusion with delay. Our investigations will be situated in the Banach space of real continuous and bounded functions on the real half axis $[0, +\infty)$. We will use Bohnenblust – Karlin's fixed theorem, combined with the Corduneanu's compactness criteria. More precisely, we will consider the following problem:

$$y'(t) - Ay(t) \in F(t, y_t), \quad \text{a.e.} \quad t \in J := [0, +\infty), \tag{1.1}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \tag{1.2}$$

where $F : J \times C([-r, 0], \to \mathcal{P}(E)$ is a multivalued map with nonempty compact values, $\mathcal{P}(E)$ is the family of all nonempty subsets of E, $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t)$, $t \in J$, $\phi : [-r, 0] \to E$ is a given continuous function, and $(E, |.|)$ is a real Banach space. For any function y defined on $[-r, +\infty)$ and any $t \in J$, we denote by y_t the element of $C([-r, 0], E)$ defined by $y_t(\theta) = y(t + \theta)$, $\theta \in [-r, 0]$. Here $y_t(.)$ represents the history of the state from time $t - r$, up to the present time t.

For modeling scientific phenomena where the delay is either a fixed constant or is given as an integral in which case is called distributed delay, we use differential delay equations or functional differential equations; see for instance the books [14, 15, 18].

An extensive theory is developed for evolution equations [1, 2, 12]. Uniqueness and existence results have been established recently for various classes of evolution problems in the papers by Baghli and Benchohra for finite and infinite delays in the Fréchet space setting in [3–6].

The aim of the present paper is to provide sufficient conditions for the existence of global mild solutions in the Banach space setting. Let us notice that most of the global existing results are given in Fréchet space setting. Thus the present results can be considered as a contribution for the global existence of mild solution of problem (1.1), (1.2).

2. Preliminaries. In this section we present briefly some notations, definition, and a theorem that are used throughout this work.

By $B(E)$ denotes the Banach space of bounded linear operators from E into E, with norm

$$\|N\|_{B(E)} = \sup\{|N(y)| : |y| = 1\}.$$

By $BC := BC([-r,+\infty))$ we denote the Banach space of all bounded and continuous functions from $[-r,+\infty)$ into $\mathbb{R}$ equipped with the standard norm

$$\|y\|_{BC} = \sup_{t\in[-r,+\infty)} |y(t)|.$$

We need the following definitions in the sequel.

Let (E,d) be a metric space. We use the following notations:

$$\mathcal{P}_{cl}(E) = \{Y \in \mathcal{P}(E) : Y \text{ closed}\}, \quad \mathcal{P}_{cv}(E) = \{Y \in \mathcal{P}(E) : Y \text{ convex}\},$$

$$\mathcal{P}_b(E) = \{Y \in \mathcal{P}(E) : Y \text{ bounded}\}.$$

Consider $H_d : \mathcal{P}(E) \times \mathcal{P}(E) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$, given by

$$H_d(\mathcal{A},\mathcal{B}) = \max\left\{\sup_{a\in\mathcal{A}} d(a,\mathcal{B}), \sup_{b\in\mathcal{B}} d(\mathcal{A},b)\right\},$$

where $d(\mathcal{A},b) = \inf_{a\in\mathcal{A}} d(a,b)$, $d(a,\mathcal{B}) = \inf_{b\in\mathcal{B}} d(a,b)$.

Let $(E,\|\cdot\|)$ be a Banach space. A multivalued map $A : E \to \mathcal{P}(E)$ has *convex (closed) values* if $A(x)$ is convex (closed) for all $x \in E$. We say that A is *bounded* on bounded sets if $A(B)$ is bounded in E for each bounded set B of E, i.e.,

$$\sup_{x\in B}\{\sup\{\|y\| : y \in A(x)\}\} < \infty.$$

F is said to be completely continuous if $F(B)$ is relatively compact for every $B \in \mathcal{P}_b(E)$. If the multivalued map F is completely continuous with non empty values, then F is u.s.c. if an only if F has a closed graph, i.e., $(x_n \to x_*, y_n \to y_*, y_n \in F(x_n)$ implies $y_* \in F(x_*))$.

Definition 2.1. *A function $F : J \times C([-r, 0]; E) \longrightarrow \mathcal{P}(E)$ is said to be a Carathéodory multivalued map if it satisfies:*

(i) $y \mapsto F(t, y)$ is upper semicontinuous for almost all $t \in J$;

(ii) $t \mapsto F(t, y)$ is measurable for each $y \in C([-r, 0]; E)$.

The multivalued map F is said to be L^1-Carathéodory if it satisfies (i), (ii) and

(iii) for every a positive constant l there exists $h_l \in L^1(J, \mathbb{R}^+)$

$$\|F(t, y)\| = \sup\{|v| : v \in F(t, y)\} \leq h_l,$$

for all $|y| \leq l$ for almost all $t \in J$.

For each $y : [-r, +\infty) \to E$ let the set $S_{F,y}$ known as the set of selections of F be defined by

$$S_{F,y} = \{v \in L^1(J; E) : v(t) \in F(t, y_t), \text{ a.e. } t \in J\}.$$

Lemma 2.1 [16]. *Let E be a Banach space. Let $F : J \times E \to \mathcal{P}_{cl,cv}(E)$ be a L^1-Carathéodory multivalued map, and let Γ be a linear continuous map from $L^1(J; E)$ into $C(J; E)$. Then the operator*

$$\Gamma \circ S_F : C(J, E) \longrightarrow \mathcal{P}_{cp,cv}(C(J, E)),$$

$$y \longmapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y})$$

is a closed graph operator in $C(J; E) \times C(J; E)$.

Finally, we say that A has a *fixed point* if there exists $x \in E$ such that $x \in A(x)$.

For more details on multivalued maps we refer to the books of Deimling [9], Denkowski et al. [10], and Hu and Papageorgiou [13].

Theorem 2.1. (Bohnenblust – Karlin fixed point [7]). *Let $B \in \mathcal{P}_{cl,cv}(E)$, and $N : B \to \mathcal{P}_{cl,cv}(B)$ be an upper semicontinuous operator and $N(B)$ be a relatively compact subset of E. Then N has at least one fixed point in B.*

Lemma 2.2 (Corduneanu [8]). *Let $D \subset BC([0, +\infty), E)$. Then D is relatively compact if the following conditions hold:*

(a) D is bounded in BC;

(b) the functions belonging to D are almost equicontinuous on $[0, +\infty)$, i.e., equicontinuous on every compact of $[0, +\infty)$;

(c) the set $D(t) := \{y(t) : y \in D\}$ is relatively compact on every compact of $[0, +\infty)$;

(d) the functions from D are equiconvergent, that is, given $\epsilon > 0$, there exists $T(\epsilon) > 0$ such that $|u(t) - \lim_{t\to+\infty} u(t)| < \epsilon$, for any $t \geq T(\epsilon)$ and $u \in D$.

3. Existence of mild solutions. Now we give our main existence result for problem (1.1), (1.2). Before stating and proving this result, we give the definition of a mild solution.

Definition 3.1. *We say that a continuous $y \in [-r, +\infty)$ is a mild solution of (1.1), (1.2) if there exists a function $f \in L^1(J, E)$ such that $f(t) \in F(t, y_t)$, a.e. on J, $y(t) = \phi(t)$, $t \in [-r, 0]$, and*

$$y(t) = T(t)\phi(t) - \int_0^t T(t - s)f(s)ds, \quad t \in J.$$

Let us introduce the following hypotheses:

(H_1) $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t), t \in J$, which is compact for $t > 0$ in the Banach space E. Let $M = \sup\{\|T(t)\|_{B(E)} : t \geq \geq 0\}$.

(H_2) The multifunction $F : J \times C([-r, 0]; E) \longrightarrow \mathcal{P}(E)$ is Carathéodory with compact and convex values.

(H_3) There exists a continuous function $k : J \to [0, +\infty)$ such that:

$$H_d(F(t, u), F(t, v)) \leq k(t)\|u - v\|,$$

for each $t \in J$ and for all $u, v \in C([-r, 0]; E)$ and

$$d(0, F(t, 0)) \leq k(t),$$

with

$$k^* := \sup_{t \in J} \int_0^t k(s)ds < \infty.$$

Theorem 3.1. *Assume that (H_1) – (H_3) hold. If $k^*M < 1$, then the problem (1.1), (1.2) has at least one mild solution on BC.*

Proof. Transform the problem (1.1), (1.2) into a fixed point problem. Consider the multivalued operator $N : BC \to \mathcal{P}(BC)$ defined by

$$N(y) := \left\{ h \in BC : h(t) = \begin{cases} \phi(t), & \text{if} \quad t \in [-r, 0], \\ T(t)\phi(0)+ & \\ +\displaystyle\int_0^t T(t-s)f(s)ds, \quad f \in S_{F,y} & \text{if} \quad t \in J \end{cases} \right\}.$$

The operator N maps BC into BC for any $y \in BC$ and $h \in N(y)$ and for each $t \in J$, we have

$$|h(t)| \leq M\|\phi\| + M\int_0^t |f(s)|ds \leq M\|\phi\| + M\int_0^t (k(s)\|y_s\| + \|F(s, 0)\|)ds \leq$$

$$\leq M\|\phi\| + M\int_0^t k(s)(\|y_s\| + 1)ds \leq M\|\phi\| + M(\|y\|_{BC} + 1)k^* := c.$$

Hence, $h(t) \in BC$.

Moreover, let $r > 0$ be such that $r \geq \dfrac{M\|\phi\| + Mk^*}{1 - Mk^*}$, and B_r be the closed ball in BC centered at the origin and of radius r. Let $y \in B_r$ and $t \in [0, +\infty)$. Then,

$$|h(t)| \leq M\|\phi\| + Mk^* + Mk^*r.$$

Thus,

$$\|h\|_{BC} \leq r,$$

which means that the operator N transforms the ball B_r into itself.

Now we prove that $N : B_r \to B_r$ satisfies the assumptions of Bohnenblust – Karlin's fixed theorem. The proof will be given in several steps.

Step 1. We shall show that the operator N is closed and convex. This will be given in two claims.

Claim 1. $N(y)$ is closed for each $y \in B_r$.

Let $(h_n)_{n\geq 0} \in N(y)$ such that $h_n \to \tilde{h}$ in B_r. Then for $h_n \in B_r$ there exists $f_n \in S_{F,y}$ such that

$$h_n(t) = T(t)\phi(0) + \int_0^t T(t-s)f_n(s)ds.$$

Since F has compact and convex values and from hypotheses (H_2), (H_3), an application of Mazur's theorem [19] implies that we may pass to a subsequence if necessary to get that f_n converges to $f \in L^1(J,E)$ and hence $f \in S_{F,y}$. More details on this matter can be found in [11]. Then for each $t \in J$,

$$h_n(t) \to \tilde{h}(t) = T(t)\phi(0) + \int_0^t T(t-s)f(s)ds.$$

So, $\tilde{h} \in N(y)$.

Claim 2. $N(y)$ is convex for each $y \in B_r$.

Let $h_1, h_2 \in N(y)$, then there exist $f_1, f_2 \in S_{F,y}$ such that, for each $t \in J$, we have

$$h_i(t) = T(t)\phi(0) + \int_0^t T(t-s)f_i(s)ds, \quad i = 1, 2.$$

Let $0 \leq \delta \leq 1$. Then, we have for each $t \in J$:

$$(\delta h_1 + (1-\delta)h_2)(t) = T(t)\phi(0) + \int_0^t T(t-s)[\delta f_1(s) + (1-\delta)f_2(s)]ds.$$

Since $F(t,y)$ is convex, one has

$$\delta h_1 + (1-\delta)h_2 \in N(y).$$

Step 2. $N(B_r) \subset B_r$; this is clear.

Step 3. $N(B_r)$ is equicontinuous on every compact interval $[0,b]$ of $[0,+\infty)$ for $b > 0$.

Let $\tau_1, \tau_2 \in [0, b]$ with $\tau_2 > \tau_1$, we obtain

$$|h(\tau_2) - h(\tau_1)| \leq \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)}\|\phi\| +$$

$$+ \int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)}|f(s)|ds +$$

$$+ \int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)}|f(s)|ds \leq \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)}\|\phi\| +$$

$$+ \int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)}(k(s)\|y_s\| + |F(s,0)|)ds +$$

$$+ \int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)}(k(s)\|y_s\| + |F(s,0)|)ds \leq$$

$$\leq \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)}\|\phi\| +$$

$$+ (r+1)\int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)}k(s)ds +$$

$$+ (r+1)\int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)}k(s)ds.$$

When $\tau_2 \to \tau_1$, the right-hand side of the above inequality tends to zero, since $T(t)$ is a strongly continuous operator and the compactness of $T(t)$, for $t > 0$, implies the continuity in the uniform operator topology (see [17]). This proves the equicontinuity.

Step 4. $N(B_r)$ is relatively compact on every compact interval of $[0, +\infty)$.

Let $t \in [0, b]$ for $b > 0$ and let ε be a real number satisfying $0 < \varepsilon < t$. For $y \in B_r$, let $h \in N(y)$ and $f \in S_{F,y}$. Define

$$h_\varepsilon(t) = T(t)\phi(0) + T(\varepsilon)\int_0^{t-\varepsilon} T(t - s - \varepsilon)f(s)ds.$$

Note that the set

$$\left\{T(t)\phi(0) + \int_0^{t-\varepsilon} T(t - s - \varepsilon)f(s)ds : y \in B_r\right\}$$

is bounded,

$$\left|T(t)\phi(0) + \int_0^{t-\varepsilon} T(t - s - \varepsilon)f(s)ds\right| \leq r.$$

Since $T(t)$ is a compact operator for $t > 0$, the set,

$$H_\varepsilon(t) = \{h_\varepsilon(t) : h_\varepsilon \in N(y), y \in B_r\}$$

is precompact in E for every ε, $0 < \varepsilon < t$. Moreover, for every $y \in B_r$ we have

$$|h(t) - h_\varepsilon(t)| \le M \int\limits_{t-\varepsilon}^{t} |f(s)|ds \le M \int\limits_{t-\varepsilon}^{t} (k(s)\|y_s\| + |F(s,0|)ds \le$$

$$\le M(1+r) \int\limits_{t-\varepsilon}^{t} k(s)ds \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Therefore, the set $H(t) = \{h(t) : h \in N(y), y \in B_r\}$ is precompact, i.e., relatively compact. Hence the set $H(t) = \{h(t) : h \in N(B_r)\}$ is relatively compact.

Step 5. N has closed graph.

Let $\{y_n\}$ be a sequence such that $y_n \to y_*$, $h_n \in N(y_n)$ and $h_n \to h_*$. We shall show that $h_* \in N(y_*)$. The relation $h_n \in N(y_n)$ means that there exists $f_n \in S_{F,y_n}$ such that

$$h_n(t) = T(t)\phi(0) + \int\limits_0^t T(t-s)f_n(s)ds, \quad t \in J.$$

We must prove that there exists f_*,

$$h_*(t) = T(t)\phi(0) + \int\limits_0^t T(t-s)f_*(s)ds, \quad t \in J.$$

Consider the linear and continuous operator $K : L^1(J,E) \to BC$ defined by

$$K(v)(t) = \int\limits_0^t T(t-s)v(s)ds.$$

We have

$$|K(f_n)(t) - K(f_*)(t)| = |(h_n(t) - T(t)\phi(0)) - (h_*(t) - T(t)\phi(0))| =$$
$$= |h_n(t) - h_*(t)| \le \|h_n - h_*\|_\infty \to 0, \quad \text{as} \quad n \to \infty.$$

From Lemma 2.1 it follows that $K \circ S_F$ is a closed graph operator and from the definition of K we have

$$h_n(t) - T(t)\phi(0) \in K \circ S_{F,y_n}.$$

As $y_n \to y_*$ and $h_n \to h_*$, there exists $f_* \in S_{F,y_*}$ such that

$$h_*(t) - T(t)\phi(0) = \int_0^t T(t-s)f_*(s).$$

Hence the multivalued operator N has closed graph, which implies that it is upper semicontinuous.

Step 6. $N(B_r)$ is equiconvergent.

Let $h \in N(y)$, there exists $f \in S_{F,y}$ such that for each $t \in [0,+\infty)$ and $y \in B_r$ we have

$$|h(t)| \le M\|\phi\| + M\int_0^t |f(s)|ds \le M\|\phi\| + Mk^* + Mr\int_0^t k(s)ds \le M\|\phi\| + Mk^* + Mrk^*.$$

Then,

$$|h(t)| \to M\|\phi\| + Mk^*(1+r), \quad \text{as} \quad t \to +\infty.$$

Hence,

$$|h(t) - h(+\infty)| \to 0, \quad \text{as} \quad t \to +\infty.$$

As a consequence of Steps 1–6, and Lemma 2.2, we conclude from Bohnenblust–Karlin's theorem that N has a fixed point y which is a mild solution of the problem (1.1), (1.2).

4. An example. Consider the functional partial differential inclusion

$$\frac{\partial}{\partial t} z(t,x) - \frac{\partial^2}{\partial x^2} z(t,x) \in F(t, z(t-r,x)), \quad x \in [0,\pi], \quad t \in J := [0,+\infty), \tag{4.1}$$

$$z(t,0) = z(t,\pi) = 0, \quad t \in J, \tag{4.2}$$

$$z(t,x) = \phi(t), \quad t \in [-r,0], \quad x \in [0,\pi], \tag{4.3}$$

where F is a given multivalued map. Take $E = L^2[0,\pi]$ and define $A : E \to E$ by $A\omega = \omega''$ with domain

$$D(A) = \{\omega \in E;\ \omega,\ \omega' \text{ are absolutely continuous},\ \omega'' \in E,\ \omega(0) = \omega(\pi) = 0\}.$$

Then,

$$A\omega = \sum_{n=1}^{\infty} n^2(\omega,\omega_n)\omega_n, \quad \omega \in D(A)$$

where $\omega_n(s) = \sqrt{\frac{2}{\pi}}\sin ns$, $n = 1, 2, \ldots$, is an orthogonal set of eigenvectors in A. It is well know (see [17]) that A is the infinitesimal generator of an analytic semigroup $T(t), t \ge 0$ in E and is given by

$$T(t)\omega = \sum_{n=1}^{\infty} \exp(-n^2 t)(\omega,\omega_n)\omega_n, \quad \omega \in E.$$

Since the analytic semigroup $T(t)$ is compact for $t > 0$, there exists a positive constant M such that

$$\|T(t)\|_{B(E)} \leq M.$$

Then the problem (1.1), (1.2) is the abstract formulation of the problem (4.1) – (4.3). If conditions (H_1) – (H_3) are satisfied, Theorem 3.1 implies that the problem (4.1) – (4.3) has at least one global mild solution on BC.

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