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TO THE PROBLEM OF COMPLEMENTABILITY OF A PERIODIC FRAME TO A PERIODIC BASIS *

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We obtain sufficient conditions (and necessary conditions in the simplest case) of complementability of a periodic frame to a periodic basis for the Euclidean space in terms of monodromy matrices of some linear system of differential equations built by using this periodic frame. We consider the problem of complementability for introducing local coordinates in a neighbourhood of a smooth m-dimensional invariant torus of a dynamic system in the Euclidean space $\mathbf{R}^n$ if the dimensions satisfy the inequality $m+1<n\leq 2m$.

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1. Introduction

Let's consider a system of differential equations,

$$\frac{dx}{dt} = X(x), \quad x \in \mathbf{R}^n, \tag{1}$$

where $X \in C^l(\mathbf{R}^n)$, $l \geq 1$, $C^l(D)$ is the space of l times continuously differentiable functions on a domain $D \subseteq \mathbf{R}^n$. Suppose system (1) has an invariant manifold

$$M : \ x = f(\varphi), \tag{2}$$

where $f(\varphi) \in C^l(\mathcal{T}_m)$, $\mathcal{T}_m$ is the m-dimensional torus, $\operatorname{rank}(\partial f(\varphi)/\partial\varphi) = m \quad \forall\varphi \in \mathcal{T}_m$.

Many works were devoted to a construction of local coordinates in a neighbourhood of invariant toroidal manifold (2) (in particular, these problems were considered in [1–8]). As shown in [1, 5], this problem is solved in the case where the system of vectors

$$\frac{\partial f(\varphi)}{\partial\varphi_i}, \quad i = \overline{1,m}, \tag{3}$$

can be complemented to a 2π-periodic basis for the space $\mathbf{R}^n$ and sufficient conditions for complementability of the system of linearly independent vectors (r-frame)

$$u_i(\varphi), \quad \varphi \in \mathcal{T}_m, \ i = \overline{1,r}, \tag{4}$$

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have been found, e. g.,

$$\{n > m + r\} \cup \{n = r + 1\}. \tag{5}$$

As it was shown in [4], in general, the system of vectors (4) can not be complemented to a 2π-periodic basis for the space $\mathbf{R}^n$. In particular, the vector

$$\begin{aligned} u_1(\varphi_1, \varphi_2) &= (2 + (2 + \sin(\varphi_1)) \sin(\varphi_2))\ (1 + 4(2 + \sin(\varphi_1))(1 + \sin(\varphi_2)))^{-1/2}, \\ u_2(\varphi_1, \varphi_2) &= (2 + \sin(\varphi_1)) \cos(\varphi_2)\ (1 + 4(2 + \sin(\varphi_1))(1 + \sin(\varphi_2)))^{-1/2}, \\ u_3(\varphi_1, \varphi_2) &= \cos(\varphi_1)(1 + 4(2 + \sin(\varphi_1))(1 + \sin(\varphi_2)))^{-1/2} \end{aligned} \tag{6}$$

can not be complemented to a 2π-periodic basis for the space $\mathbf{R}^3$.

The possibility of introducting local coordinates in a neighbourhood of an invariant torus in the case where conditions (5) are not fulfilled, namely,

$$m + 1 < n < 2m + 1, \tag{7}$$

was investigated in [7, 8]. It was suggested to complement system (1) with p equations

$$\frac{dy}{dt} = Y(y), \quad y \in \mathbf{R}^p, \quad Y \in C^l(\mathbf{R}^n), \tag{8}$$

such that system (1), (8) has an invariant torus,

$$M_1 : \ x = f(\varphi), \quad y = f_1(\varphi). \tag{9}$$

It is possible to introduce local coordinates $\varphi \in \mathcal{T}_m$, $h \in \mathbf{R}^{n-m+p}$, in a neighbourhood of M_1, where the dimension p satisfies certain conditions and $\|h\| = \left(\sum_{i=1}^{m+1} h_i^2\right)^{1/2} < \delta$, δ is a small positive number, by defining

$$x = f(\varphi) + B(\varphi)h, \quad y = f_1(\varphi) + b(\varphi)h, \tag{10}$$

where the matrices $B(\varphi)$, $b(\varphi)$ are taken from the space $C^r(\mathcal{T}_m)$ and such that

$$\det \begin{bmatrix} \partial f(\varphi)/\partial\varphi & B(\varphi) \\ \partial f_1(\varphi)/\partial\varphi & b(\varphi) \end{bmatrix} \neq 0 \quad \forall \varphi \in \mathcal{T}_m. \tag{11}$$

If the condition $Y(0) = 0$ is fulfilled for the right-hand side of (8), then system (1), (8) always has an invariant manifold,

$$M_0 : \ x = f(\varphi), \quad y = 0,$$

and the construction of local coordinates in a neighbourhood of the torus M_0 is simpler than in general. In this case, the minimal number of equtions in system (8) sufficient to introduce local coordinates was calculated in [8]. The following question arises: can we find a system (8) smaller than obtained in [8] and such that the condition $Y(0) = 0$ is not fulfilled, but system (1), (8) can be written in local coordinates. In this article we'll answer this question in the negative.

System (1) with the smallest dimension such that condition (7) is fulfilled is a system of 4 differential equations that has 2-dimensional invariant torus. Thus the problem of introducing local coordinates depends on the problem of complementability of the frame for the dimensions $n = 4, m = 2, r = m = 2$, to a periodic basis. This problem can be simplified [1, 8] and reduced to the problem of complementability of the frame with the dimensions $n = 3$, $m = 2$, $r = 1$. But example (6) shows that this can not be done in the general case. One of sufficient conditions for complementability of one vector to a periodic basis is that the set of values of this vector is not everywhere dense in the orb $S_1(0)$ [5]. But this condition is not necessary, because, for example, the vector $u(\varphi) = (\sin\varphi_1 \sin\varphi_2,\ \cos\varphi_1 \sin\varphi_2,\ \cos\varphi_2)$ is everywhere dense in orb and it is obvious that this vector can be complemented. That's why it is important to get another necessary and sufficient conditions of complementability.

This paper continues [8]. In Section 2 we'll get sufficient conditions of complementability of an r-frame $U(\varphi)$, $\varphi \in \mathcal{T}_m$, with the help of the monodromy matrix of a system of linear differential equations constructed from vectors of this frame so that these conditions will be necessary in the simplest case ($n = 3$, $m = 2$, $r = 1$). In Section 3 we'll improve the results obtained in Section 2 in paper [8].

2. Conditions of Complementability

Let's consider a system of linearly independent vectors in the space $\mathbf{R}^n$,

$$U(\varphi) = (u_1(\varphi), \ldots, u_r(\varphi)), \quad \operatorname{rank} U(\varphi) = r, \quad \forall\varphi \in \mathcal{T}_m, 1 \le r < n,$$

where $U(\varphi) \in C^l(\mathcal{T}_m)$. We say that the r-frame $U(\varphi)$ can be complemented to a periodic basis of the space $\mathbf{R}^n$ if there is an $(n-r)$-frame $V(\varphi)$ in the space $C^l(\mathcal{T}_m)$ such that

$$\det(U(\varphi), V(\varphi)) \ne 0 \quad \forall\varphi \in \mathcal{T}_m.$$

Lemma 1. *A system of vectors $U(\varphi) \in C^l(\mathcal{T}_m)$ has a maximum subsystem of p vectors ($n - m - 1 \le p \le r - 1$) that can be complemented to a periodic basis of the space $\mathbf{R}^n$ (any subsystem of $(p + 1)$ vectors can not be complemented) if and only if the r-frame $U(\varphi)$ can be represented in the form*

$$U(\varphi) = M_1(\varphi) \begin{pmatrix} E_p & 0 \\ 0 & \overline{U}(\varphi) \end{pmatrix} M_2(\varphi),$$

where the matrices $M_1(\varphi)$, $M_2(\varphi)$, $\overline{U}(\varphi) \in C^l(\mathcal{T}_m)$ have the dimensions $n\times n$, $r\times r$, $(n-p)\times(r-p)$ and $\det M_k(\varphi) \ne 0\ \forall\varphi \in \mathcal{T}_m$, $k = \overline{1,2}$, and any vector-column of the matrix $\overline{U}(\varphi)$ can not be complemented to a periodic basis of the space $\mathbf{R}^{n-p}$, E_p is the p-dimensional identity matrix.

This Lemma is a corollary of Lemma 1 from [8]. Besides, due to the proof of the last lemma we can conclude that the complementability of the frame can be reduced to that of complementability of separate vectors to a periodic basis. The algorithm of this simlification is given in [8]. That's why we'll consider the given problem only for a single vector

$u(\varphi_1, \dots, \varphi_m) \in C^l(\mathcal{T}_m)$. Without loss of generality we can assume that $u^*(\varphi)u(\varphi) = 1$ (D^* is the transpose of the matrix D). Also suppose that $3 \leq n \leq m+1$, so that the sufficient condition (5) is not fulfilled.

Let's denote $\varphi = (\varphi_i', \varphi_i'')$, $\varphi_i' = (\varphi_1, \dots, \varphi_i)$, $\varphi_i'' = (\varphi_{i+1}, \dots, \varphi_m)$, $i = \overline{1,m}$. Consider the vector $u(\varphi_j', 0)$ for some fixed $j \leq m$. Suppose that this vector can be complemented to a periodic basis if one of the variables is fixed (we may assume that this variable is φ_j). Then there exists a square matrix $O_j(\varphi_{j-1}') = \left[Z_j(\varphi_{j-1}'), u(\varphi_{j-1}', 0)\right] \in C^l(\mathcal{T}_{j-1})$, $O_j^*(\varphi_{j-1}')O_j(\varphi_{j-1}') = E_n$. The vector $\overline{u}_j(\varphi_j') = O_j^*(\varphi_{j-1}')u(\varphi_j, 0) \in C^l(\mathcal{T}_j)$ satisfies the condition $\overline{u}_j(\varphi_{j-1}', 0) = \mathrm{colon}(0, \dots, 0, 1)$.

Let's construct the following matrix by using the vector $\overline{u}_j(\varphi_j')$,

$$K_j(\varphi_j') = \frac{\partial \overline{u}_j(\varphi_j')}{\partial \varphi_j}\overline{u}_j^*(\varphi_j') - \overline{u}_j(\varphi_j')\frac{\partial \overline{u}_j^*(\varphi_j')}{\partial \varphi_j} \in C^{l-1}(\mathcal{T}_j),$$

and consider the system of differential equations

$$\frac{\partial h}{\partial \varphi_j} = K_j(\varphi_j')h. \tag{12}$$

The coefficients of this system are periodic functions of the variables φ_j and the parameters φ_{j-1}'. This system has the following properties:

1. $K_j(\varphi_j')$ is a skew-symmetric matrix.

2. The vector $\overline{u}_j(\varphi_j')$ is a solution of system (12). Indeed, taking into account that the vector $\overline{u}_j(\varphi_j')$ has lenght one, we obtain

$$K_j\overline{u}_j = \frac{\partial \overline{u}_j}{\partial \varphi_j}\overline{u}_j^*\overline{u}_j - \overline{u}_j\frac{\partial \overline{u}_j^*}{\partial \varphi_j}\overline{u}_j = \frac{\partial \overline{u}_j}{\partial \varphi_j}.$$

3. The matriciant $\Omega_0^{\varphi_j}(\varphi_{j-1}')$ of system (12) is a periodic function of the parameters φ_j'.

4. The matriciant of system (12) is orthonormalized $\left(\Omega_0^{\varphi_j}(\varphi_{j-1}')\right)^*\Omega_0^{\varphi_j}(\varphi_{j-1}') = E_n$. This follows from the first property.

Combining the second and the third properties we get that the matriciant of system (12) can be represented in the form $\Omega_0^{\varphi_j}(\varphi_{j-1}') = \left[H_j(\varphi_j'), \overline{u}_j(\varphi_j')\right]$, where $H_j(\varphi_j') \in C^{l-1}(\mathcal{T}_{j-1} \times \mathbf{R})$.

Taking into account that the matriciant $\Omega_0^{\varphi_j}(\varphi_{j-1}')$ is orthonormalized and the vector $\overline{u}_j(\varphi_j')$ is a 2π-periodic function of φ_j, the monodromy matrix of system (12) can be represented in the form

$$\Omega_0^{2\pi}(\varphi_{j-1}') = \begin{pmatrix} \overline{H}_j(\varphi_{j-1}', 2\pi) & 0 \\ 0 & 1 \end{pmatrix},$$

where $\overline{H}_j(\varphi_j')$ is the first $(n-1)$ rows of the matrix $H_j(\varphi_j')$. Note that $\det \overline{H}_j(\varphi_{j-1}', 2\pi) \neq 0 \quad \forall \varphi_{j-1}' \in \mathcal{T}_{j-1}$.

Suppose there exists a real-valued logarithm of the monodromy matrix $A_j(\varphi'_{j-1}) = \ln\left(\overline{H}_j(\varphi'_{j-1}, 2\pi)\right)$ in the space $C^{l-1}(\mathcal{T}_{j-1})$. Let us remark that such a logarithm does not always exist. Some sufficient conditions for existence of this logarithm are obtained in the papers [5, 9]. In this case, Floquet's matrix of system (12) can be represented in the form

$$\Phi(\varphi'_j) = \Phi(\varphi'_{j-1}, \varphi_j) = \Omega_0^{\varphi_j}(\varphi'_{j-1}) \exp\left\{-(2\pi)^{-1}\varphi_j \ln \Omega_0^{2\pi}(\varphi'_{j-1})\right\}$$

$$= \left[H_j(\varphi'_j), \overline{u}_j(\varphi'_j)\right] \operatorname{diag}\left\{\exp\left\{-(2\pi)^{-1}A(\varphi'_{j-1})\varphi_j\right\}, 1\right\} = \left[H'_j(\varphi'_j), \overline{u}_j(\varphi'_j)\right],$$

where $H'_j(\varphi'_j) = H_j(\varphi'_j) \exp\left\{-(2\pi)^{-1}A(\varphi'_{j-1})\varphi_j\right\}$.

The matrix $\Phi(\varphi'_j)$ is continuously differentiable of order $l-1$ and is a periodic function of the parameters φ'_{j-1} (being a superposition of a periodic matrix of parameters) and the variable φ_j (as follows from the Floquet theory). In other words, $\Phi(\varphi'_j)$ is an element of the class $C^{l-1}(\mathcal{T}_j)$ and, thus, the matrix $H'_j(\varphi'_j)$ is an element of this class, too. The last matrix is a complement of the vector $\overline{u}_j(\varphi'_j)$ to a periodic basis of the space $\mathbf{R}^n$, because $\Phi(\varphi'_j)$ is a nondegenerate matrix.

In the same way, the vector $\overline{u}_{j+1}(\varphi'_{j+1})$, where $j+1 \leq m$, under the given conditions, can be complemented if there exists a matrix $A_{j+1}(\varphi_j)$ from the space $C^{l-1}(\mathcal{T}_j)$ (this is a sufficient condition). In this case, we can take $H'_j(\varphi'_j)$ as the matrix $Z_{j+1}(\varphi'_j)$. And so on. Finally we get the following theorem.

Theorem 1. *Suppose a vector $u(\varphi) \in C^l(\mathcal{T}_m)$ can be complemented to a periodic basis if some $m-s$ variables are fixed (without loss of generality, we can take them to be $\varphi_{s+1}, \ldots, \varphi_m$). Besides, there exist matrices $A_j(\varphi'_{j-1}) \in C^{l-1}(\mathcal{T}_{j-1})$, $j = \overline{s+1, m}$. Then the vector $u(\varphi)$ can be complemented by vectors from the class $C^{l-1}(\mathcal{T}_m)$ to a periodic basis of the space $\mathbf{R}^n$.*

Note 1. If $s \leq n-2$, then the first condition of the previous theorem is fulfilled. Indeed, conditions (5) are fulfilled in this case.

Note 2. Suppose a vector $u(\varphi)$ can be complemented by vectors from the class $C^{l-1}(\mathcal{T}_m)$ using the previous theorem; then taking into account Theorem 1 from [7], there also exist complementing vectors from the space $C^l(\mathcal{T}_m)$.

Now we look at the case of 3-dimension vectors of two variables in more details

$$u(\varphi) = \operatorname{colon}(u_1(\varphi), u_2(\varphi), u_3(\varphi)) \in C^l(\mathcal{T}_2).$$

Let $u(\varphi)$ be orthonormalized. It can easily be checked that for this vector the first condition of Theorem 1 is satisfied for any variables φ_1, φ_2. Then without loss of generality we assume that $u(\varphi_1, 0) = \operatorname{colon}[0; 0; 1]$. Let us construct the matrix $K_2(\varphi_1, \varphi_2)$ from this vector. So, we get the matriciant $\Omega_0^{\varphi_2}(\varphi_1) = [h'(\varphi), h''(\varphi), u(\varphi)]$ of the corresponding system (12). Taking into account that this matriciant is orthonormalized, we can represent the monodromy matrix in the form

$$\Omega_0^{2\pi}(\varphi_1) = \begin{pmatrix} a(\varphi_1) & -b(\varphi_1) & 0 \\ b(\varphi_1) & a(\varphi_1) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{where} \quad a^2(\varphi_1) + b^2(\varphi_1) \equiv 1. \tag{13}$$

Let's denote by $z(\varphi_1)$ the complex-valued function

$$z(\varphi_1) = a(\varphi_1) + ib(\varphi_1).$$

Thus, the logarithm of this monodromy matrix is always real-valued for any φ_1, namely,

$$\operatorname{Ln}\Omega_0^{2\pi}(\varphi_1) = \begin{pmatrix} 0 & -\operatorname{Arg}(z(\varphi_1)) & 0 \\ \operatorname{Arg}(z(\varphi_1)) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let's prove the following lemma.

Lemma 2. *Let $z(\varphi_1) \in C^{l-1}(\mathcal{T}_1)$ $(l \geq 2)$, $z(\varphi) \neq 0$. There exists a logarithm $\ln z(\varphi_1)$ from the space $C^{l-1}(\mathcal{T}_1)$ if and only if the point $z(\varphi_1)$ moving along a closed curve C about the origin makes the same number of clockwise and counterclockwise revolutions as the variable φ_1 ranges over* $[0; 2\pi]$.

Proof. It is clear that $\ln|z(\varphi_1)| \in C^{l-1}(\mathcal{T}_1)$. Because $z(\varphi_1) \in C^{l-1}(\mathcal{T}_1)$, we get that the point $z(\varphi_1)$ moves continuesly along some closed curve C while the variable φ_1 ranges over $\mathbf{R}$. Let's fix some point $\varphi_1^{(0)} \in [0; 2\pi]$ and denote by $\omega(\varphi_1) : \mathbf{R} \to \mathbf{R}$ the continuous function that is equal to the angle between the vector $z(\varphi_1)$ and the positive real half axis such that $\omega(\varphi_1^{(0)}) = \omega_0 \in [0; 2\pi]$. The function $\omega(\varphi_1)$ belongs to the same space as $z(\varphi_1)$, because

$$\frac{d\omega(\varphi_1)}{d\varphi_1} = \frac{db(\varphi_1)}{d\varphi_1}a(\varphi_1) - \frac{da(\varphi_1)}{d\varphi_1}b(\varphi_1).$$

Thus, $\operatorname{Arg} z(\varphi_1)$ consists of all smoth curves $\omega(\varphi_1) + 2\pi k$, $k \in \mathbf{Z}$. If the vector $z(\varphi_1)$ makes a complete turn around zero moving clockwise (counterclockwise) from $z(\varphi_1^{(0)})$ then the value of the function $\omega(\varphi_1)$ changes from ω_0 to $\omega_0 + 2\pi$ $(\omega_0 - 2\pi)$. Hence, $\omega(\varphi_1)$ is a 2π-periodic function (that is, $\omega(\varphi_1^{(0)} + 2\pi) = \omega_0$) if and only if the point $z(\varphi_1)$ makes the same numbers of complete turns around zero in each of the two directions while $\varphi \in [\varphi_1^{(0)}; \varphi_1^{(0)} + 2\pi]$. Replacing variable $\overline{\varphi_1} = \varphi_1 - \varphi_1^{(0)}$ we complete the proof.

Let's choose

$$\Lambda(\varphi_1) = \frac{1}{2\pi}\begin{pmatrix} 0 & -\omega(\varphi_1) & 0 \\ \omega(\varphi_1) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\omega(\varphi_1) \in C^{l-1}(\mathcal{T}_1)$ is the function from the previous lemma. In this case the Floquet's matrix of system (12) can be represented in the form

$$\Phi(\varphi) = \Omega_0^{\varphi_2}(\varphi_1)\exp\{-\Lambda(\varphi_1)\varphi_2\}$$

$$= \left[h'(\varphi), h''(\varphi), u(\varphi)\right] \begin{pmatrix} \cos(\omega(\varphi_1)\varphi_2/2\pi) & \sin(\omega(\varphi_1)\varphi_2/2\pi) & 0 \\ -\sin(\omega(\varphi_1)\varphi_2/2\pi) & \cos(\omega(\varphi_1)\varphi_2/2\pi) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \left[w'(\varphi), w''(\varphi), u(\varphi)\right],$$

where

$$w'(\varphi) = \cos(\omega(\varphi_1)\varphi_2/2\pi)\, h'(\varphi) - \sin(\omega(\varphi_1)\varphi_2/2\pi)\, h''(\varphi),$$

$$w''(\varphi) = \sin(\omega(\varphi_1)\varphi_2/2\pi)\, h'(\varphi) + \cos(\omega(\varphi_1)\varphi_2/2\pi)\, h''(\varphi).$$

Therefore, from the above-stated properties, the vectors $w'(\varphi)$, $w''(\varphi)$ are orthonormalized, belong to the class $C^{l-1}(\mathcal{T}_2)$, and complete the vector $u(\varphi)$ to a periodic basis of the space $\mathbf{R}^3$. The main sufficient condition for constructing these vectors is existence of a continuous 2π-periodic function $\omega(\varphi_1)$, which is an equivalent to the condition of Lemma 2. Now we show that this condition is necessary.

Let's consider any 3×3-matrix $\overline{\Psi}(\varphi) \in C^l(\mathcal{T}_2)$ such that the columns make an orthonormal periodic basis of the space $\mathbf{R}^3$, that is $\overline{\Psi}^*(\varphi)\overline{\Psi}(\varphi) \equiv E_3$. Then the columns of the matrix $\Psi(\varphi) = \overline{\Psi}^*(\varphi_1, 0)\overline{\Psi}(\varphi)$ also make an orthonormal periodic basis of the space $\mathbf{R}^3$ and the condition $\Psi(\varphi_1, 0) = E_3$ is satisfied. Denoting $\Psi(\varphi) = [v'(\varphi), v''(\varphi), u(\varphi)]$ and using the given vector $u(\varphi)$, we construct system (12). Then the matrix $\overline{\Omega}_0^{\varphi_2}(\varphi_1) = \Psi(\varphi)$ is a matriciant of the system

$$\frac{\partial h}{\partial \varphi_2} = K_2(\varphi)h + h\Gamma(\varphi), \quad h \in \mathbf{R}^3, \tag{14}$$

where $K_2(\varphi)$ is the matrix of coefficients in system (12) and

$$\Gamma(\varphi) = \begin{pmatrix} 0 & -\gamma(\varphi) & 0 \\ \gamma(\varphi) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\gamma(\varphi) = (v''(\varphi))^* \partial v'(\varphi)/\partial\varphi_2 = -(v'(\varphi))^* \partial v''(\varphi)/\partial\varphi_2 \in C^{l-1}(\mathcal{T}_2)$.

It is easy to prove by a direct calculation that the matrix $\Gamma(\varphi)$ and the integral of this matrix with respect to φ_2 commute. Then, the Lappo–Danilevsky condition for the system of differential equations

$$\frac{\partial h}{\partial \varphi_2} = h\Gamma(\varphi), \quad h \in \mathbf{R}^3, \tag{15}$$

is filfilled and the matriciant of this system is the following:

$$\widetilde{\Omega}_0^{\varphi_2}(\varphi_1) = \exp\left\{ \int\limits_0^{\varphi_2} \Gamma(\varphi_1, \tau) d\tau \right\}. \tag{16}$$

Hence, the matriciant of system (14) can be written as a product of the matriciants of systems (12), (14),

$$\overline{\Omega}_0^{\varphi_2}(\varphi_1) = \Omega_0^{\varphi_2}(\varphi_1)\widetilde{\Omega}_0^{\varphi_2}(\varphi_1). \tag{17}$$

Let's show that the matrix $\operatorname{Ln}\Omega_0^{2\pi}(\varphi_1)$, which consists of zero elements and continuously differential functions $\omega(\varphi_1)$ (as was shown in Lemma 2) is 2π-periodic. Consider the matrix $L(\varphi_1) = \Phi^*(\varphi)\Psi(\varphi)$. Using (16), (17), the form of Floquet's matrix $\Phi(\varphi)$ of system (12), and commutativity of the matrices $\Lambda(\varphi_1)$ i $\Gamma(\varphi)$, we obtain

$$L(\varphi_1) = \exp\left\{\int\limits_0^{\varphi_2}\Gamma(\varphi_1,\tau)d\tau + \Lambda(\varphi_1)\varphi_2\right\}. \tag{18}$$

This matrix belongs to the class $C^{l-1}(\mathbf{R}^2)$ and is a periodic function of the variable φ_2, because $\Phi(\varphi)$ and $\Psi(\varphi)$ have such properties. Taking into account the form of the matrix in the exponent of the exponential function in the right-hand side of (18), we can write

$$L(\varphi) = \begin{pmatrix} \cos l(\varphi) & -\sin l(\varphi) & 0 \\ \sin l(\varphi) & \cos l(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where

$$l(\varphi) = \int\limits_0^{\varphi_2}\gamma(\varphi_1,\tau)d\tau + \frac{1}{2\pi}\omega(\varphi_1)\varphi_2.$$

Since $L(\varphi)$ is a 2π-periodic function of the second variable, we get $\sin l(\varphi_1,2\pi) = \sin l(\varphi_1,0) = 0$ and

$$l(\varphi_1,2\pi) = \int\limits_0^{2\pi}\gamma(\varphi_1,\tau)d\tau + \omega(\varphi_1) = k(\varphi_1)\pi, \quad k(\varphi_1) \in \mathbf{Z}.$$

The functions $\gamma(\varphi), \omega(\varphi_1)$ are continuous in the parameter φ_1 ($\gamma(\varphi)$ is a continuous function of the variable φ_2). That's why, for all $\varphi_1 \in \mathbf{R}$, the function $l(\varphi_1,2\pi)$ is continuous. This implies that $k(\varphi_1) = k = \text{const} \quad \forall\varphi_1 \in \mathbf{R}$. Thus, we get

$$\omega(\varphi_1+2\pi) - \omega(\varphi_1) = l(\varphi_1+2\pi,2\pi) - \int\limits_0^{2\pi}\gamma(\varphi_1+2\pi,\tau)d\tau - l(\varphi_1,2\pi)$$

$$+ \int\limits_0^{2\pi}\gamma(\varphi_1,\tau)d\tau = l(\varphi_1+2\pi,2\pi) - l(\varphi_1,2\pi) = k\pi - k\pi = 0,$$

which proves that the matrix $\operatorname{Ln}\Omega_0^{2\pi}(\varphi_1)$ belongs to the class $C^{l-1}(\mathcal{T}_1)$.

So, we can formulate the next theorem.

Theorem 2. *A necessary and sufficient condition for a unit vector $u(\varphi) \in C^l(\mathcal{T}_2)$, $l \geq 1$, to be complemented to a periodic basis of the space $\mathbf{R}^3$ is that the point $z(\varphi_1) = a(\varphi_1) + ib(\varphi_1)$, moving along some curve C encircling the origin, makes the same number of turns in the clockwise and counterclockwise directions while the variable φ_1 ranges over $[0; 2\pi]$. Here functions $a(\varphi_1)$, $b(\varphi_1)$ are coefficients of the monodromy matrix (13) of system (12) constructed from the vector $u(\varphi)$. The vectors that complement this vector belong to the space $C^{l-1}(\mathcal{T}_2)$.*

Note 3. According to Lemma 1, the problem of complementability of a vector $u(\varphi) \in C^l(\mathcal{T}_2)$ to a periodic basis of the space $\mathbf{R}^3$ is the main condition for introducing local coordinates in a neighbourhood of the invariant 2-dimensional torus in the space $\mathbf{R}^4$ (as was shown above that this is the simplest case when this is impossible). That's why, the previous theorem is very important.

3. Complementability of a Frame to a Periodic Basis in the Extended Space

Let us consider a system of differential equation (1) that has invariant manifold (2) under condition (7). Suppose there exists a system (8) such that system (1), (8) has invariant torus (9). We'll investigate the problem of constructing local coordinates in system (1), (8) in the case when manifold (9) does not coincide with M_0. The following lemmas about constructing a periodic basis for the extended space are needed in the sequel.

Lemma 3. *Suppose $U(\varphi) = [u_1(\varphi), \ldots, u_r(\varphi)]$, where each of the vectors $u_j(\varphi)$, $j = \overline{1,r}$, can not be complemented to a periodic basis of the space $\mathbf{R}^n$, $U^*(\varphi)U(\varphi) = E_r$, belongs to the class $C^l(\mathcal{T}_m)$, $r < n$. Then for any r-dimensional square matrix $N(\varphi) \in C^l(\mathcal{T}_m)$ there exists a periodic basis in the space $\mathbf{R}^{n+r}$,*

$$F(\varphi) = \begin{pmatrix} U(\varphi) & E_n - U(\varphi)U^*(\varphi) - U(\varphi)N^*(\varphi)T(\varphi)U^*(\varphi) \\ N(\varphi) & T(\varphi)U^*(\varphi) \end{pmatrix} \in C^l(\mathcal{T}_m), \qquad (19)$$

where $T(\varphi) \in C^l(\mathcal{T}_m)$ is a triangular r-dimensional square matrix used in the Gramm – Schmidt orthonormalization process for the system of vectors $[U^*(\varphi), N(\varphi)]^*$.

Proof. By a direct calculation we get the following product:

$$F^*(\varphi)F(\varphi) = \operatorname{diag}\{E_r + N^*(\varphi)N(\varphi),\ E_n - U(\varphi)U^*(\varphi)$$

$$+\ U(\varphi)T^*(\varphi)(N(\varphi)N^*(\varphi) + E_r)T(\varphi)U^*(\varphi)\}.$$

According to the conditions of the lemma, the system of vectors $P(\varphi) = [U^*(\varphi), N(\varphi)]^*\, T(\varphi)$ is orthonormalized. So, $P^*(\varphi)P(\varphi) = T^*(\varphi)(E_r + N(\varphi)N^*(\varphi))T(\varphi) = E_r$. Besides, taking into account that the quadratic form $\eta^*(E_r + N^*(\varphi)N(\varphi))\eta = \|\eta\|^2 + \|N(\varphi)\eta\|^2 \geq \|\eta\|^2$, $\eta \in \mathbf{R}^r$, is positive definite, we obtain $\det(E_r + N^*(\varphi)N(\varphi)) > 0 \quad \forall\varphi \in \mathcal{T}_m$. Thus, $\det F^*(\varphi)F(\varphi) = \det(E_r + N^*(\varphi)N(\varphi)) > 0 \quad \forall\varphi \in \mathcal{T}_m$. This means that the matrix $F(\varphi)$ is a periodic basis in the space $\mathbf{R}^{n+r}$, which proves the lemma.

The previous lemma has useful corollaries.

Corollary 1. *Suppose an n-dimensional vector $u(\varphi) \in C^l(\mathcal{T}_m)$ can not be complemented to a periodic basis. Then for any function $\widetilde{u}(\varphi) \in C^l(\mathcal{T}_m)$ there exists the following periodic basis in the space $\mathbf{R}^{n+1}$:*

$$\widetilde{F}(\varphi) = \begin{pmatrix} u(\varphi) & E_n - \left(1 + \widetilde{u}(\varphi)\left(1 + \widetilde{u}^2(\varphi)\right)^{-1/2}\right) u(\varphi)u^*(\varphi) \\ \widetilde{u}(\varphi) & (1 + \widetilde{u}^2(\varphi))^{-1/2} u^*(\varphi) \end{pmatrix}.$$

Corollary 2. *Suppose the conditions for the matrix $U(\varphi)$ in Lemma 3 are fulfilled. Then there exists an orthonormalized periodic basis in space $\mathbf{R}^{n+r}$,*

$$\overline{F}(\varphi) = \begin{pmatrix} U(\varphi) & E_n - U(\varphi)U^*(\varphi) \\ O_r & U^*(\varphi) \end{pmatrix} \in C^l(\mathcal{T}_m), \tag{20}$$

there O_r is an r-dimensional square zero matrix.

The last corollary gives the simplest constructive way to complement the frame $U(\varphi)$ under the condition of Lemma 3 to a periodic basis. Besides, using Proposition 1 from paper [8], we get that in this case r is the smallest, which means that if we add k coordinates to every vector of this frame, where $k < r$, then the given frame can not be complemented to a periodic basis in the extended space. The following question arises: can we add, to every vector of the frame $U(\varphi)$, q *nonzero* coordinates, where $q < r$, such that the extended frame can be complemented to a periodic basis in the space $\mathbf{R}^{n+q}$. This question is important in the theory of invariant toroidal manifolds, because the answer in the affirmative would allow to decrease the number of additional equations in system (8) to construct local coordinates [7, 8]. Now we'll show that we can answer in the negative.

Lemma 4. *Suppose the matrix*

$$\overline{\overline{F}}(\varphi) = \begin{pmatrix} U(\varphi) & B(\varphi) \\ N(\varphi) & b(\varphi) \end{pmatrix} \in C^l(\mathcal{T}_m)$$

consists of vectors that make a periodic basis in the space $\mathbf{R}^n$,

$$\det \overline{\overline{F}}(\varphi) \neq 0 \quad \forall \varphi \in \mathcal{T}_m,$$

where $N(\varphi)$, $B(\varphi)$ are square matrices of dimensions r and n, $r < n$. Besides, let the systems of vectors $[U^*(\varphi),\ N^*(\varphi)]^*$ and $[B^*(\varphi),\ b^*(\varphi)]^*$ belong to two orthogonal subspaces of the space $\mathbf{R}^n$. Then any vector from the frame $U(\varphi)$ can not be complemented to a periodic basis in the space $\mathbf{R}^{n-k}$ if and only if any row of the matrix $b(\varphi)$ can not be complemented to a periodic basis in the space $\mathbf{R}^{n-k}$.

Proof. First we shall prove this lemma for the matrix $F(\varphi)$ from Lemma 3 with $b(\varphi) = T(\varphi)U^*(\varphi)$. Using the proof Lemma 3, it can be easily checked that all conditions of Lemma 4 are satisfied for the matrix $F(\varphi)$. Suppose any vector of the frame $U(\varphi)$ can not be complemented to a periodic basis in the space $\mathbf{R}^n$. Taking into account properties of the matrix $T(\varphi) = \{T_{kj}(\varphi)\}_{k,j=1}^r$ from Lemma 3 we obtain that for any coefficient on the diagonal $T_{jj}(\varphi) \neq 0 \quad \forall\varphi \in \mathcal{T}_m$, and the rows of the matrix $b(\varphi)$ can be written as

$$\widetilde{u}_k^*(\varphi) = \sum_{j=k}^{r} T_{kj}(\varphi)u_j^*(\varphi), \quad k = \overline{1,r}.$$

Suppose there exists some vector $\widetilde{u}_{k_0}(\varphi)$ that can be complemented to a periodic basis in the space $\mathbf{R}^n$. Then there exists an orthonormalized $n \times (n-1)$-matrix $\widetilde{B}_{k_0}(\varphi)$ such that $\det\left[\widetilde{u}_{k_0}(\varphi),\ \widetilde{B}_{k_0}(\varphi)\right] \neq 0$, $\widetilde{u}_{k_0}^*(\varphi)\widetilde{B}_{k_0}(\varphi) = 0 \quad \forall\varphi \in \mathcal{T}_m$. Thus, we get

$$\begin{pmatrix} u_{k_0}^*(\varphi) \\ \widetilde{B}_{k_0}^*(\varphi) \end{pmatrix} \left[\widetilde{u}_{k_0}(\varphi),\ \widetilde{B}_{k_0}(\varphi)\right] = \begin{pmatrix} T_{kk}(\varphi) & u_{k_0}^*(\varphi)\widetilde{B}_{k_0}(\varphi) \\ 0 & E_{n-1} \end{pmatrix}.$$

It follows from the last identity that $\left[u_{k_0}(\varphi),\ \widetilde{B}_{k_0}(\varphi)\right]$ is a nondegenerate matrix for all $\varphi \in \mathcal{T}_m$, because the other matrices in this identity are nondegenerate. Thus, there exists a vector $u_{k_0}(\varphi)$ that can be complemented to a periodic basis in the space $\mathbf{R}^n$, which contradicts the assumed properties of the matrix $U(\varphi)$.

In the same way, we can prove that if any row of the matrix $b(\varphi)$ can not be complemented then any vector from $U(\varphi)$ can not be complemented (it is sufficient to transpose the matrix $F(\varphi)$ with respect to the second diagonal and use $U^*(\varphi) = T^{-1}(\varphi)T(\varphi)U^*(\varphi)$, where $T^{-1}(\varphi)$ has the same properties as $T^{-1}(\varphi)$).

Now note that the matrix $\overline{\overline{F}}(\varphi)$ can be represented in the form $\overline{\overline{F}}(\varphi) = F(\varphi)S(\varphi)$, where $S(\varphi) \in C^l(\mathcal{T}_m)$ is an $n+r$-dimensional nondegenerate square matrix. Taking into account that the matrices $\overline{\overline{F}}(\varphi)$ and $F(\varphi)$ are orthonormalized, we get

$$S(\varphi) = (F^*(\varphi)F(\varphi))^{-1} F^*(\varphi)\overline{\overline{F}}(\varphi) = \operatorname{diag}\{E_r;\ \overline{S}(\varphi)\},$$

where $\overline{S} = (E_n - UU^* - UT^*NU^*)B + UT^*b$. Hence, the matrix $b(\varphi)$ can be represented in the form $b(\varphi) = T(\varphi)U^*(\varphi)\overline{S}(\varphi)$, where $\overline{S}(\varphi)$ is a nondegenerate matrix . Suppose there exists a row $b_j(\varphi)$ in the matrix $b(\varphi)$ that can be complemented to a periodic basis in the space $\mathbf{R}^n$. Then there exists an $(n-1) \times n$–matrix $B_j(\varphi) \in C^l(\mathcal{T}_m)$ such that $\det\left[b_j^*(\varphi),\ B_j^*(\varphi)\right]^* \neq 0$ for all $\varphi \in \mathcal{T}_m$. For the j-th column of the matrix $T(\varphi)U^*(\varphi)$, we get

$$\det\begin{pmatrix} (T(\varphi)U^*(\varphi))_j \\ B_j(\varphi)\overline{S}^{-1}(\varphi) \end{pmatrix} = \det\begin{pmatrix} (T(\varphi)U^*(\varphi))_j\,\overline{S}(\varphi) \\ B_j(\varphi) \end{pmatrix}\overline{S}^{-1}(\varphi) = \det\begin{pmatrix} b_j(\varphi) \\ B_j(\varphi) \end{pmatrix}\overline{S}^{-1}(\varphi) \neq 0$$

for all $\varphi \in \mathcal{T}_m$. This shows that one row of the matrix $b(\varphi)$ can be complemented to a periodic basis in the space $\mathbf{R}^n$ and we get the contradiction.

Similarly, if any row of the matrix $b(\varphi)$ can not be complemented to a periodic basis, then any row of the matrix $T(\varphi)U^*(\varphi)$ can not be complemented to a periodic basis, and as above, it follows that any vector from $U(\varphi)$ can not be complemented to a periodic basis. This completes the proof.

Lemma 5. *Suppose the matrix $U(\varphi)$ satisfies the conditions of Lemma 3. Then for any $q \times r$-dimensional matrix $\widehat{N}(\varphi) \in C^l(\mathcal{T}_m)$, $q < r$, the r-frame $\left[U^*(\varphi),\ \widehat{N}^*(\varphi)\right]^*$ can not be complemented to a periodic basis in the space $\mathbf{R}^{n+q}$.*

Proof. Assume the converse. Then there exist matrices $\widehat{B}(\varphi), \widehat{b}(\varphi) \in C^l(\mathcal{T}_m)$ of dimensions $n \times (n+q-r)$ and $q \times (n+q-r)$ such that

$$\det \widehat{F}(\varphi) \neq 0 \quad \forall \varphi \in \mathcal{T}_m, \quad \text{where} \quad \widehat{F}(\varphi) = \begin{pmatrix} U(\varphi) & \widehat{B}(\varphi) \\ \widehat{N}(\varphi) & \widehat{b}(\varphi) \end{pmatrix}.$$

Let us consider the matrix $\widehat{F}_1(\varphi) = \operatorname{diag}\left\{\widehat{F}_1(\varphi),\ E_{r-q}\right\}$. We orthonormalize this matrix using the Gramm – Schmidt procedure. So, we obtain

$$\widehat{F}_1(\varphi) = \begin{pmatrix} \widehat{U}_1(\varphi) & \widehat{B}_1(\varphi) \\ \widehat{N}_1(\varphi) & \widehat{b}_1(\varphi) \end{pmatrix} = \begin{pmatrix} U(\varphi) & \widehat{B}(\varphi) \\ \widehat{N}_0(\varphi) & \widehat{b}_0(\varphi) \end{pmatrix} \begin{pmatrix} R_{11}(\varphi) & R_{12}(\varphi) \\ 0 & R_{22}(\varphi) \end{pmatrix},$$

where $R_{11}(\varphi)$ and $R_{22}(\varphi)$ are the nondegenerate square matrices of dimensions r and n,

$$\widehat{N}_0(\varphi) = \begin{pmatrix} \widehat{N}(\varphi) \\ 0 \end{pmatrix}, \quad \widehat{b}_0(\varphi) = \begin{pmatrix} \widehat{b}(\varphi) & 0 \\ 0 & E_{r-q} \end{pmatrix}.$$

From the last identity we obtain

$$\widehat{b}_0 = -\widehat{N}_1 R_{11}^{-1} R_{12} R_{22}^{-1} + \widehat{b}_1 R_{22}^{-1} = \left(-\widehat{N}_0 R_{12} + \widehat{b}_1\right) R_{22}^{-1}.$$

Denote by $\widetilde{b}_0, \widetilde{b}_1, \widetilde{N}_0$ the last $r-q$ rows of the matrices $\widehat{b}_0, \widehat{b}_1, \widehat{N}_0$. Since $\widetilde{N}_0 = 0$, we get

$$\widetilde{b}_0(\varphi) = \widetilde{b}_1(\varphi) R_{22}^{-1}(\varphi). \tag{21}$$

The matrix $\widehat{F}_1(\varphi)$ satisfies all conditions of the previous lemma. That's why any row of $\widehat{b}_1(\varphi)$ and also any row of $\widetilde{b}_1(\varphi)$ can not be complemented to a periodic basis in the space $\mathbf{R}^n$. As in the proof of Lemma 4, we obtain that any row of the matrix $\widetilde{b}_1(\varphi)R_{22}^{-1}(\varphi)$ can not be complemented to a periodic basis. But according to (21), this matrix is represented in the form $\widetilde{b}_0(\varphi) = [0,\ E_{r-q}]$, where any row can be complemented to a periodic basis in a trivial way. This contradiction concludes the proof.

Using Lemmas 1, 4, 5 and the proof of Theorem 1 from [8] we obtain the following conclusion.

Theorem 3. *Suppose system (1) has invariant torus (2) of dimension satisfying condition (7). Besides, let system of vectors (3) can not be complemented to a periodic basis in the space $\mathbf{R}^n$, but have a maximal subsystem of $m-p$ vectors ($1 \leq p \leq 2m+1-n$) that can be complemented to a periodic basis in the space $\mathbf{R}^n$ (any subsystem of $(m-p+1)$ vectors can not be complemented). Then in the extended system (1), (8), where $y \in \mathbf{R}^p$, with invariant manifold (9), it is possible to introduce local coordinates $\varphi \in \mathcal{T}_m$, $h \in \mathbf{R}^l$, $\|h\| < \delta$, $l = n-m+p$, according (10) such that (11) is fulfilled, where the matrices $B(\varphi)$ and $b(\varphi)$ of dimension $n \times l$ and $p \times l$ belong to the class $C^r(\mathcal{T}_m)$, any row of the matrix $b(\varphi)$ can not be complemented to a periodic basis in the space $\mathbf{R}^l$. Besides, p is the smallest number of differential equations in system (8).*

Note 4. Theorem 3 shows that if system (8) is such that the extended system (1), (8) has invariant manifold (9) that doesn't coincide with M_0, then the dimension p of the variable y is not less then in [8]. That's why the construction of local coordinates as given in Theorem 1 in [8] is optimal.

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