UDC 517.9

OSCILLATION OF NEUTRAL DIFFERENTIAL EQUATIONS WITH DISTRIBUTED DEVIATING ARGUMENTS

ОСЦИЛЯЦІЯ НЕЙТРАЛЬНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ З РОЗПОДІЛЕНИМИ ВІДХИЛЕНИМИ АРГУМЕНТАМИ

T. Candan

Niğde Univ.
51200 Niğde, Turkey
e-mail: tcandan@nigde.edu.tr

B. Karpuz

Afyon Kocatepe Univ.
03200 Afyonkarahisar, Turkey
e-mail: bkarpuz@gmail.com

Ö. Öcalan

Afyon Kocatepe Univ.
03200 Afyonkarahisar, Turkey
e-mail: ozkan@aku.edu.tr

In this article, we study oscillatory nature of all solutions of a class of neutral differential equations with distributed deviating arguments. Some examples are also included to show the applicability of the results.

Вивчається коливна природа розв'язків класу нейтральних диференціальних рівнянь з розподіленими відхиленими аргументами. Наведено приклади застосування результатів.

1. Introduction. We consider the neutral differential equation with distributed deviating arguments

$$[x(t) - R(t)x(t-\rho)]' + \int\limits_{\tau_1}^{\tau_2} [P(t,\zeta)x(t-\zeta) - Q(t,\zeta)x(t-\zeta+\sigma)]d\zeta = 0 \quad \text{for} \quad t \geq t_0, \tag{1}$$

where $R \in C([t_0,\infty),\mathbb{R}_0^+)$, $P,Q \in C([t_0,\infty)\times[\tau_1,\tau_2],\mathbb{R}_0^+)$. Here $[\tau_1,\tau_2]$ is a positive interval in reals and $\sigma \in [0,\tau_1)$. Clearly, when $Q \equiv 0$, (1) reduces to the following form:

$$[x(t) - R(t)x(t-\rho)]' + \int\limits_{\tau_1}^{\tau_2} P(t,\zeta)x(t-\zeta)d\zeta = 0 \quad \text{for} \quad t \geq t_0$$

whose oscillatory nature has been investigated in [2]. Note that a similar situation appears when $\sigma = 0$, thus we suppose that $\sigma > 0$ holds for the rest of the paper.

The neutral differential equation with concentrated delays, which corresponds to (1), has the form

$$[x(t) - R(t)\,x(t-\rho)]' + P(t)\,x(t-\tau) - Q(t)x(t-\sigma) = 0 \quad \text{for} \quad t \geq t_0. \tag{2}$$

Oscillation and nonoscillation of this equation has been investigated in the literature extensively (see [5–11]). Also the neutral difference equation corresponding to (1) is of the form

$$\Delta[x(n) - R(n)\,x(n-\rho)] + P(n)x(n-\tau) - Q(n)x(n-\sigma) = 0 \quad \text{for} \quad n \in \mathbb{N},$$

which has been discussed in the papers [6, 12, 13]. We refer the readers to [1, 3] for the fundamental results of the oscillation theory.

The results of this study are motivated by the fundamental results on neutral differential equations involving concentrated delays with positive and negative coefficients.

As it is customary, by a *solution* of (1), we mean a function $x \in C([t_0 - \max\{\rho, \tau_2\}, \infty), \mathbb{R})$ such that $x(t) - R(t)x(t-\rho)$ is continuously differentiable for all $t \geq t_0$ and identically satisfies (1) on $[t_0, \infty)$. A solution of (1) is called *nonoscillatory* if it is eventually of constant sign; otherwise, it is called *oscillatory*.

2. Preliminaries. In this section, we give some known results in the literature, which will be used in the latter sections.

Consider the following first-order delay differential inequality

$$x'(t) + A(t)x(t-\tau) \leq 0 \quad \text{for} \quad t \geq t_0, \tag{3}$$

where $A \in C([t_0, \infty), \mathbb{R}^+)$ and $\tau > 0$.

Theorem 2.1 ([3], Theorem 2.3.1). *Assume that*

$$\liminf_{t \to \infty} \int_{t-\tau}^{t} A(\mu)\,d\mu > \frac{1}{e}.$$

Then (3) has no eventually positive solutions.

Theorem 2.2 ([7]. Theorem 1). *Assume that*

$$\int^{\infty} A(\mu) \ln \left\{ e \int_{\mu}^{\mu+\tau} A(\lambda) d\lambda \right\} d\mu = \infty \quad \textit{and} \quad \int_{t}^{t+\tau} A(\mu)\,d\mu > 0 \quad \textit{for all large} \quad t.$$

Then (3) has no eventually positive solutions.

Now consider the second-order delay differential inequality

$$x''(t) + A(t)x(t) \leq 0 \quad \text{for} \quad t \geq t_0 \tag{4}$$

with $A \in C([t_0, \infty), \mathbb{R}^+)$.

Theorem 2.3 [4]. *Assume that*

$$\liminf_{t\to\infty} t\int\limits_t^\infty A(\mu)d\mu > \frac{1}{4}.$$

Then (4) has no eventually positive solutions.

3. Main results. This section is dedicated to the study of (1) under the following primary assumptions:

(A1) $H \in C([t_0,\infty)\times[\tau_1,\tau_2],\mathbb{R}^+)$ defined by $H(t,s) := P(t,s) - Q(t-\sigma,s)$, is not identically zero,

(A2) $\rho_1 := \begin{cases} \rho, & Q\equiv 0, \\ \min\{\tau_1-\sigma,\rho\}, & \text{otherwise,}\end{cases}$ and $\rho_2 := \begin{cases} \tau_2, & R\equiv 0, \\ \max\{\tau_2,\rho\}, & \text{otherwise.}\end{cases}$

Throughout the paper, we assume that (A1) and (A2) hold.

Lemma 3.1. *Assume that*

$$R(t) + \int\limits_{t-\sigma}^{t}\int\limits_{\tau_1}^{\tau_2} Q(\mu,\zeta)\,d\zeta d\mu \le 1 \quad \textit{for all large} \quad t. \tag{5}$$

If x is an eventually positive solution of (1), then the companion function z defined by

$$z(t) := x(t) - R(t)x(t-\rho) - \int\limits_{t-\sigma}^{t}\int\limits_{\tau_1}^{\tau_2} Q(\mu,\zeta)x(\mu-\zeta+\sigma)d\zeta\,d\mu \quad \textit{for} \quad t \ge t_0+\rho_2, \tag{6}$$

satisfies

$$z'(t)\le 0 \quad \textit{and} \quad z(t)>0 \quad \textit{for all large} \quad t.$$

Proof. Let x be an eventually positive solution of (1). Then there exists a $t_1 \ge t_0$ such that $x(t-\rho_2) > 0$ and (5) hold for all $t \ge t_1$. From (1) and (6), we obtain

$$z'(t) = [x(t) - R(t)x(t-\rho)]' - \left(\int\limits_{t-\sigma}^{t}\int\limits_{\tau_1}^{\tau_2} Q(\mu,\zeta)x(\mu-\zeta+\sigma)d\zeta d\mu\right)' =$$

$$= [x(t) - R(t)x(t-\rho)]' - \int\limits_{\tau_1}^{\tau_2} Q(t,\zeta)x(t-\zeta+\sigma)d\zeta + \int\limits_{\tau_1}^{\tau_2} Q(t-\sigma,\zeta)x(t-\zeta)d\zeta =$$

$$= -\int\limits_{\tau_1}^{\tau_2} H(t,\zeta)x(t-\zeta)d\zeta \le 0 \tag{7}$$

for all $t \ge t_1$. Therefore, eventually nonincreasing z is either eventually positive or negative. Suppose that z is eventually negative. Thus there exists a $t_2 \ge t_1$ such that $z(t) < 0$ for all $t \ge t_2$. Now we have the following possible cases.

(C1) Let x be unbounded. Then there exists $T > t_2 + \rho_2$ satisfying $x(T) = \max\{x(t) : t \in \in [t_2, T]\}$. From (5) and (6), we see that

$$x(T) = z(T) + R(T)x(T-\rho) + \int\limits_{T-\sigma}^{T}\int\limits_{\tau_1}^{\tau_2} Q(\mu,\zeta)x(\mu-\zeta+\sigma)d\zeta d\mu \le$$

$$\le z(t_2) + \left(R(T) + \int\limits_{T-\sigma}^{T}\int\limits_{\tau_1}^{\tau_2} Q(\mu,\zeta)d\zeta d\mu\right)x(T) \le z(t_2) + x(T),$$

which yields a contradiction because of $z(t_2) < 0$.

(C2) Let x be bounded. That is $L := \limsup_{t\to\infty} x(t) < \infty$. Thus there exists an increasing divergent sequence of reals $\{\xi_n\}_{n=1}^{\infty}$ satisfying $\lim_{n\to\infty} x(\xi_n) = L$. Let $\{\eta_n\}_{n=1}^{\infty}$ be a sequence of divergent real numbers satisfying $x(\eta_n) = \max\{x(t) : t \in [\xi_n - \rho_2, \xi_n - \rho_1]\}$. Clearly, we have $\limsup_{n\to\infty} x(\eta_n) \le L$. Thus from (5) and (6), we have

$$x(\xi_n) = z(\xi_n) + R(\xi_n)x(\xi_n-\rho) + \int\limits_{\xi_n-\sigma}^{\xi_n}\int\limits_{\tau_1}^{\tau_2} Q(\mu,\zeta)x(\mu-\zeta+\sigma)d\zeta d\mu \le$$

$$\le z(t_2) + \left(R(\xi_n) + \int\limits_{\xi_n-\sigma}^{\xi_n}\int\limits_{\tau_1}^{\tau_2} Q(\mu,\zeta)d\zeta d\mu\right)x(\eta_n) \le z(t_2) + x(\eta_n)$$

for all $n \in \mathbb{N}$. By taking upper limit on both sides of the above inequality, we obtain $L \le \le z(t_2) + L$, which yields a contradiction and therefore the proof is complete.

We obtain contradictions in both of the possible cases (C1) and (C2). Therefore z is eventually positive.

Lemma 3.1 is proved.

We state below the first main result of the paper.

Theorem 3.1. *Assume that (5) holds and*

$$x'(t) + \int\limits_{\tau_1}^{\tau_2} H(t,\zeta)x(t-\zeta)d\zeta \le 0 \quad \textit{for} \quad t \ge t_0$$

has no eventually positive solutions. Then every solution of (1) is oscillatory.

Proof. Suppose to the contrary that (1) has an eventually positive solution x. From Lemma 3.1, we obtain $x \ge z > 0$ eventually. Substituting z into (7), we see that the desired inequality holds and leads to a contradiction with the positive nature of z.

Corollary 3.1. *Assume that*

$$\liminf_{t\to\infty} \int\limits_{t-\tau_1}^{t}\int\limits_{\tau_1}^{\tau_2} H(\mu,\zeta)d\zeta d\mu > \frac{1}{e} \tag{8}$$

or

$$\int^{\infty}\int_{\tau_1}^{\tau_2} H(\mu,\zeta)\,d\zeta \ln\left\{ e\int_{\mu}^{\mu+\tau_1}\int_{\tau_1}^{\tau_2} H(\lambda,\zeta)\,d\zeta\,d\lambda\right\} d\mu=\infty,$$

$$\int_{t}^{t+\tau_1}\int_{\tau_1}^{\tau_2} H(\mu,\zeta)d\zeta d\mu>0 \quad \textit{for all large} \quad t. \tag{9}$$

Then every solution of (1) is oscillatory.

Proof. Assume on the contrary that (1) has a nonoscillatory solution. Without loss of generality, we assume that x is an eventually positive solution on $[t_1,\infty)$. Then we have by Lemma 3.1 that $z(t)$ is positive and nonincreasing for all $t\geq t_2$, where $t_2\geq t_1$ is sufficiently large. Considering (6) and (7), we see that

$$0=z'(t)+\int_{\tau_1}^{\tau_2} H(t,\zeta)x(t-\zeta)d\zeta\geq z'(t)+\int_{\tau_1}^{\tau_2} H(t\zeta)z(t-\zeta)d\zeta\geq$$

$$\geq z'(t)+\left(\int_{\tau_1}^{\tau_2} H(t,\zeta)d\zeta\right)z(t-\tau_1) \tag{10}$$

for all $t\geq t_2$. But in view of (8) or (9), the inequality (10) has no eventually positive solutions by Theorem 2.1 and Theorem 2.2. This is the contradiction which completes the proof.

Almost all of the results for the oscillation of the neutral differential equation with concentrated delay (2) makes use of the condition

$$R(t)+\int_{t-\tau+\sigma}^{t} Q(\mu)d\mu\leq 1 \quad \text{for all large} \quad t,$$

which can be regarded as the analogue of (5). For instance, see [3] (Theorem 2.6.1) and [8] (Theorem 3.1).

Remark 3.1. Consider (1) with $Q\equiv 0$. In this case, Corollary 3.1 still improves [2] (Theorem 1) by dropping the condition

$$\int^{\infty}\int_{\tau_1}^{\tau_2} P(\mu,\zeta)d\zeta\,d\mu=\infty.$$

Now we give the following example.

Example 3.1. Consider the following neutral differential equation with distributed delays:

$$[x(t)-e^{-2\pi}x(t-\pi)]'+\int_{\frac{5}{2}\pi}^{3\pi}\left[e^{\zeta}x(t-\zeta)-e^{\zeta-4\pi}x(t-\zeta+\pi)\right]d\zeta=0 \quad \text{for} \quad t\geq 0. \tag{11}$$

Here $R(t) = e^{-2\pi}$, $\rho = \pi$, $P(t,s) = e^{s}$, $Q(t,s) = e^{s-4\pi}$, $\sigma = \pi$, $\tau_1 = 5\pi/2$ and $\tau_2 = 3\pi$. Hence we have

$$e^{-2\pi} + \int\limits_{t-\pi}^{t} \int\limits_{\frac{5}{2}\pi}^{3\pi} e^{\zeta-4\pi} d\zeta d\mu = e^{-\frac{3}{2}\pi} \left[e^{-\frac{\pi}{2}} + \pi (e^{\frac{\pi}{2}} - 1) \right] < 1 \quad \text{for all} \quad t \geq 0$$

and

$$\liminf_{t\to\infty} \int\limits_{t-\frac{5}{2}\pi}^{t} \int\limits_{\frac{5}{2}\pi}^{3\pi} \left(1 - e^{-4\pi}\right) e^{\zeta} d\zeta d\mu = \frac{5}{2}\pi e^{-\frac{3}{2}\pi} \left(e^{4\pi} - 1\right) \left(e^{\frac{1}{2}\pi} - 1\right) > \frac{1}{e}.$$

Therefore every solution of (11) is oscillatory by Corollary 3.1. Direct substitution shows that $x(t) = e^{t}\sin(t)$ is such an oscillatory solution.

Lemma 3.2. *Assume that*

$$R(t) + \int\limits_{t-\sigma}^{t} \int\limits_{\tau_1}^{\tau_2} Q(\mu,\zeta) d\zeta d\mu \geq 1 \quad \textit{for all large} \quad t. \tag{12}$$

If x is an eventually positive solution of (1) and the second-order differential inequality

$$x''(t) + \left(\frac{1}{\rho_2} \int\limits_{\tau_1}^{\tau_2} H(t,\zeta) d\zeta \right) x(t) \leq 0 \quad \textit{for} \quad t \geq t_0 \tag{13}$$

has no eventually positive solutions, then

$$z'(t) \leq 0 \quad \textit{and} \quad z(t) < 0 \quad \textit{for all large} \quad t.$$

Proof. Let x be an eventually positive solution of (1). Say $x(t-\rho_2) > 0$ for all $t \geq t_1$. As in the proof of Lemma 3.1, we have $z'(t) \leq 0$ for all $t \geq t_2$, where $t_2 \geq t_1$. Thus $z(t)$ is of constant sign and (7) holds for all $t \geq t_3$, where $t_3 \geq t_2$. We shall prove that $z(t) < 0$ for all $t \geq t_3$. On the contrary assume that $z(t) > 0$ for all $t \geq t_3$. Now set $M := \min\{x(t) : t \in \in [t_3 - \rho_2, t_3]\}/2 \geq z(t_3)/2 > 0$. We claim that $x(t) > M$ for all $t \geq t_3 - \rho_2$. If not, there exists $T > t_3$ such that $x(t) > M$ for all $t \in [t_3 - \rho_2, T)$ and $x(T) = M$. From (6) and (12), we see that

$$x(T) = z(T) + R(T)x(T-\rho) + \int\limits_{T-\sigma}^{T} \int\limits_{\tau_1}^{\tau_2} Q(\mu,\zeta) x(\mu - \zeta + \sigma) d\zeta d\mu >$$

$$> M \left(R(T) + \int\limits_{T-\sigma}^{T} \int\limits_{\tau_1}^{\tau_2} Q(\mu,\zeta) d\zeta d\mu \right) \geq M = x(T),$$

which is a contradiction. Now let $L := \lim_{t\to\infty} z(t)$ and consider the following cases.

(C1) Let $L = 0$. There exists a $T \geq t_3$ satisfying $z(t) < M/2$ for all $t \geq T$. Then we have

$$x(t) > M = \frac{1}{\rho_2}\frac{M}{2}2\rho_2 \geq \frac{1}{\rho_2}\frac{M}{2}(t+\rho_2-T) > \frac{1}{\rho_2}\int\limits_{T}^{t+\rho_2} z(\lambda)d\lambda$$

for all $t \in [T, T+\rho_2]$.

(C2) Let $L > 0$. Since $z'(t) \leq 0$, we have $z(t) > L$ for all $t \geq t_3$. From (6) and (12), we obtain

$$x(t) > L + R(t)x(t-\rho) + \int\limits_{t-\sigma}^{t}\int\limits_{\tau_1}^{\tau_2} Q(\mu,\zeta)x(\mu-\zeta+\sigma)d\zeta\, d\mu \geq$$

$$\geq L + \left(R(t) + \int\limits_{t-\sigma}^{t}\int\limits_{\tau_1}^{\tau_2} Q(\mu,\zeta)d\zeta\, d\mu\right) M \geq L + M$$

for all $t \geq t_3$, which by induction yields to

$$x(t) \geq nL + M \quad \text{for all} \quad t \geq t_3 + (n-1)\rho_2 \quad \text{and} \quad n \in \mathbb{N},$$

and thus $\lim_{t\to\infty} x(t) = \infty$. Therefore there exists a $T \geq t_3$ satisfying $x(t) > 2z(T)$ for all $t \geq T$. Hence

$$x(t) > 2z(T) \geq \frac{t+\rho_2-T}{\rho_2}z(T) \geq \frac{1}{\rho_2}\int\limits_{T}^{t+\rho_2} z(\lambda)\, d\lambda$$

for all $t \in [T, T+\rho_2]$.

Combining the cases (C1) and (C2) ensures existence of a $t_4 \geq t_3$ satisfying

$$x(t) > \frac{1}{\rho_2}y(t+\rho_2) \tag{14}$$

for all $t \in [t_4, t_4+\rho_2]$, where

$$y(t) := \int\limits_{t_4}^{t} z(\lambda)\, d\lambda \quad \text{for} \quad t \geq t_4.$$

Note that y is positive and increasing. Now we claim that (14) holds for all $t \geq t_4$. If not, there exists a $T > t_4+\rho_2$ such that (14) holds for all $t \in [t_4, T)$ and $x(T) = y(T+\rho_2)/\rho_2$. Considering

(6), (12) and (14), we get

$$x(T) > z(T) + \frac{1}{\rho_2}\left(R(T)y(T-\rho+\rho_2) + \int\limits_{T-\sigma}^{T}\int\limits_{\tau_1}^{\tau_2} Q(\mu,\zeta)y(\mu-\zeta+\rho_2+\sigma)\,d\zeta\,d\mu\right) \geq$$

$$\geq z(T) + \frac{1}{\rho_2}\left(R(t)y(T) + y(T-\tau_2+\rho_2)\int\limits_{T-\sigma}^{T}\int\limits_{\tau_1}^{\tau_2} Q(\mu,\zeta)d\zeta d\mu\right) \geq$$

$$\geq \frac{1}{\rho_2}\int\limits_{T}^{T+\rho_2} z(\lambda)d\lambda + \frac{1}{\rho_2}\left(R(t) + \int\limits_{T-\sigma}^{T}\int\limits_{\tau_1}^{\tau_2} Q(\mu,\zeta)d\zeta\,d\mu\right)y(T) \geq$$

$$\geq \frac{1}{\rho_2}\left(\int\limits_{T}^{T+\rho_2} z(\lambda)\,d\lambda + y(T)\right) = \frac{1}{\rho_2}y(T+\rho_2) = x(T),$$

which is a contradiction. Hence, (14) holds on $[t_4,\infty)$, which implies

$$x(t-\zeta) \geq \frac{1}{\rho_2}y(t-\zeta+\rho_2) \geq \frac{1}{\rho_2}y(t) \quad \text{for all} \quad \zeta\in[\tau_1,\tau_2] \quad \text{and} \quad t\geq t_4. \tag{15}$$

Then $y>0$, $y'=z>0$ and $y''=z'\leq 0$ hold on $[t_4,\infty)$. By taking (15) into account and substituting y into (7), we obtain

$$0 \geq y''(t) + \left(\frac{1}{\rho_2}\int\limits_{\tau_1}^{\tau_2} H(t,\zeta)d\zeta\right)y(t \quad \text{for all} \quad t\geq t_4,$$

which shows that (13) has an eventually positive solution. This contradiction proves the claim that z is eventually negative.

The following is another main result of the paper.

Theorem 3.2. *Assume that*

$$R(t) + \int\limits_{t-\sigma}^{t}\int\limits_{\tau_1}^{\tau_2} Q(\mu,\zeta)\,d\zeta\,d\mu \equiv 1 \quad \textit{for all large} \quad t \tag{16}$$

and (13) has no eventually positive solutions. Then every solution of (1) is oscillatory.

Proof. For the sake of contradiction, let x be an eventually positive solution of (1), then the companion function z is eventually positive by Lemma 3.1 while it is eventually negative by Lemma 3.2.

Corollary 3.2. *Suppose that (16) holds. If*

$$\liminf_{t\to\infty} t\int\limits_{t}^{\infty}\int\limits_{\tau_1}^{\tau_2} H(\mu,\zeta)\,d\zeta\,d\mu > \frac{\rho_2}{4}, \tag{17}$$

then every solution of (1) is oscillatory.

The conditions

$$R(t)+\int\limits_{t-\tau+\sigma}^{t} Q(\mu)\,d\mu\equiv 1 \quad \text{for all large} \quad t \tag{18}$$

and

$$\liminf_{t\to\infty} t\int\limits_{t}^{\infty}\left[P(\mu)-Q(\mu-\tau+\sigma)\right]\,d\mu>\frac{\rho_2}{4}, \tag{19}$$

which are analogues of (16) and (17), respectively, appear in recent results for the oscillation of the neutral differential equation with concentrated delays (2), see [8] (Theorem 3.2), [10] (Theorem 3.1) and [11] (Theorem 1), see also [10] (Theorem 3.1), where (18) is still required but instead of (19) the following is assumed

$$\int\limits_{t_0}^{\infty}\left[P(\mu)-Q(\mu-\tau+\sigma)\right]\exp\left\{\frac{1}{\rho_2}\int\limits_{t_0}^{\mu}\left[P(\zeta)-Q(\zeta-\tau+\sigma)\right]\,d\zeta\right\}d\mu=\infty.$$

We would like to mention that the condition (18) is replaced with a weaker one in [9] (Theorem 1).

Now, we proceed with an example.

Example 3.2. Consider the following neutral differential equation with distributed deviating arguments:

$$\left[x(t)-\frac{1}{4}x(t-1)\right]'+\int\limits_{1}^{4}\left[\left(\frac{1}{t^2\sqrt{\zeta}}+\frac{1}{4}\right)x(t-\zeta)-\frac{1}{4}\,x(t-\zeta+1)\right]\,d\zeta=0 \quad \text{for} \quad t\geq 5. \tag{20}$$

In this equation, $R(t)=1/4$, $\rho=1$, $P(t,s)=1/(t^2\sqrt{s})+1/4$, $Q(t,s)=1/4$, $\sigma=1$, $\tau_1=1$ and $\tau_2=4$. Then we have

$$\frac{1}{4}+\int\limits_{t-1}^{t}\int\limits_{1}^{4}\frac{1}{4}\,d\zeta\,d\mu\equiv 1 \quad \text{for all} \quad t\geq 5$$

and

$$\liminf_{t\to\infty} t\int\limits_{t}^{\infty}\int\limits_{1}^{4}\frac{1}{\mu^2\sqrt{\zeta}}\,d\zeta\,d\mu=2>1.$$

Therefore every solution of (20) is oscillatory.

4. Iterative results. In this section, we advance the results given in the previous section by using a recursive method.

To this end, we need to introduce

$$H_n(t) := \begin{cases} 1, & n = 0, \\ R(t)H_{n-1}(t-\rho) + \int\limits_{t-\sigma}^{t} \int_{\tau_1}^{\tau_2} Q(\mu,\zeta)H_{n-1}(\mu-\zeta+\sigma)\,d\zeta\,d\mu, & n \in \mathbb{N}. \end{cases}$$

Lemma 4.1. *Assume that assumptions of Lemma 3.1 hold. Then, for any* $n \in \mathbb{N}$, z *is an eventually positive solution of the inequality*

$$x'(t) + \int\limits_{\tau_1}^{\tau_2} H(t,\zeta) \sum_{k=0}^{n} H_k(t-\zeta)x(t-\zeta)\,d\zeta \le 0 \quad \textit{for} \quad t \ge t_0. \tag{21}$$

Proof. From Lemma 3.1, we see that there exists a $t_1 \ge t_0$ such that $x(t) \ge z(t) > 0$ and $z'(t) \le 0$ hold for all $t \ge t_1$. We claim that there exists an increasing divergent sequence of reals $\{\xi_n\}_{n=1}^{\infty} \subset [t_1,\infty)$ satisfying

$$x(t) \ge \sum_{k=0}^{n} H_k(t)\,z(t) \tag{22}$$

for all $t \ge \xi_n$ and $n \in \mathbb{N}$. We use mathematical induction. For $n = 1$, we see from (6) that

$$x(t) \ge z(t) + R(t)z(t-\rho) + \int\limits_{t-\sigma}^{t} \int\limits_{\tau_1}^{\tau_2} Q(\mu,\zeta)z(\mu-\zeta+\sigma)\,d\zeta\,d\mu \ge$$

$$\ge \left(1 + R(t) + \int\limits_{t-\sigma}^{t} \int\limits_{\tau_1}^{\tau_2} Q(\mu,\zeta)\,d\zeta\,d\mu\right) z(t) = \sum_{k=0}^{1} H_k(t)\,z(t)$$

for all $t \ge \xi_1 \ge t_1 + \rho_2$. Now suppose that (22) holds for some n. Then we have

$$x(t) = z(t) + R(t)x(t-\rho) + \int\limits_{t-\sigma}^{t} \int\limits_{\tau_1}^{\tau_2} Q(\mu,\zeta)x(\mu-\zeta+\sigma)\,d\zeta\,\mu \ge$$

$$\ge z(t) + R(t)\sum_{k=0}^{n} H_k(t-\rho)z(t-\rho) +$$

$$+ \int\limits_{t-\sigma}^{t} \int\limits_{\tau_1}^{\tau_2} Q(\mu,\zeta)\sum_{k=0}^{n} H_k(\mu-\zeta+\sigma)z(\mu-\zeta+\sigma)\,ds\zeta\,d\mu \ge$$

$$\ge \left(1 + R(t)\sum_{k=0}^{n} H_k(t-\rho) + \int\limits_{t-\sigma}^{t} \int\limits_{\tau_1}^{\tau_2} Q(\mu,\zeta)\sum_{k=0}^{n} H_k(\mu-\zeta+\sigma)\,d\zeta\,d\mu\right) z(t) = \sum_{k=0}^{n+1} H_k(t)z(t),$$

for all $t \geq \xi_{n+1} \geq \xi_n + \rho_2$, which shows that the claim is also true for $(n+1)$. Substituting (22) into (7), we see that z is an eventually positive solution of (21).

Lemma 4.1 is proved.

The last main result of the paper is the following.

Theorem 4.1. *Assume that assumptions of Lemma 3.1 hold and there exists* $n_0 \in \mathbb{N}$ *such that*

$$x'(t) + \int_{\tau_1}^{\tau_2} H(t,\zeta) \sum_{k=0}^{n_0} H_k(t-\zeta) x(t-\zeta)\, d\zeta \leq 0 \quad \textit{for} \quad t \geq t_0$$

has no eventually positive solutions. Then every solution of (1) is oscillatory.

Proof. Proof is trivial and is omitted.

As an immediate consequence of Theorem 4.1, we have the following corollary.

Corollary 4.1. *Assume that there exists* $n_0 \in \mathbb{N}$ *satisfying*

$$\liminf_{t\to\infty} \int_{t-\tau_1}^{t} \int_{\tau_1}^{\tau_2} H(\mu,\zeta) \sum_{k=0}^{n_0} H_k(\mu-\zeta)\, d\zeta\, d\mu > \frac{1}{e}$$

or

$$\int_{t_0}^{\infty} \int_{\tau_1}^{\tau_2} H(\mu,\zeta) \sum_{k=0}^{n_0} H_k(\mu-\zeta)\, d\zeta \ln \left\{ e \int_{\mu}^{\mu+\tau} \int_{\tau_1}^{\tau_2} H(\lambda,\zeta) \sum_{k=0}^{n_0} H_k(\lambda-\zeta)\, d\zeta\, d\lambda \right\} d\mu = \infty,$$

$$\int_{t}^{t+\tau_1} \int_{\tau_1}^{\tau_2} H(\mu,\zeta) \sum_{k=0}^{n_0} H_k(\mu-\zeta)\, d\zeta\, d\mu > 0 \quad \textit{for all large} \quad t.$$

Then all solutions of (1) are oscillatory.

As an application of Corollary 4.1, we present two simple examples below.

Example 4.1. Consider the following neutral differential equation with distributed delays:

$$\left[x(t) - \frac{3}{8}x(t-1)\right]' + \int_{1}^{2} \left[\left(\frac{1}{t} + \frac{3}{8}\right) x(t-\zeta) - \frac{1}{8}x(t-\zeta+1)\right] d\zeta = 0 \quad \text{for} \quad t \geq 5. \tag{23}$$

For this equation $R(t) = 3/8$, $\rho = 1$, $P(t,s) = 1/t + 1/8$, $Q(t,s) = 1/8$, $\sigma = 1$, $\tau_1 = 1$ and $\tau_2 = 2$. Then we have

$$H(t,s) = \frac{1}{t} + \frac{1}{4} \quad \text{and} \quad H_n(t) = \frac{1}{2^n} \quad \text{for} \quad n \in \mathbb{N},$$

and so

$$L(n) := \liminf_{t\to\infty} \int_{t-1}^{t} \int_{1}^{2} \left(\frac{1}{\mu} + \frac{1}{4}\right) \sum_{k=0}^{n} \frac{1}{2^k}\, d\zeta\, d\mu = \left(1 - \frac{1}{2^{(n+1)}}\right) \frac{1}{2}.$$

Note that $L(0) < 1/e$, while $L(1) > 1/e$. Hence every solution of (23) is oscillatory by Corollary 4.2.

Example 4.2. Consider the following neutral differential equation:

$$\left[x(t) - \frac{13}{16}x(t-1)\right]' + \int_1^2 \left[\left(\frac{1}{t} + \frac{1}{2}\right)x(t-\zeta) - \frac{1}{16}x(t-\zeta+1)\right]d\zeta = 0 \quad \text{for} \quad t \geq 5. \tag{24}$$

For this equation $R(t) = 13/16$, $\rho = 1$, $P(t,s) = 1/t + 1/16$, $Q(t,s) = 1/16$, $\sigma = 1$, $\tau_1 = 1$ and $\tau_2 = 2$. Then we have

$$H(t,s) = \frac{1}{t} + \frac{7}{16} \quad \text{and} \quad H_n(t) = \left(\frac{7}{8}\right)^n \quad \text{for} \quad n \in \mathbb{N},$$

and so

$$L(n) := \liminf_{t\to\infty} \int_{t-1}^{t} \int_1^2 \left(\frac{1}{\mu} + \frac{7}{16}\right) \sum_{k=0}^{n} \left(\frac{7}{8}\right)^k d\zeta\, d\mu = \left(1 - \left(\frac{7}{8}\right)^{(n+1)}\right) \frac{7}{16}.$$

One can check that $L(n) < 1/e$ for $n = 0, 1, \ldots, 12$, while $L(13) > 1/e$. Hence every solution of (24) is oscillatory by Corollary 4.1.

1. *Agarwal R. P., Grace S. R., O'Regan D.* Oscillation theory for difference and functional differential equations. — Dordrecht: Kluwer Acad. Publ., 2000.
2. *Candan T.* Oscillation of first-order neutral differential equations with distributed deviating arguments // Comput. Math. Appl. — 2008. — **55**, № 3. — P. 510–515.
3. *Győri I., Ladas G.* Oscillation theory of delay differential equations with applications. — Oxford: Oxford Sci. Publ., 1991.
4. *Hille E.* Non-oscillation theorems // Trans. Amer. Math. Soc. — 1948. — **64**, № 2. — P. 234–252.
5. *Karpuz B., Öcalan Ö.* Oscillation criteria for some classes of linear delay differential equations of first-order // Bull. Inst. Math. Acad. Sin. (N.S.). — 2008. — **3**, № 2. — P. 293–314.
6. *Karpuz B.* Some oscillation and nonoscillation criteria for neutral delay difference equations with positive and negative coefficients // Comput. Math. Appl. — 2009. — **57**, № 4. — P. 633–642.
7. *Li B.* Oscillation of first order delay differential equations // Proc. Amer. Math. Soc. — 1996. — **124**, № 12. — P. 3729–3737.
8. *Luo Z. G., Shen J. H.* Oscillation and nonoscillation of neutral differential equations with positive and negative coefficients // Czech. Math. J. — 2004. — **54 (129)**, № 1. — P. 79–93.
9. *Öcalan Ö.* Oscillation of neutral differential equations with positive and negative coefficients // J. Math. Anal. and Appl. — 2007. — **331**, № 1. — P. 644–654.
10. *Shen J. H., Debnath L.* Oscillations of solutions of neutral differential equations with positive and negative coefficients // Appl. Math. Lett. — 2001. — **14**, № 6. — P. 775–781.
11. *Shen J. H., Stavroulakis I. P., Tang X. H.* Hille type oscillation and nonoscillation criteria for neutral equations with positive and negative coefficients // Stud. Univ. Žilina Math. Ser. — 2001. — **14**, № 1. — P. 45–59.
12. *Tang X. H., Yu J. S., Peng D. H.* Oscillation and nonoscillation of neutral difference equations with positive and negative coefficients // Comput. Math. Appl. — 2000. — **39**, № 7–8. — P. 169–181.
13. *Tian C. J., Cheng S. S.* Oscillation criteria for delay neutral difference equations with positive and negative coefficients // Bol. Soc. paran. mat. (3). — 2003. — **21**, № 1–2. — P. 19–30.

Received 05.03.10,
after revision — 17.03.11