UDC 517.9

ON OSCILLATION OF SOLUTIONS OF SECOND ORDER NONLINEAR DIFFERENCE EQUATIONS*

ПРО КОЛИВАННЯ РОЗВ'ЯЗКІВ НЕЛІНІЙНИХ РІЗНИЦЕВИХ РІВНЯНЬ ДРУГОГО ПОРЯДКУ

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The difference equation

$$\Delta^2 u(k) + p(k)\,|u(\sigma(k))|^\lambda \operatorname{sign} u(\sigma(k)) = 0,$$

is considered, where $0 < \lambda < 1$, $p : N \to R_+$, $\sigma : N \to N$, $\sigma(k) \ge k+1$ *for* $k \in N$ *and the difference operator is defined by* $\Delta u(k) = u(k+1) - u(k)$, $\Delta^2 = \Delta \circ \Delta$. *Necessary conditions are obtained for the above equation to have a positive solution. In addition, oscillation criteria of new type are obtained.*

Розглядається різницеве рівняння

$$\Delta^2 u(k) + p(k)\,|u(\sigma(k))|^\lambda \operatorname{sign} u(\sigma(k)) = 0,$$

де $0 < \lambda < 1$, $p : N \to R_+$, $\sigma : N \to N$, $\sigma(k) \ge k+1$ *для* $k \in N$, *різницевий оператор визначається як* $\Delta u(k) = u(k+1) - u(k)$ *та* $\Delta^2 = \Delta \circ \Delta$. *Отримано необхідні умови для існування додатного розв'язку наведеного рівняння. Також встановлено нові критерії коливання розв'язку.*

1. Introduction. Consider the difference equation

$$\Delta^2 u(k) + p(k)|u(\sigma(k))|^\lambda \operatorname{sign} u(\sigma(k)) = 0, \tag{1.1}$$

where

$$p : N \to R_+, \quad \sigma : N \to N \tag{1.2}$$

are functions defined on the set of natural numbers $N = \{1, 2, \ldots\}$, $\Delta u(k) = u(k+1) - u(k)$ and $\Delta^2 = \Delta \circ \Delta$. Everywhere below it is assumed that

$$\sigma(k) \ge k+1 \quad \text{for} \quad k \in N, \quad 0 < \lambda < 1, \tag{1.3}$$

* The work was supported by the Georgian National Science Foundation (Grant No. GNSF/ST09-81-3-101).

$$\sup\{p(k) : k \geq i\} > 0 \quad \text{for} \quad i \in N. \tag{1.4}$$

For each $n \in N$ denote $N_n = \{n, n+1, \dots\}$.

Definition 1.1. *Let $n_0 \in N$. We will call a function $u : N_{n_0} \to R$ a proper solution of the equation (1.1) if it satisfies (1.1) on N_{n_0} and*

$$\sup\Big\{|u(i)| : i \geq k\Big\} > 0 \quad \textit{for any} \quad k \in N_{n_0}.$$

Remark 1.1. Let the condition (1.3) be fulfilled, $k_0 \in N$ and $u : N_{k_0} \to R$ be a nontrivial solution of (1.1). Then u is a proper solution. Indeed, if we assume the contrary, then there exists $k_1 > k_0$ such that $u(k) = 0$ when $k \geq k_1$ and $u(k_1 - 1) \neq 0$. Since $\sigma(k_1 - 1) \geq k_1$, from the equality

$$u(k+2) - 2u(k+1) + u(k) = p(k)|u(\sigma(k))|^{\lambda} \operatorname{sign} u(\sigma(k)) = 0$$

we have $u(k_1 - 1) = 0$. The obtained contradiction proves that u is a proper solution of (1.1). Therefore if the condition (1.3) is fulfilled, then the set of nontrivial solutions of (1.1) coincides with the set of proper solutions. On the other hand, we can give examples of difference equations with a nontrivial solution which is its nonproper solution.

Definition 1.2. *We say that a proper solution $u : N_{n_0} \to R$ of (1.1) is oscillatory if for any $k \in N_{n_0}$ there exist $n_1, n_2 \in N_k$ such that $u(n_1)\, u(n_2) \leq 0$. Otherwise the solution is called nonoscillatory.*

The problem of oscillation of solutions of linear difference equations has been studied by several authors, see [1 – 6] and references therein.

As to investigation of the analogous problem for equations of type (1.1) ($0 < \lambda < 1$), to our knowledge for them there have not been obtained results analogous to those known for ordinary differential equations (see [7, 8]). In this paper we will try to fill this gap for second order difference equations with advanced argument.

Everywhere below it is assumed that the condition

$$\sum_{k=1}^{+\infty} k\, p(k) = +\infty \tag{1.5}$$

is fulfilled.

Remark 1.2. The existence of positive proper solutions of rather general functional-differential equations of higher order is well known (see e.g. Lemma 4.1 from [9]). We can analogously prove the fact that if condition (1.5) is not fulfilled, then equation (1.1) has proper solution u satisfying the condition $\lim_{k\to+\infty} u(k) = c \neq 0$.

Analogous results for n-th order Emden – Fowler type differential equations are given in [8].

2. Some anxiliary statement.

Lemma 2.1. *Let (1.2) – (1.4) be fulfilled and $u : N_{n_0} \to R$ be a nonoscillatory proper solution of (1.1). Then there exists $k_0 \in N_{n_0}$ such that*

$$u(k)\, \Delta u(k) > 0 \quad \textit{for} \quad k \in N_{k_0}. \tag{2.1}$$

Lemma 2.2. *Let (1.2), (1.5) hold and* $u : N_{n_0} \to R$ *be a nonoscillatory solution of (1.1). Then*

$$\lim_{k\to+\infty} |u(k)| = +\infty, \quad \limsup_{k\to+\infty} \frac{|u(k)|}{k} < +\infty. \tag{2.2}$$

We refer the reader to [3] for the proof of Lemmas 2.1 and 2.2.

Lemma 2.3. *Let (1.2), (1.3), (1.5) be fulfilled and* $u : N_{n_0} \to (0,+\infty)$ *be a positive proper solution of (1.1). Then for any* $s \in N$ *there exists* $k_0 \in N_{n_0}$ *such that*

$$u(k) \ge \rho_{s,k_0}(k) \quad \textit{for} \quad k \ge k_0, \quad s = 1, 2, \ldots, \tag{2.3}$$

where

$$\rho_{1,k_0}(k) = \left((1-\lambda) \sum_{i=k_0}^{k} \sum_{j=i}^{+\infty} p(j) \right)^{\frac{1}{1-\lambda}}, \tag{2.4}$$

$$\rho_{s,k_0}(k) = \sum_{i=k_0}^{k} \sum_{j=i}^{+\infty} p(j) \left(\rho_{s-1,k_0}(\sigma(i))\right)^{\lambda}, \quad s = 2, 3, \ldots. \tag{2.5}$$

Proof. Let $u : N_{n_0} \to (0,+\infty)$ be a positive solution of (1.1). Then according to Lemma 2.1 there exists $k_0 \in N_{n_0}$ such that (2.1) holds. Therefore, from (1.1) we have

$$\Delta u(k) \ge \sum_{i=k}^{+\infty} p(i)\, u^{\lambda}(\sigma(i)) \quad \text{for} \quad k \in N_{k_0}. \tag{2.6}$$

Consequently, by (1.3) and Lemma 2.1,

$$\Delta u(k) \ge u^{\lambda}(k+1) \sum_{i=k}^{+\infty} p(i) \quad \text{for} \quad k \in N_{k_0}.$$

Thus, we have

$$\sum_{i=k_0}^{k} \frac{\Delta u(i)}{u^{\lambda}(i+1)} \ge \sum_{i=k_0}^{k} \left(\sum_{j=i}^{+\infty} p(j) \right). \tag{2.7}$$

Taking into account that

$$\Delta u(i) = \int_{u(i)}^{u(i+1)} ds \quad \text{and} \quad u^{-\lambda}(i+1) \le s^{-\lambda} \quad \text{for} \quad u(i) \le s \le u(i+1),$$

from (2.7) we have

$$\sum_{i=k_0}^{k} \int\limits_{u(i)}^{u(i+1)} s^{-\lambda} ds \ge \sum_{i=k_0}^{k} \left(\sum_{j=i}^{+\infty} p(j) \right) \quad \text{for} \quad k \in N_{k_0}.$$

Therefore

$$\int\limits_{u(k_0)}^{u(k+1)} s^{-\lambda}\, ds \ge \sum_{i=k_0}^{k} \left(\sum_{j=i}^{+\infty} p(j) \right) \quad \text{for} \quad k \in N_{k_0}.$$

Hence it is clear that

$$u^{1-\lambda}(k+1) - u^{1-\lambda}(k_0) \ge (1-\lambda) \sum_{i=k_0}^{k} \left(\sum_{j=i}^{+\infty} p(j) \right) \quad \text{for} \quad k \in N_{k_0}.$$

Thus, since $\lambda \in (0,1)$, from the last equality we have

$$u(k+1) \ge \left((1-\lambda) \sum_{i=k_0}^{k} \left(\sum_{j=i}^{+\infty} p(j) \right) \right)^{\frac{1}{1-\lambda}} \quad \text{for} \quad k \in N_{k_0}. \tag{2.8}$$

On the other hand from (2.6) we find

$$u(k+1) \ge \sum_{i=k_0}^{k} \left(\sum_{j=i}^{+\infty} p(j)\, u^{\lambda}(\sigma(j)) \right).$$

Therefore, (2.8) and (1.3) obviously imply (2.5) for any $k \ge k_0$ and $s = 2, 3, \ldots$.

The lemma is proved.

Let $k_0 \in N$. Denote by $\mathbf{U}_{k_0}$ the set of all proper solutions of (1.1) satisfying the condition $u(k) > 0$ for $k \in N_{k_0}$.

Lemma 2.4. *Let $k_0 \in N$ and $\mathbf{U}_{k_0} \neq \varnothing$. Then for any $\Delta \in [0, \lambda]$ and $s \in N$,*

$$\sum_{k=1}^{+\infty} k^{\lambda-\Delta} \left(\rho_s(\sigma(k))\right)^{\Delta} p(k) < +\infty, \tag{2.9}$$

where

$$\rho_1(k) = \left((1-\lambda) \sum_{i=1}^{k} \left(\sum_{j=i}^{+\infty} p(j) \right) \right)^{\frac{1}{1-\lambda}}, \tag{2.10}$$

$$\rho_s(k) = \sum_{i=1}^{k} \left(\sum_{j=i}^{+\infty} p(j) \rho_{s-1}^{\lambda}(\sigma(j)) \right), \quad s = 2, 3, \ldots. \tag{2.11}$$

Proof. Let $n_0 \in N$ and $\mathbf{U}_{n_0} \neq \varnothing$. Then (1.1) has a proper positive solution $u : N_{n_0} \to \to (0, +\infty)$. According to Lemma 2.3, there exist $k_0 \in N_{n_0}$ such that the condition (2.3) is fulfilled, where ρ_{s,k_0}, $s = 1, 2, \ldots$, is defined by (2.4) and (2.5). Therefore, by (2.1) and (2.6) we have

$$u(k+1) \geq \sum_{i=k_0}^{k} \sum_{j=i}^{+\infty} p(j)\, u^{\Delta}(\sigma(j))\, u^{\lambda-\Delta}(\sigma(j)) \geq \sum_{i=k_0}^{k} \sum_{j=i}^{+\infty} p(j)\, \rho^{\Delta}_{s,k_0}(\sigma(j))\, u^{\lambda-\Delta}(\sigma(j)).$$

Hence, by (1.3) we get

$$u(k+1) \geq \sum_{i=k_0}^{k} \sum_{j=i}^{+\infty} p(j)\, \rho^{\Delta}_{s,k_0}(\sigma(j))\, u^{\lambda-\Delta}(j+1) \geq (k-k_0) \sum_{j=k}^{+\infty} p(j)\, \rho^{\Delta}_{s,k_0}(\sigma(j))\, u^{\lambda-\Delta}(j+1), \tag{2.12}$$

$$k = k_0 + 1, k_0 + 2, \ldots .$$

If $\Delta = \lambda$, by (2.2) and (2.12) we have

$$\sum_{j=k}^{+\infty} p(j)\, \rho^{\lambda}_{s,k_0}(\sigma(j)) \leq \frac{u(k+1)}{k-k_0} < +\infty. \tag{2.13}$$

Let $\Delta < \lambda$. Then by (2.12) we have

$$u^{\lambda-\Delta}(k+1) \geq (k-k_0)^{\lambda-\Delta} \left(\sum_{j=k}^{+\infty} p(j)\, \rho^{\Delta}_{s,k_0}(\sigma(j))\, u^{\lambda-\Delta}(j+1) \right)^{\lambda-\Delta}.$$

Hence

$$\sum_{j=k_0}^{k} \frac{u^{\lambda-\Delta}(j+1)\, \rho^{\lambda}_{s,k_0}(\sigma(j))\, p(j)}{\left(\sum\limits_{i=j}^{+\infty} u^{\lambda-\Delta}(i+1)\, \rho^{\Delta}_{s,k_0}(\sigma(i))\, p(i) \right)^{\lambda-\Delta}} \geq \sum_{j=k_0}^{k} (j-k_0)^{\lambda-\Delta} \rho^{\Delta}_{s,k_0}(\sigma(j))\, p(j). \tag{2.14}$$

Denote

$$a_j = \sum_{i=j}^{+\infty} u^{\lambda-\Delta}(i+1)\, \rho^{\Delta}_{s,k_0}(\sigma(i))\, p(i).$$

Then from (2.14) we have

$$\sum_{j=k_0}^{k} \frac{a_j - a_{j+1}}{a_j^{\lambda-\Delta}} \geq \sum_{j=k_0}^{k} (j-k_0)^{\lambda-\Delta} p(j)\, \rho^{\Delta}_{s,k_0}(\sigma(j)). \tag{2.15}$$

On the other hand,

$$\sum_{j=k_0}^{k} \frac{a_j - a_{j+1}}{a_j^{\lambda-\Delta}} = \sum_{j=k_0}^{k} a_j^{\Delta-\lambda} \int\limits_{a_{j+1}}^{a_j} ds \le \sum_{j=k_0}^{k} \int\limits_{a_{j+1}}^{a_j} s^{\Delta-\lambda}\, ds =$$

$$= \int\limits_{a_k+1}^{a_{k_0}} s^{\Delta-\lambda}\, ds = \frac{1}{1-\lambda+\Delta}\left(a_{k_0}^{1-\lambda+\Delta} - a_{k+1}^{1-\lambda+\Delta}\right) \to$$

$$\to \frac{1}{1-\lambda+\Delta}\, a_{k_0}^{1-\lambda+\Delta} \quad \text{for} \quad k \to +\infty.$$

Therefore, from (2.15) we have

$$\sum_{j=k_0}^{+\infty} (j-k_0)^{\lambda-\Delta} p(j)\, \rho_{s,k_0}^{\Delta}(\sigma(j)) < +\infty \quad \text{for any} \quad \Delta \in [0,\lambda). \tag{2.16}$$

According to (2.13) and (2.16), for any $\Delta \in [0,\lambda]$ and $s \in N$ (2.16) holds. Therefore, since

$$\lim_{k\to+\infty} \frac{\rho_s(k)}{\rho_{s,k_0}(k)} = 1 \quad \text{for any} \quad k_0 \in N \quad \text{and} \quad s \in N,$$

by (2.16) it is obvious that for any $\Delta \in [0,\lambda]$ and $s \in N$ (2.9) holds, which proves the validity of the lemma.

3. Sufficient conditions for oscillation.

Theorem 3.1. *Let the conditions (1.2) – (1.5) be fulfilled and for some $\Delta \in [0,\lambda]$ and $s \in N$,*

$$\sum_{k=1}^{+\infty} k^{\lambda-\Delta}\, \rho_s^{\Delta}(\sigma(k))\, p(k) = +\infty. \tag{3.1}$$

Then all solutions of (1.1) are oscillatory, where ρ_s is defined by (2.10) and (2.11).

Proof. Assume the contrary. Then there exists $k_0 \in N$ such that (1.1) has a proper solution $u : N_{k_0} \to (0,+\infty)$ (the case $u(k) < 0$ is similar). Since the conditions of the Lemma 2.4 are fulfilled, for any $\Delta \in [0,\lambda]$ and $s \in N$ (2.9) holds, which condradicts (3.1). The obtained condradiction proves the validity of the theorem.

Corollary 3.1. *Let the conditions (1.2) – (1.4) be fulfilled and*

$$\sum_{k=1}^{+\infty} k^{\lambda}\, p(k) = +\infty. \tag{3.2}$$

Then all solutions of (1.1) are oscillatory.

Proof. To prove the corollary, it suffices to note that according to (3.2) the condition (3.1) holds for $\Delta = 0$.

Corollary 3.2. *Let the conditions (1.2) – (1.5) be fulfilled and for some* $s_0 \in N$,

$$\sum_{k=1}^{+\infty} p(k)\, \rho_{s_0}^{\lambda}(\sigma(k)) = +\infty. \tag{3.3}$$

Then all solutions of (1.1) are oscillatory.

Proof. It is obvious that by (3.3) for $\Delta = \lambda$ and $s = s_0$ the condition (3.1) holds.

Corollary 3.3. *Let the conditions (1.2) and (1.3) be fulfilled and for some* $\gamma \in (0,1)$

$$\liminf_{n\to+\infty} k^{\gamma} \sum_{k=1}^{+\infty} p(j) > 0 \tag{3.4}$$

and

$$\sum_{k=1}^{+\infty} p(k)(\sigma(k))^{\frac{\lambda(1-\gamma)}{1-\lambda}} = +\infty. \tag{3.5}$$

Then all solutions of (1.1) are oscillatory.

Proof. It suffices to show that for $s_0 = 1$ (3.3) is satisfied. Indeed, according (3.4), there exist $k_0 \in N$ and $c > 0$ such that

$$\sum_{j=k}^{+\infty} p(j) \geq c\, k^{-\gamma} \quad \text{for} \quad k \in N_{k_0}.$$

Therefore, by (2.10) we have

$$\rho_1(k) \geq \left(c(1-\lambda) \sum_{i=1}^{k} i^{-\gamma} \right)^{\frac{1}{1-\lambda}} = \left(c(1-\lambda) \sum_{i=1}^{k} i^{-\gamma} \int_{i}^{i+1} ds \right)^{\frac{1}{1-\lambda}} \geq$$

$$\geq \left(c(1-\lambda) \sum_{i=1}^{k} \int_{i}^{i+1} s^{-\gamma}\, ds \right)^{\frac{1}{1-\lambda}} = \left(c(1-\lambda) \int_{1}^{k+1} s^{-\gamma}\, ds \right)^{\frac{1}{1-\lambda}} =$$

$$= \left(\frac{c(1-\lambda)}{1-\gamma} \left((k+1)^{1-\gamma} - 1 \right) \right)^{\frac{1}{1-\lambda}} \frac{(k+1)^{\frac{1-\gamma}{1-\lambda}}}{2} \left(\frac{c(1-\lambda)}{1-\gamma} \right)^{\frac{1}{1-\lambda}} \quad \text{for } k \geq k_1,$$

where $k_1 > k_0$ is a sufficiently large natural number. Thus, according to (3.5) it is obvious that for $s_0 = 1$ (3.3) holds, which proves the corollary.

Corollary 3.4. *Let the conditions (1.2) and (1.3) be fulfilled and*

$$\liminf_{n\to+\infty} k \sum_{j=k}^{+\infty} p(j) > 0, \tag{3.6}$$

$$\sum_{k=1}^{+\infty} p(k)\,(\ln\sigma(k))^{\frac{\lambda}{1-\lambda}} = +\infty. \tag{3.7}$$

Then all solutions of (1.1) are oscillatory.

Proof. To prove the corollary, it is sufficient to note that according to (3.6), (3.7) implies (3.3) for $s_0 = 1$.

Theorem 3.2. *Let the conditions (1.2) – (1.5) and (3.4) be fulfilled and there exist* $\alpha \in (1,+\infty)$ *such that*

$$\liminf_{n\to+\infty} \frac{\sigma(k)}{k^{\alpha}} > 0. \tag{3.8}$$

If, moreover, at least one the conditions

$$\alpha\,\lambda \ge 1 \tag{3.9}$$

or

$$\alpha\,\lambda < 1$$

and for some $\varepsilon > 0$

$$\sum_{k=1}^{+\infty} k^{\frac{\alpha\lambda(1-\gamma)}{1-\alpha\lambda}-\varepsilon} p(k) = +\infty \tag{3.10}$$

is fulfilled, then all solutions of (1.1) are oscillatory.

Proof. It suffices to show that the condition (3.3) is satisfied for some $s_0 \in N$. Indeed, according to (3.4) and (3.8) there exist $\alpha > 1$, $\gamma \in (0,1)$, $c > 0$ and $k_0 \in N$ such that

$$\sum_{j=k}^{+\infty} p(j) \ge c\,k^{-\gamma} \quad \text{for} \quad k \in N_{k_0} \tag{3.11}$$

and

$$\sigma(k) \ge c\,k^{\alpha} \quad \text{for} \quad k \in N_{k_0}. \tag{3.12}$$

According to (3.11) and (2.10), it is obvious that $\lim_{k\to+\infty}\rho_1(k) = +\infty$. Therefore, without loss of generality we can assume that $\rho_1(k) \ge 1$ for $k \in N_{k_0}$. Thus, by (2.10), (2.11) and (3.11) we have

$$\rho_2(k) \ge c\sum_{j=k_0}^{k} j^{-\gamma} = c\sum_{j=k_0}^{k} j^{-\gamma}\int_j^{j+1} ds \ge c\sum_{j=k_0}^{k}\int_j^{j+1} s^{-\gamma} ds =$$

$$= c\int_{k_0}^{k+1} s^{-\gamma}\,ds = \frac{c}{1-\gamma}\left((k+1)^{1-\gamma} - k_0^{1-\gamma}\right).$$

Choose $k_1 > k_0$ such that

$$\rho_2(k) \ge \frac{c}{2(1-\gamma)}\, k^{1-\gamma} \quad \text{for} \quad k \ge k_1.$$

Therefore, by (2.11), (3.11) and (3.12) for $s = 2$, if $\lambda\,\alpha(1-\gamma)-\gamma < 0$ we have

$$\rho_3(k) = \sum_{i=k_0}^{k} \left(\frac{c\, i^{\alpha(1-\gamma)}}{2(1-\gamma)}\right)^{\lambda} \sum_{j=i}^{+\infty} p(j) \ge$$

$$\ge \frac{c^{1+\lambda}}{(2(1-\gamma))^{\lambda}} \sum_{i=k_0}^{k} i^{\alpha\lambda(1-\gamma)-\gamma} =$$

$$= \frac{c^{1+\lambda}}{(2(1-\gamma))^{\lambda}} \sum_{i=k_0}^{k} i^{\alpha\lambda(1-\gamma)-\gamma} \int_{i}^{i+1} ds \ge$$

$$\ge \frac{c^{1+\lambda}}{(2(1-\gamma))^{\lambda}} \sum_{i=k_0}^{k} \int_{i}^{i+1} s^{\alpha\lambda(1-\gamma)-\gamma}\, ds =$$

$$= \frac{c^{1+\lambda}}{(2(1-\gamma))^{\lambda}} \int_{k_0}^{k+1} s^{(1-\gamma)\alpha\lambda-\gamma}\, ds =$$

$$= \frac{c^{1+\lambda}}{(2(1-\gamma))^{\lambda}(1-\gamma)(1+\alpha\lambda)} \Big((k+1)^{(1-\gamma)(1+\alpha\lambda)} - k_0^{(1-\gamma)(1+\alpha\lambda)}\Big) =$$

$$= \frac{c^{1+\lambda} k^{(1-\gamma)(1+\alpha\lambda)}}{2(2(1-\gamma))^{1+\lambda}(1+\alpha\lambda)} \ \text{ for } \ k \ge k_2, \tag{3.13}$$

where $k_2 > k_1$ is a sufficiently large natural number.

If $\lambda\alpha(1-\gamma)-\gamma \ge 0$, then we have

$$\rho_3(k) \ge \sum_{i=k_0}^{k} \left(\frac{c}{2(1-\gamma)}\right)^{\lambda} i^{\lambda\alpha(1-\gamma)-\gamma} \int_{i-1}^{i} ds \ge$$

$$\ge c \left(\frac{c}{2(1-\gamma)}\right)^{\lambda} \int_{k_0-1}^{k} s^{\lambda\alpha(1-\gamma)-\gamma}\, ds =$$

$$= \left(\frac{c}{2(1-\gamma)}\right)^{\lambda} \frac{\varepsilon}{(1-\gamma)(1+\alpha\lambda)} \Big(k^{(1-\gamma)(1+\alpha\lambda)} - (k_0-1)^{(1-\gamma)(1+\alpha\lambda)}\Big) \ge$$

$$\ge \left(\frac{c}{2(1-\gamma)}\right)^{1+\lambda} \frac{k^{(1-\gamma)(1+\alpha\lambda)}}{1+\alpha\lambda} \ \text{ for } \ k \ge k_2', \tag{3.14}$$

where $k_2' > k_1$ is sufficiently large.

Thus by (3.13) and (3.14) we have

$$\rho_3(k) \geq \left(\frac{c}{2(1-\gamma)}\right)^{1+\lambda} \frac{k^{(1-\gamma)(1+\alpha\lambda)}}{1+\alpha\lambda} \quad \text{for} \quad k \geq k_3,$$

where $k_3 = \max\{k_2, k_2'\}$. Therefore, for any $s_0 \in N$ there exists $k_{s_0} \in N$ such that

$$\rho_{s_0}(k) \geq \left(\frac{c}{2(1-\gamma)(1+\alpha\lambda+\ldots+(\alpha\lambda)^{s_0-2}}\right)^{1+\lambda+\ldots+\lambda^{s_0-2}} k^{(1-\gamma)(1+\alpha\lambda+\ldots+(\alpha\lambda)^{s_0-2})} \quad \text{for} \quad k \geq k_{s_0}. \tag{3.15}$$

Assume that (3.4) holds. Choose $s_0 \in N$ such that $(1-\gamma)(s_0-1) \geq 1$. Then, according to (1.5) and (3.15) it is obvious that (3.3) holds. In the case where (3.9) holds, the validity of the theorem is proved. Assume now that $\alpha\lambda < 1$ and for some $\varepsilon > 0$ (3.10) is fulfilled. Choose $s_0 \in N$ such that

$$1+\alpha\lambda+\ldots+(\alpha\lambda)^{s_0-2} > \frac{1}{1-\alpha\lambda} - \frac{\varepsilon}{\alpha(1-\gamma)}.$$

Then from (3.15) we have

$$\rho_{s_0}(k) \geq c_0\, k^{\frac{1-\gamma}{1-\alpha\lambda}-\frac{\varepsilon}{\alpha}} \quad \text{for} \quad k \geq k_{s_0},$$

where $c_0 > 0$. Therefore by (3.12)

$$\rho_{s_0}^{\lambda}(\sigma(k)) \geq c_1\, k^{\frac{\alpha\lambda(1-\gamma)}{1-\alpha\gamma}-\varepsilon} \quad \text{for} \quad k \geq k_{s_0},$$

where $c_1 > 0$. Consequently, according to (3.10), it is obvious that (3.3) holds.

The theorem is proved.

In a similar manner we can prove the following theorem.

Theorem 3.3. *Let the conditions (1.2)–(1.5) and* (3.6) *be fulfilled and for some* $\alpha > 0$

$$\liminf_{k\to+\infty} k^{-\alpha} \ln \sigma(k) > 0.$$

Then all solutions of (1.1) are oscillatory.

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Received 25.01.11,
after revision — 04.05.11