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BOUNDARY-VALUE PROBLEMS OF DIFFERENTIAL INCLUSIONS WITH RIEMANN – LIOUVILLE FRACTIONAL DERIVATIVE

ГРАНИЧНІ ЗАДАЧІ ДЛЯ ДИФЕРЕНЦІАЛЬНИХ ВКЛЮЧЕНЬ З ДРОБОВОЮ ПОХІДНОЮ РІМАНА – ЛІУВІЛЛЯ

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In this paper, we establish sufficient conditions for the existence of solutions for a class of boundary value problem for fractional differential inclusions involving the Riemann-Liouville fractional derivative. The cases of both convex and nonconvex valued right-hand side are considered. The topological structure of the set of solutions is also examined.

Отримано достатні умови існування розв'язків для класу граничних задач для диференціальних включень з похідною дробового порядку, включаючи дробову похідну Рімана – Ліувілля. Розглянуто випадки опуклої та неопуклої правої частини. Також вивчено топологічну структуру множини розв'язків.

1. Introduction. This paper deals with the existence of solutions for the boundary-value problem (BVP for short) for fractional order differential inclusions of the form

$$D^{\alpha}y(t) \in F(t, y(t)), \quad \text{for a.e.} \quad t \in J, \quad 1 < \alpha \leq 2, \tag{1}$$

$$y(0) = 0, \quad y'(T) = 0, \tag{2}$$

where D^{α} is the Riemann – Liouville fractional derivative, $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, and $J = [0, T]$. Differential equations of fractional order have recently been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic. A significant development in fractional differential and partial differential equations has appeared in recent years; we refer to the monographs by Kilbas et al. [15], Podlubny [17], Samko et al. [18] and the papers by Agarwal et al. [1], Benchohra and Hamani [2], Furati and Tatar [9], and Ouahab [16], and the references therein. In [3], the authors studied the existence and uniqueness of solutions of classes of initial value problems for functional differential equations with infinite delay and fractional order. The aim of this work is the study of a BVP for differential inclusion with a Riemann – Liouville fractional derivative.

This paper is organized as follows. In Section 2, we introduce some preliminary results needed in the subsequent sections. In Section 3, using the nonlinear alternative of Leray and Schauder, we present an existence result for problem (1), (2) when the right-hand side is convex-valued. In Section 4, two results, for the case of nonconvex-valued right-hand side, are given. The first one is based upon a fixed point theorem for contraction multivalued maps due to Covitz and Nadler while the second one employs the nonlinear alternative of Leray and Schauder for single-valued maps [11], combined with a selection theorem due to Bressan – Colombo [4] for lower semicontinuous multivalued maps with decomposable values. The topological structure of the solutions set is also considered in Section 5. These results extend to the multivalued case some results from the above cited literature, and constitute a contribution to this emerging field of research.

2. Preliminaries. In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper. Let $C(J,\mathbb{R})$ be the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : 0 \le t \le T\},$$

and let $L^1(J,\mathbb{R})$ denote the Banach space of functions $y : J \longrightarrow \mathbb{R}$ that are Lebesgue integrable with norm

$$\|y\|_{L^1} = \int\limits_0^T |y(t)|dt.$$

$AC(J,\mathbb{R})$ is the space of functions $y : J \to \mathbb{R}$, which are absolutely continuous. Given a topological vector space X, let $\mathcal{P}(X)$ be the set of all nonempty subsets of X. Denote by $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$, $P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$, $P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$, $P_{cp,cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact and convex}\}$, $P_{cl,cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed and convex}\}$ and so on. A multivalued map $G : X \to P(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(A) = \Cup_{x\in A} G(x)$ is bounded in X for all $A \in P_b(X)$, i.e. $\sup_{x\in A}\{\sup\{|y| : y \in G(x)\}\} < \infty$. G is called upper semicontinuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subseteq N$, Equivalently, G is u.s.c. if the set $\{x \in X : G(x) \subset B\}$ is open for any open set B in X. G is lower semicontinuous (l.s.c.) if the set $\{x \in X : G(x) \cap B \neq \varnothing\}$ is open for any open set B in X. G is said to be completely continuous if $G(A)$ is relatively compact for every $A \in P_b(X)$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., ($x_n \longrightarrow x_*$, $y_n \longrightarrow y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $Fix\,G$. A multivalued map $G : J \to \mathcal{P}_{cl}(\mathbb{R})$ is said to be measurable provided for every open $U \subset \mathbb{R}$, the set $\{t \in J,\ G(t) \subset U\}$ is Lebesgue measurable in J. We have the following lemma.

Lemma 2.1 (see [5, 10]). *G is measurable if and only if for each $x \in \mathbb{R}$, the function $\zeta : J \to [0,+\infty)$ defined by $\zeta(t) = \operatorname{dist}(x, G(t)) = \inf\{\|x-y\|,\ y \in G(t)\}, \quad t \in J$ is Lebesgue measurable.*

The following lemma is known as the Kuratowski – Ryll – Nardzewski selection theorem.

Lemma 2.2 (see [10], Theorem 19.7 or [5], Theorem III.6). *Let E be a separable metric space and $G : [a,b] \to \mathcal{P}(E)$ a measurable multivalued map with closed values. Then G has a measurable selection.*

The following one is taken from [20], Lemma 3.2.

Lemma 2.3. *Let $G : [0,b] \to \mathcal{P}(E)$ be a measurable multifunction and $u : [0,b] \to E$ a measurable function. Then, for any measurable $v : [0,b] \to (0,+\infty)$, there exists a measurable selection g_v of G such that, for a.e. $t \in [0,b]$,*

$$|u(t) - g_v(t)| \le d(u(t), G(t)) + v(t).$$

Definition 2.1. *A multivalued map $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if*

(a) *the map $t \longmapsto F(t,u)$ is measurable for each $u \in \mathbb{R}$;*

(b) *the map $u \longmapsto F(t,u)$ is upper semicontinuous for a.e. $t \in J$;*

(c) *it is L^1-Carathéodory if it is further integrably bounded, i.e., there exists $h \in L^1(J,\mathbb{R}^+)$ such that*

$$\|F(t,x)\|_{\mathcal{P}} \le h(t) \quad \textit{for a.e.} \quad t \in J \quad \textit{and all} \quad x \in \mathbb{R},$$

where $\|F(t,u)\|_{\mathcal{P}} = \sup\{|v| : v \in F(t,u)\}$.

For each $y \in C(J,\mathbb{R})$, define the set of selections of F by

$$S_{F,y} = \{v \in L^1(J,\mathbb{R}) : v(t) \in F(t,y(t)), \quad \text{a.e.} \quad t \in J\}.$$

Remark 2.1. From ([19], Theorem 5.10), we know that $S_{F,y}$ is nonempty if and only if the mapping $t \mapsto \{\inf\|v\| : v \in F(t,y(t))\}$ belongs to $L^1(J)$. It is bounded if and only if the mapping $t \mapsto \|F(t,y(t))\|_{\mathcal{P}} = \{\sup\|v\| : v \in F(t,y(t))\}$ belongs to $L^1(J)$; this particularly holds true when F is integrably bounded.

Let (X,d) be a metric space. Define the Hausdorff pseudometric distance $H_d : \mathcal{P}(X) \times \times\mathcal{P}(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ by

$$H_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\},$$

where $d(A,b) = \inf_{a\in A} d(a,b)$ and $d(a,B) = \inf_{b\in B} d(a,b)$. Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space (see [14]).

Definition 2.2. *A multivalued operator $N : X \to P_{cl}(X)$ is called* *(a) γ-Lipschitz if there exists $\gamma > 0$ such that*

$$H_d(N(x), N(y)) \leq \gamma d(x,y), \quad \textit{for each} \quad x, y \in X,$$

(b) a contraction if it is γ-Lipschitz with $0 < \gamma < 1$.

The following result is known as the Covitz – Nadler fixed point theorem.

Lemma 2.4 [6]. *Let (X, d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then $Fix\, N \neq \varnothing$.*

For more details about multivalued maps, we refer to the books by Deimling [7], Górniewicz [10], and Kisielewicz [14]. We end this section with the definitions of fractional order integral and derivative (see [15, 17]).

Definition 2.3. *The fractional (arbitrary) order integral of a function $h \in L^1([a,b], \mathbb{R}^+)$ of order $\alpha > 0$ is defined by*

$$I_a^\alpha h(t) = \frac{1}{\gamma(\alpha)} \int\limits_a^t (t-s)^{\alpha-1} h(s) ds,$$

where γ is the Gamma function. When $a = 0$, we write $I^\alpha h(t) = h(t) * \varphi_\alpha(t)$, where $\varphi_\alpha(t) = = \dfrac{t^{\alpha-1}}{\gamma(\alpha)}$ for $t > 0$ and $\varphi_\alpha(t) = 0$ for $t \leq 0$. Note φ_α behaves as the Delta function when $\alpha \to 0$. Indeed, it is shown (see e.g. (2.89) in [17], p. 65) that $\lim_{\alpha \to 0} I^\alpha h(t) = h(t)$ whenever h is continuous.

Definition 2.4. *For a function h defined on the interval $[a,b]$, the $\alpha - th$ Riemann – Liouville fractional-order derivative of h is given by*

$$(D_{a+}^\alpha h)(t) = \frac{1}{\gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int\limits_a^t (t-s)^{n-\alpha-1} h(s)\, ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

3. The convex case. In this section, we are concerned with the existence of solutions to problem (1), (2) when the right-hand side takes convex, compact values. Let us start by defining what we mean by a solution of problem (1), (2).

Definition 3.1. *A function $y \in AC(J, \mathbb{R})$ is said to be a solution of problem (1), (2), if there exists a function $v \in L^1(J, \mathbb{R})$ with $v(t) \in F(t, y(t))$, for a.e. $t \in J$, such that*

$$D^\alpha y(t) = v(t), \quad \textit{a.e.} \quad t \in J, \quad 1 < \alpha \leq 2,$$

and the function y satisfies the conditions (2).

For the existence of solutions for problem (1), (2), we need an auxiliary lemma:

Lemma 3.1 [15]. *Let $\alpha > 0$. If we assume $h \in C((0,T), \mathbb{R}) \bigcap L((0,T), \mathbb{R})$, then the fractional differential equation*

$$D^\alpha h(t) = 0$$

has solutions

$$h(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_n t^{\alpha-n}, \quad \textit{for} \quad c_i \in \mathbb{R}, \quad i = 1, 2, \ldots, n.$$

Lemma 3.2 [15]. *Assume* $h \in C((0,T),\mathbb{R}) \bigcap L((0,T),\mathbb{R})$ *with a fractional derivative of order* $\alpha > 0$. *Then*

$$I^\alpha D^\alpha h(t) = h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_n t^{\alpha-n}$$

for some constants c_i, $i = 1, 2, \ldots, n$.

As a consequence of Lemmas 3.1 and 3.2, we have the following result which provides the integral formulation for problem (1), (2).

Lemma 3.3. *Let* $1 < \alpha \leq 2$ *and let* $\sigma : J \to \mathbb{R}$ *be continuous. A function* y *is a solution of the fractional integral equation*

$$y(t) = \int_0^T G(t,s)\,\sigma(s)\,ds, \tag{3}$$

where

$$G(t,s) = \begin{cases} \dfrac{(t-s)^{\alpha-1}}{\gamma(\alpha)} - \dfrac{t^{\alpha-1}(T-s)^{\alpha-2}}{(\alpha-1)T^{\alpha-2}\gamma(\alpha-1)}, & 0 \leq s \leq t, \\[2ex] -\dfrac{t^{\alpha-1}(T-s)^{\alpha-2}}{(\alpha-1)T^{\alpha-2}\gamma(\alpha-1)}, & t \leq s < T, \end{cases} \tag{4}$$

if and only if y *is a solution of the fractional BVP*

$$D^\alpha y(t) = \sigma(t), \quad t \in J, \tag{5}$$

$$y(0) = 0, \quad y'(T) = 0. \tag{6}$$

Proof. Assume that y satisfies (5); then Lemma 3.2 implies that

$$y(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \frac{1}{\gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\sigma(s)\,ds.$$

From (6), a simple calculation yields $c_2 = 0$ and

$$c_1 = \frac{-1}{(\alpha-1)T^{\alpha-2}\gamma(\alpha-1)} \int_0^T (T-s)^{\alpha-2}\sigma(s)\,ds,$$

whence equation (3). Conversely, it is clear that if y satisfies equation (3), then by Lemmas 3.2, 3.3, equations (5), (6) hold.

Remark 3.1. The function $t \in J \mapsto \int_0^T |G(t,s)|ds$ is continuous on $[0,T]$, hence bounded. Let

$$G^* = \sup\left\{\int\limits_0^T |G(t,s)|ds,\ t \in J\right\}.$$

Our first existence result is based on the nonlinear alternative of Leray – Schauder type for multivalued maps [10, 11] which we recall for the reader's convenience:

Lemma 3.4. *Let $(X, \|\cdot\|)$ be a Banach space and $F\colon X \to \mathcal{P}_{cl,cv}(X)$ a compact, u.s.c. multivalued map. Then either one of the following conditions hold:*

(a) *F has at least one fixed point,*

(b) *the set $\mathcal{M} := \{x \in X,\ x \in \lambda F(x),\ \lambda \in (0,1)\}$ is unbounded.*

We have the following theorem.

Theorem 3.1. *Assume the following hypotheses hold:*

$(\mathcal{H}_1)$ *$F : J \times \mathbb{R} \longrightarrow \mathcal{P}_{cp,cv}(\mathbb{R})$ is a Carathéodory multivalued map.*

$(\mathcal{H}_2)$ *There exist $p \in L^\infty(J,\mathbb{R})$ and $\psi : [0,\infty) \to (0,\infty)$ continuous and nondecreasing such that*

$$\|F(t,u)\|_{\mathcal{P}} \leq p(t)\psi(|u|) \quad \textit{for} \quad t \in J \quad \textit{and each} \quad u \in \mathbb{R}.$$

$(\mathcal{H}_3)$ *There exists a constant $M > 0$ such that*

$$\frac{M}{p^* G^* \psi(M)} > 1, \tag{7}$$

where

$$p^* = \|p\|_{L^\infty}.$$

Then problem (1), (2) has at least one solution on J.

Proof. In order to transform problem (1), (2) into a fixed point problem, consider the multivalued operator $N : C(J,\mathbb{R}) \longrightarrow \mathcal{P}(C(J,\mathbb{R}))$ defined by

$$N(y) = \left\{h \in C(J,\mathbb{R}) : h(t) = \int\limits_0^T G(t,s)v(s)ds,\ v \in S_{F,y}\right\}$$

where the Green function $G(t,s)$ is given by (4). Clearly, from Lemma 3.3, the fixed points of N are solutions of (1) – (3). We shall show that N satisfies the assumptions of Lemma 3.4. The proof will be given in four steps. First, since $S_{F,y}$ is convex (because F has convex values), then *$N(y)$ is convex for each $y \in C(J,\mathbb{R})$.*

Step 1: N maps bounded sets into bounded sets in $C(J,\mathbb{R})$. Let $B_{\eta^*} = \{y \in C(J,\mathbb{R}) : \|y\|_\infty \leq \eta^*\}$ be a bounded set in $C(J,\mathbb{R})$ and $y \in B_{\eta^*}$. Then, for each $h \in N(y)$ and $t \in J$, there exists $v \in S_{F,y}$ such that, by $(\mathcal{H}_2)$ we have

$$|h(t)| \leq \int_0^T G(t,s)|v(s))|ds \leq p^* G^* \psi(\|y\|_\infty).$$

Thus

$$\|h\|_\infty \leq p^* G^* \psi(\eta^*).$$

Step 2: N maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$. Let $t_1, t_2 \in J, t_1 < t_2$, B_{η^*} be a bounded set of $C(J,\mathbb{R})$, $y \in B_{\eta^*}$ and $h \in N(y)$. As in Step 1, we have

$$|h(t_2) - h(t_1)| \leq \int_0^T |G(t_2,s) - G(t_1,s)|\,|v(s)|\,ds + p^*\psi(\eta^*) \sup_{s\in J} \left| \int_0^T G(t_2,s) - G(t_1,s) \right| ds.$$

As $t_1 \longrightarrow t_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1, 2 together with the Arzéla – Ascoli theorem, we conclude that N is completely continuous.

Step 3: N has a closed graph. Let $y_n \to y_*$, $h_n \in N(y_n)$ and $h_n \to h_*$. We need to show that $h_* \in N(y_*)$. $h_n \in N(y_n)$ means that there exists $v_n \in S_{F,y_n}$ such that, for each $t \in J$,

$$h_n(t) = \int_0^T G(t,s)\, v_n(s)\, ds.$$

We must show that there exists $v_* \in S_{F,y_*}$ such that, for each $t \in J$,

$$h_*(t) = \int_0^T G(t,s)\, v_*(s)\, ds.$$

Since $F(t,\cdot)$ is upper semicontinuous, for every $\varepsilon > 0$, there exists $n_0(\varepsilon) \geq 0$ such that for every $n \geq n_0$, we have

$$v_n(t) \in F(t, y_n(t)) \subset F(t, y_*(t)) + \varepsilon B(0,1), \quad \text{a.e.} \quad t \in J.$$

Since $F(\cdot,\cdot)$ has compact values, there exists a subsequence $v_{n_m}(\cdot)$ such that

$$v_{n_m}(\cdot) \to v_*(\cdot), \quad \text{as} \quad m \to \infty$$

and then

$$v_*(t) \in F(t, y_*(t)), \quad \text{a.e.} \quad t \in J.$$

Since

$$\left| h_*(t) - \int_0^T G(t,s)\, v_*(s)\, ds \right| \leq |h_*(t) - h_{n_m}(t)| + \left| h_{n_m}(t) - \int_0^T G(t,s) v_*(s) ds \right| \leq$$

$$\leq |h_*(t) - h_{n_m}(t)| + \left| \int_0^T G(t,s) |v_{n_m}(s) - v_*(s)|\, ds \right|,$$

our claim follows from the Lebesgue dominated convergence theorem.

Step 4: A priori bounds on solutions. Let y be such that $y \in \lambda N(y)$ for $\lambda \in [0,1]$. Then, there exists $v \in S_{F,y}$ such that, for each $t \in J$,

$$|y(t)| \leq \int_0^T G(t,s) p(s) \psi(|y(s)|) ds \leq p^* G^* \psi(\|y\|_\infty).$$

Thus

$$\frac{\|y\|_\infty}{p^* G^* \psi(\|y\|_\infty)} \leq 1.$$

Condition (7) implies that $\|y\|_\infty \neq M$. Given

$$U = \{y \in C(J, \mathbb{R}) : \|y\|_\infty < M\},$$

there is no $y \in \partial U$ such that $y \in \lambda N(y)$ for some $\lambda \in (0,1)$. Moreover, the operator $N : \overline{U} \to$ $\to \mathcal{P}(C(J, \mathbb{R}))$ is upper semicontinuous and completely continuous. Therefore, with Lemma 3.4, we deduce that N has a fixed point y in $\overline{U}$, a solution of problem (1), (2).

Theorem 3.1 is proved.

4. The nonconvex case. In this section, two existence results for problem (1), (2) are given when the right-hand side takes nonconvex values.

4.1. A first result.

Theorem 4.1. *Assume that*

($\mathcal{H}_4$) $F : J \times \mathbb{R} \longrightarrow \mathcal{P}_{cp}(\mathbb{R})$ *is integrably bounded and* $F(\cdot, u) : J \to \mathcal{P}_{cp}(\mathbb{R})$ *is measurable for each* $u \in \mathbb{R}$.

($\mathcal{H}_5$) *There exists* $l \in L^\infty(J, \mathbb{R})$ *such that* $d(0, F(t,0)) \leq l(t)$, *a.e.* $t \in J$ *and*

$$H_d(F(t,u), F(t,\overline{u})) \leq l(t)|u - \overline{u}| \quad \textit{for every} \quad u, \overline{u} \in \mathbb{R}.$$

If further

$$\|l\|_{L^\infty} G^* < 1, \tag{8}$$

then problem (1), (2) has at least one solution on J.

Proof. For each $y \in C(J,\mathbb{R})$, the set $S_{F,y}$ is nonempty by $(\mathcal{H}_4)$ (see Remark 2.1). By Lemma 2.2, F has a measurable selection. We shall show that N satisfies the assumptions of Lemma 2.4. The proof will be given in two steps.

Step 1: $N(y) \in P_{cl}(C(J,\mathbb{R}))$ *for each* $y \in C(J,\mathbb{R})$. Indeed, let $(y_n)_{n\geq 0} \in N(y)$ be such that $y_n \longrightarrow \widetilde{y}$ in $C(J,\mathbb{R})$. Then, $\widetilde{y} \in C(J,\mathbb{R})$ and there exists $v_n \in S_{F,y}$ such that, for each $t \in J$,

$$y_n(t) = \int\limits_0^T G(t,s)\, v_n(s)\, ds. \tag{9}$$

By Assumption $(\mathcal{H}_4)$, the sequence v_n is integrably bounded. Moreover F has compact values. Then by the Dunford-Pettis theorem (see [13], Proposition 4.2.1), we may pass to a subsequence, if necessary, to get that $(v_n)_{n\in\mathbb{N}}$ converges weakly to v in $L^1_w(J,\mathbb{R})$ (the space endowed with the weak topology). Define the linear operator

$$\gamma : L^1(J,\mathbb{R}) \longrightarrow C(J,\mathbb{R})$$

by $(\gamma v)(t) = \int_0^T G(t,s)v(s)ds$. The Ascoli–Arzéla lemma implies that γ is completely continuous. As a consequence, $(v_n)_{n\in\mathbb{N}}$ admits a subsequence $(v_{n_k})_{k\in\mathbb{N}}$ such that $\gamma(v_{n_k})$ converges strongly to $\gamma(v)$ in $C(J,\mathbb{R})$. Passing to the limit in (9) with $n = n_k$, as $k \to +\infty$, yields that, for each $t \in J$

$$\widetilde{y}(t) = \int\limits_0^T G(t,s)\, v(s)\, ds,$$

hence $\widetilde{y} \in N(y)$ and $N(y)$ is closed.

Step 2: There exists $\gamma < 1$ *such that*

$$H_d(N(y), N(\overline{y})) \leq \gamma \|y - \overline{y}\|_\infty, \quad \textit{for each} \quad y, \overline{y} \in C(J,\mathbb{R}).$$

Let $y, \overline{y} \in C(J,\mathbb{R})$ and $h_1 \in N(y)$. Then, there exists $v_1 \in S_{F,y}$ such that for each $t \in J$,

$$h_1(t) = \int\limits_0^T G(t,s) v_1(s) ds, \quad \text{a.e.} \quad t \in J.$$

From $(\mathcal{H}_5)$, we deduce

$$H_d(F(t,y(t)), F(t,\overline{y}(t))) \leq l(t)|y(t) - \overline{y}(t)|.$$

Hence, there exists $w \in F(t,\overline{y}(t))$ such that

$$|v_1(t) - w| \leq l(t)|y(t) - \overline{y}(t)|, \quad t \in J.$$

Consider the multivalued map $U : J \to \mathcal{P}(\mathbb{R})$ defined by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq l(t)|y(t) - \overline{y}(t)|\} := \mathcal{B}(v_1(t), \gamma(t)),$$

where $\gamma(t) = l(t)|y(t) - \overline{y}(t)|$. Assumptions $(\mathcal{H}_4)$ and $(\mathcal{H}_5)$ imply that the multimap $t \mapsto$ $\mapsto F(t, \overline{y}(t))$ is measurable. Since v_1 and γ are measurable, Theorem III.4.1 in [5] tells us that the closed ball $\mathcal{B}$ is measurable. Moreover, the set $V(t) = U(t) \cap F(t, \overline{y}(t))$ is nonempty. Indeed, taking the measurable function $v = 0$ in Lemma 2.3, we obtain a measurable selection u of $F(t, \overline{y}(t))$ such that

$$|u(t) - v_1(t)| \leq d(v_1(t), F(t, \overline{y}(t)) \leq \gamma(t).$$

Then $u \in U(t)$, hence $u \in V(t)$, proving our claim. Finally, since the multivalued operator V defined by $V(t) = U(t) \cap F(t, \overline{y}(t))$ is measurable (see [5, 10]), there exists, by Lemma 2.2, a function which is a measurable selection for V. So $v_2(t) \in F(t, \overline{y}(t))$, and for each $t \in J$,

$$|v_1(t) - v_2(t)| \leq l(t)|y(t) - \overline{y}(t)|.$$

Let us define for a.e. $t \in J$,

$$h_2(t) = \int_0^T G(t,s)\, v_2(s)\, ds.$$

Then, for a.e. $t \in J$

$$|h_1(t) - h_2(t)| \leq G^* \|l\|_{L^\infty} \|y - \overline{y}\|_\infty \leq \|l\|_{L^\infty} G^* \|y - \overline{y}\|_\infty.$$

Hence

$$\|h_1 - h_2\|_\infty \leq \|l\|_{L^\infty} G^* \|y - \overline{y}\|_\infty.$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, we get

$$H_d(N(y), N(\overline{y})) \leq \|l\|_{L^\infty} G^* \|y - \overline{y}\|_\infty.$$

Finally, condition (8) implies that N is a contraction and thus, by Lemma 2.4, N has a fixed point y, solution to problem (1), (2).

Theorem 4.1 is proved.

4.2. A second result. Now, we present a result for problem (1), (2) in the spirit of the nonlinear alternative of Leray – Schauder type [11] for single-valued maps, combined with a selection theorem due to Bressan – Colombo [4] for lower semicontinuous multivalued maps with decomposable values. Details on multivalued maps with decomposable values and their properties can be found in the book by Fryszkowski [8]. Let A be a subset of $J \times \mathbb{R}$.

Definition 4.1. *(a) A is called $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $I \times D$ where I is Lebesgue measurable in J and D is Borel measurable in $\mathbb{R}$.*

(b) A subset $A \subset L^1(J, \mathbb{R})$ is decomposable if for all $u, v \in A$ and for every Lebesgue measurable set $I \subset J$, $u\chi_I + v\chi_{J \setminus I} \in A$, where χ stands for the characteristic function.

Let $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty closed values. Assign to F the multivalued operator $\mathcal{F} : C(J, \mathbb{R}) \to \mathcal{P}(L^1(J, \mathbb{R}))$ defined by $\mathcal{F}(y) = S_{F,y}$ and let $\mathcal{F}(t, y) =$ $= S_{F,y}(t)$, $t \in J$, $y \in C(J, \mathbb{R})$. The operator $\mathcal{F}$ is called the Nemyts'kĭi operator associated to F.

Definition 4.2. *Let $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say that F is of lower semicontinuous type (l.s.c. type) if its associated Nemyts'kĭi operator $\mathcal{F}$ is lower semicontinuous and has nonempty closed and decomposable values.*

Lemma 4.1 [4]. *Let X be a separable metric space and let E be a Banach space. Then every l.s.c. multivalued operator $N : X \to \mathcal{P}_{cl}(L^1([0,T], E))$ with nonempty closed decomposable values has a continuous selection, i.e. there exists a continuous single-valued function $f : X \to L^1(J, E)$ such that $f(x) \in N(x)$ for every $x \in X$.*

Let us introduce the following hypotheses:

$(\mathcal{H}_6)$ $F : [0,T] \times \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ is a nonempty compact-valued multivalued map such that:
(a) the map $(t,u) \mapsto F(t,u)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable;
(b) the map $y \mapsto F(t,y)$ is lower semicontinuous for a.e. $t \in [0,T]$.

$(\mathcal{H}_7)$ F is locally integrably bounded, i.e., for each $q > 0$, there exists a function $h_q \in L^1([0,T], \mathbb{R}^+)$ such that

$$\|F(t,u)\|_{\mathcal{P}} \leq h_q(t), \quad \text{for a.e.} \quad t \in [0,T] \quad \text{and for} \quad y \in \mathbb{R} \quad \text{with} \quad |u| \leq q.$$

The following lemma is crucial in the proof of our main existence theorem. The second one is the classical Nonlinear Alternative of Leray and Schauder for single-valued mappings.

Lemma 4.2 [7]. *Let $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty, compact values. Assume that $(\mathcal{H}_6)$ and $(\mathcal{H}_7)$ hold. Then F is of l.s.c. type.*

Lemma 4.3 [11]. *Let X be a Banach space and $C \subset X$ a nonempty bounded, closed, convex subset. Assume U is an open subset of C with $0 \in U$ and let $G : \bar{U} \to C$ be a a continuous compact map. Then*
(a) either there is a point $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda G(u)$,
(b) or G has a fixed point in $\overline{U}$.

Theorem 4.2. *Suppose that Assumptions $(\mathcal{H}_2)$, $(\mathcal{H}_3)$, $(\mathcal{H}_6)$, $(\mathcal{H}_7)$ are satisfied. Then problem (1), (2) has at least one solution.*

Proof. $(\mathcal{H}_6)$ and $(\mathcal{H}_7)$ together with Lemma 4.2 imply that F is of lower semi-continuous type. Then, from Theorem 4.1, there exists a continuous function $f : C([0,T], \mathbb{R}) \to L^1([0,T], \mathbb{R})$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in C([0,T], \mathbb{R})$. Consider the problem

$$^cD^{\alpha} y(t) = f(y)(t), \quad \text{for a.e.} \quad t \in J, \quad 1 < \alpha \leq 2, \tag{10}$$

$$y(0) = 0, \quad y'(T) = 0. \tag{11}$$

If $y \in AC([0,T], \mathbb{R})$ is a solution of problem (10), (11), then y is a solution to problem (1), (2). Problem (10), (11) is then reformulated as a fixed point problem for the single-valued operator $N_1 : C([0,T], \mathbb{R}) \to C([0,T], \mathbb{R})$ defined by

$$N_1(y)(t) = \int\limits_0^T G(t,s)\, f(y)(s)\, ds,$$

where the functions G is given by (4). Using $(\mathcal{H}_2)$, $(\mathcal{H}_3)$, we can easily show (using similar argument as in Theorem 3.1) that the operator N_1 satisfies all conditions of Lemma 4.3, which completes the proof of Theorem 4.2.

5. Topological structure. In this section, we present a result on the topological structure of the set of solutions of problem (1), (2).

Theorem 5.1. *Assume that* $(\mathcal{H}_1)$ – $(\mathcal{H}_3)$ *hold. Then the solution set for problem (1), (2) is nonempty and compact in* $C(J,\mathbb{R})$.

Proof. Let

$$S = \{y \in C(J,\mathbb{R}) : y \text{ is solution of problem } (1), (2)\}.$$

From Theorem 3.1, $S \neq \varnothing$; thus we only prove that S is a compact set. Let $(y_n)_{n\in\mathbb{N}} \in S$, then there exists $v_n \in S_{F,y_n}$ such that for $t \in J$

$$y_n(t) = \int_0^T G(t,s)\, v_n(s)\, ds,$$

where the function $G(t,s)$ is given by (4). From $(\mathcal{H}_2)$, we can prove that there exists a constant $M_1 > 0$ such that

$$\|y_n\|_\infty \leq M_1, \quad \text{for every} \quad n \geq 1.$$

As in Step 2 in Theorem 3.1, we can easily show that the set $\{y_n : n \geq 1\}$ is equicontinuous in $C(J,\mathbb{R})$. By the Arzéla – Ascoli theorem we can conclude that there exists a subsequence of $\{y_n\}$ and still denoted again by $\{y_n\}$, such that y_n converges to some limit y in $C(J,\mathbb{R})$. We shall show that there exists $v(\cdot) \in F(\cdot, y(\cdot))$ such that

$$y(t) = \int_0^T G(t,s)\, v(s)\, ds.$$

Since $F(t,.)$ is upper semicontinuous, then for every $\varepsilon > 0$, there exists $n_0(\varepsilon) \geq 0$ such that for every $n \geq n_0$, we have

$$v_n(t) \in F(t, y_n(t)) \subset F(t, y(t)) + \varepsilon B(0,1), \quad \text{a.e.} \quad t \in J.$$

Since $F(.,.)$ has compact values, there exists subsequence $v_{n_m}(.)$ such that

$$v_{n_m}(.) \to v(.), \quad \text{as} \quad m \to \infty$$

and

$$v(t) \in F(t, y(t)), \quad \text{a.e.} \quad t \in J.$$

Moreover

$$|v_{n_m}(t)| \leq p(t), \quad \text{a.e.} \quad t \in J.$$

By the Lebesgue dominated convergence theorem, we conclude that $v \in L^1(J, \mathbb{R})$ which implies that $v \in S_{F,y}$. Thus

$$y(t) = \int_0^T G(t,s)\,v(s)\,ds, \quad t \in J,$$

that is $y \in S$. Hence $S \in \mathcal{P}_{cp}(C(J, \mathbb{R}))$, as claimed.

6. An example. As an application of our results, consider the fractional differential inclusion

$$D^\alpha y(t) \in F(t,y), \quad \text{a.e.} \quad t \in J = [0,1], \quad 1 < \alpha \le 2, \tag{12}$$

$$y(0) = 0, \quad y'(1) = 0, \tag{13}$$

where $F(t,y) = \{v \in \mathbb{R} : f_1(t,y) \le v \le f_2(t,y)\}$ and $f_1, f_2 : J \times \mathbb{R} \to \mathbb{R}$ are two single-valued functions. Assume that for each $t \in J$, the function $f_1(t,\cdot)$ is lower semicontinuous (i.e., the set $\{y \in \mathbb{R} : f_1(t,y) > \mu\}$ is open for each $\mu \in \mathbb{R}$), and that for each $t \in J$, the function $f_2(t,\cdot)$ is upper semicontinuous (i.e., the set $\{y \in \mathbb{R} : f_2(t,y) < \mu\}$ is open for each $\mu \in \mathbb{R}$). Assume further that there are $p \in L^\infty(J, \mathbb{R}^+)$ and $\psi : [0,\infty) \to (0,\infty)$ continuous and nondecreasing such that

$$\max(|f_1(t,y)|, |f_2(t,y)|) \le p(t)\psi(|y|), \quad t \in J \quad \text{and} \quad y \in \mathbb{R}.$$

From (4), the Green function G is given by

$$G(t,s) = \begin{cases} \dfrac{(t-s)^{\alpha-1}}{\gamma(\alpha)} - \dfrac{t^{\alpha-1}(1-s)^{\alpha-2}}{(\alpha-1)\gamma(\alpha-1)}, & 0 \le s \le t, \\[2ex] -\dfrac{t^{\alpha-1}(1-s)^{\alpha-2}}{(\alpha-1)\gamma(\alpha-1)}, & t \le s \le 1. \end{cases}$$

Then, simple computations show that

$$G^* = \sup\left\{ \int_0^1 |G(t,s)|\,ds : t \in [0,1] \right\} = \frac{1}{\alpha\gamma(\alpha)} + \frac{1}{(\alpha-1)^2\gamma(\alpha-1)}.$$

It is clear that F is compact and convex valued, and it is upper semicontinuous (see [7]). If there exists a constant $M > 0$ such that

$$\frac{M}{p^* G^* \psi(M)} > 1,$$

then all the conditions of Theorem 3.1 are met. As a consequence, BVP (12), (13) has at least one solution y on J.

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