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Fractal properties of critical current of a self-similar system of Josephson junctions connected in parallel

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A fractal value (the local cell dimension D_C) has been found to characterize the dependence of the maximum Josephson current square I^2 on the magnetic field in self-similar systems of Josephson junctions connected in parallel and separated by distances distributed like Cantor set of segments. The cell dimension has been shown to be related to the fractal system dimension D by the relationship $D_C \approx 2-D$, which had been checked for systems with the different fractal dimensions.

It has been shown [1] that in percolative Sn–Ge thin films in the resistive state the supercurrent interference occurs on various-size loops containing Josephson junctions and belonging to the skeleton of an infinite percolating cluster along which the current propagates. The infinite cluster has a self-similar (fractal) structure [2]. The geometrical characteristic of a self-similar structure is its fractal dimension D found from the relation $M \propto L^D$, where M is the cluster mass and L is its size. In [3] we have considered a simple model for investigating dependences of the maximum Josephson current I_c on a magnetic field H in a self-similar system with Josephson junctions. It has been shown that the behavior of a square of a critical current vs. H is analogous to the dependence on H of light intensity that is scattered at a wave vector q by a fractal grating [4]. Both calculations and experiments of an optical diffraction on fractals have shown that the diffraction patterns possessed the same symmetry properties as objects in real space and they exhibited self-similarity because of the fractal nature of the gratings [4]. In the present paper we check these suggestions for the interference of critical currents and consider the self-similarity of the $I^2(H)$ dependences.

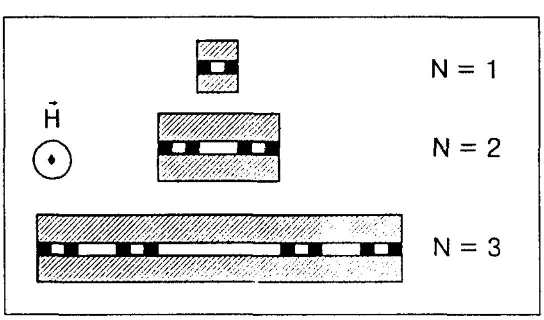

Fig. 1. Stages of forming a self-similar system with Josephson junctions (shown in black). The superconducting material is shaded.

Let us consider a system consisting of two superconductors separated by layers, impenetrable to supercurrent, and layers, thin enough for weak link formation. The stages of forming this system are shown in Fig. 1. At the N stage of the formation, the structure shaped at the previous stage is repeated three times and layers of the middle third in the newly-shaped structure penetrable for the supercurrent are replaced by impenetrable ones. The set of separations between the impenetrable layers is similar to a Cantor set of segments, the number of penetrable layers N_l satisfy the relation $N_l \propto L^D$, where $L = 3^N s$ is the structure length (s is the width of the junction); $D = \lg 2/\lg 3$ is the fractal dimension of the set of the penetrable layers.

The dependence of a maximum Josephson current of the triadic Cantor system, after the Nth stage of formation, on a magnetic field applied normally to the system (Fig. 1) is equal to [3]

$$I(k) = \frac{I_c(k)}{I_c(0)} = \left| \prod_{n=0}^{N-1} \cos\,(2\pi 3^n ks) \right| , \qquad (1)$$

where $k = Hd/\Phi_0$, $\Phi_0 = 2{,}07\cdot 10^{-15}$ Wb is a quantum of the magnetic flux; d is the sum of the layer thickness and the London depths of penetration of the magnetic field into the superconductors. The analogous relation was found for the light intensity on the grating of the triadic Cantor set of segments [4].

The fractal dimension of such a triadic system after the Nth stage of formation can be found from the slope of the mean square of the $I(k)$ [3]:

$$\langle I^2(k)\rangle = \frac{1}{k}\int_0^k I^2(k')\,dk' \propto k^{-D}. \quad (2)$$

We shall now show that the relation (2) is available for the dependences of $I^2(k)$ for quasi-one-dimensional systems with any fractal dimension (for example, constructed, according to the above-described rule of forming Cantor sets of segments). A critical current of a quasi-one-dimensional system of Josephson junctions is determined by a modulus of a Fourier transformation of a distribution function of the Josephson current density $P(x)$ [5]:

$$I_c(k) = \left|\int_{-\infty}^{\infty} P(x)\,e^{ikx}\,dx\right|. \quad (3)$$

Then the square of $I_c(k)$ is proportional to

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(x)P(x')\,e^{ik(x'-x)}\,dx dx' = \int_{-\infty}^{\infty} g(x'')\,e^{ikx''}\,dx'', \quad (4)$$

where $x'' = x' - x$, $g(x'') \propto \int P(x'')P(x''+x)\,dx$ is a correlation function of the current density. The mean correlation function of the current density for our system is proportional to the mean current density on length x or mean density of penetrable layers $\langle g(x)\rangle \propto x^{-a}$, $a = 1 - D$ [2]. This relation is correct for the Cantor set of segments with any fractal dimension. Then the Fourier transformation of the correlation function is $\propto k^{-D}$ and we find that $\langle I^2(k)\rangle \propto k^{-D}$.

Thus, the Eq. (2) is related only to the power correlation function of the distribution of the Josephson junctions between superconductors, and therefore is correct for the Cantor set of segments of any fractal dimension. For instance, in order to form a system with fractal dimension $D = 0.5$, one must take at $N = 1$ the current penetrable layers in two quarters of the system's length (at the edges), then, at the next stage of the formation, this system is repeated four times and the penetrable for the superconductive current layers in the second and third quarters are replaced by penetrable ones. The expression for $I(k)$ for this system is following:

$$I(k) = \left|\prod_{n=0}^{N-1}\cos(3\pi 4^n s)\right|, \quad (5)$$

and $\langle I^2(k)\rangle \propto k^{-D}$. The expressions for the $I(k)$ for systems with other fractal dimensions can be obtained in a similar way. The values of D for such systems (with the division at 3, 4, ..., 7 parts) are presented in Table.

Table

Fractal dimension D of Josephson junctions systems, fractal properties of the function $I^2(k)$ for such systems ($N = 7$) and values of $(2 - D)$

D	y	D_c	$2 - D$
0.631	0.63±0.02	1.37±0.03	1.369
0.500	0.51±0.02	1.49±0.02	1.500
0.431	0.438±0.009	1.54±0.04	1.569
0.387	0.39±0.01	1.62±0.01	1.613
0.356	0.356±0.008	1.67±0.04	1.644

Now, let us consider the fractal properties of the function $I^2(k)$. There are some characteristics which show that the function has a self-similarity. One of the characteristics is the exponent y which connect the mean amplitude of the curve ($\langle R\rangle$) with the interval Δk over which it is calculated [2]:

$$\langle R\rangle = \langle|I^2_{\max} - I^2_{\min}|\rangle = |\Delta k|^y, \quad (6)$$

where $\langle\ldots\rangle$ denotes averaging over all Δk within the chosen part of the curve $I^2(k)$. Since $\langle I^2(k)\rangle \propto k^{-D}$, for large enough Δk $\langle R\rangle \propto \Delta k^D$, i.e., $y \approx D$ (we note that y is not a Holder exponent). We have calculated y for dependences $I^2(k)$ for five fractal dimensions (Table). The results show the validity of this relation as well as the validity of (2) for the Cantor systems with different fractal dimensions.

Another fractal characteristic of the curves $I^2(k)$ is the local cell dimension [2] D_c that is found from the relation $D_c = d\lg(N_2(b; t, a))/d\lg b$, where N_2 is the number of $bt\times ba$ cells covering the graph curve; here b is a number, while t and a are the dimensions of the graph «elementary cell» along the coordinate axes. In the course of the local cell dimension calculations, the value of t is taken to be of the order of the step over k, while the value of a is taken with account of the condition $a \ll R$. To cover the dependence $I^2(k)$ by cells of size $[bt; ba]$ T/bt cells are needed along the abscissa (T being the length of the entire segment $[k_{\min}, k_{\max}]$, on which D_c is calculated) and about $\langle R\rangle/ba$ cells along the ordinate ($\langle R\rangle \propto b^y$), so that

$$D_c \propto (T/bt)(\langle R\rangle/ba) \propto b^{y-2} = b^{-D_c}; \quad (7)$$

$$D_c = 2 - y \approx 2 - D$$

(the last relation is true because $y \approx D$). This relation has been checked numerically for different fractal dimensions. In the process of calculations of D_c the maximum of D_c at different a has been found for the $N_2(b; t, a) = f(b)$ dependences ($t = 1$). The slope of the dependence $\lg N_2 = f(\lg b)$ for the curve $I^2(k)$ of the triadic Cantor set of Josephson junctions is

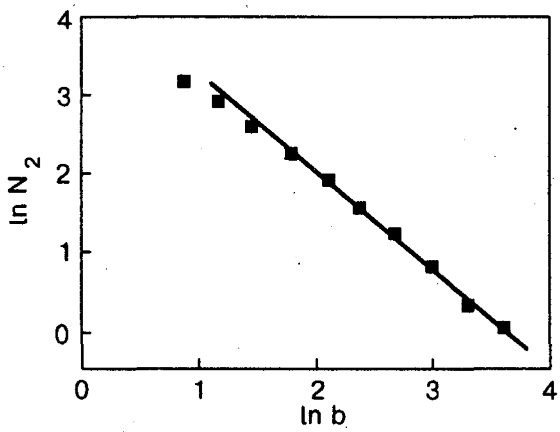

Fig. 2. Dependence lg N_2 on lg b for curve $I^2(k)$, $N = 7$. The line slope on the graph equals $-D_c = -1.37\pm0.03$.

$-D_c = -1.37\pm0.03$ (see Table). This cell dimension can be obtained in processing both the entire curve $I^2(k)$ and every part of it, provided that the part contains sufficiently many maxima. One can see the validity of the relation (7) because of $2 - D \approx 1.37$ (see Table, Fig. 2). The calculations of D_c for the other fractal dimensions also has confirmed the relation (7).

Thus, the local cell dimension of the dependence $I^2(k)$ is determined by the fractal dimension of the system containing Josephson junctions. The cell dimension of rectified voltage oscillations in percolating superconducting Sn–Ge films has been calculated [1]. The local oscillation cell dimension was assumed to be related to the dimension of the skeleton of the percolating infinite cluster where the current propagates. The results of the present work show the existence of a relation between D_c and D in the framework of a simple quasi-one-dimensional model. It may be expected that the relation between D_c and the dimension of the skeleton of an infinite cluster is expressed in a more complicated way than in the simple model described here.

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