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ON SOLUTIONS OF GENERAL NONLINEAR INITIAL BOUNDARY-VALUE PROBLEM OF INVISCID FLUID'S DYNAMICS IN MOVING VESSEL

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The problem of integrating the Laplace equation in a changing 3-dimensional region, with the initial and boundary conditions, is investigated. The paper is mainly devoted to the problem arising in dynamics of an inviscid incompressible fluid which partially fills a moving vessel and is in irrotational absolute motion. In this case the considered space region is bounded by the rigid vessel's walls and the unknown free surface of fluid. The boundary conditions consist of the Neyman conditions on the rigid walls and the nonlinear kinematic and dynamic conditions on the free surface. Besides, the condition of a constancy of the region's volume is imposed.

The concept of a solution of this problem is analyzed. One distinguishes a certain class of solutions and proves their existence. An example of such a solution is given.

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1. The Initial Boundary-Value Problem

One considers the following nonlinear initial boundary-value problem for the Laplace equation in the changing region $\Omega(t)$ (t denotes time) of the Euclidean space R^3:

$$\triangle u(x,y,z,t) = 0, \quad (x,y,z) \in \Omega(t), \quad t \in [t_0,t_1], \tag{1}$$

$$\frac{\partial u}{\partial n} = h_1(x,y,z,t,n), \quad (x,y,z) \in S(t), \tag{2}$$

$$\frac{\partial u}{\partial n} = h_2(x,y,z,t,n), \quad (x,y,z) \in \Sigma(t), \tag{3}$$

$$\Phi(x,y,z,t,\nabla u,u_t) = 0, \quad (x,y,z) \in \Sigma(t), \tag{4}$$

$$\int\limits_{\Omega(t)} d\Omega(t) = \text{const} \quad \forall t \in [t_0,t_1], \tag{5}$$

$$u(x,y,z,t_0) = u_0(x,y,z), \quad \zeta(x,y,t_0) = \zeta_0(x,y) \quad (\triangle u_0 = 0, \quad (x,y,z) \in \Omega(t_0)). \tag{6}$$

Here the operators $\triangle$, ∇ act with respect to the variables x, y, z which are the coordinates of any point of the space R^3 in a certain coordinate system $Oxyz$; the normal derivatives are taken in the direction of the unit vector $n = n(x,y,z,t)$ of the outward normal to the boundary $\Gamma(t)$ of the region $\Omega(t)$; the letter indices denote the partial derivatives with respect to the corresponding variables. The segment $[t_0,t_1]$ is finite.

As we shall see below, the boundary functions h_1, h_2 depend on the normal vector n explicitly. In the general case of the unsmooth boundary surface $\Gamma(t)$, this circumstance makes the boundary conditions (2), (3) irregular; thus it is essential in theoretical sense. To underline this, here and further the vector n is pointed out among the arguments of the functions h_1, h_2.

The domain $\Omega(t)$ is simply connected. It is inserted in the domain Π which is bounded by the fixed closed surface $\partial\Pi$ and is simply connected too. Thus the following inclusion is valid:

$$\Omega(t) \subseteq \Pi \quad \forall t \in [t_0, t_1].$$

The surface $\partial\Pi$ is known. $S(t)$ is a surface of the intersection of the unknown boundary $\Gamma(t)$ and the known surface $\partial\Pi$. In the designation $S(t)$, the symbol t reflects the motion of $S(t)$ along the surface $\partial\Pi$. Those cases are eliminated when the set $S(t)$ becomes empty.

The boundary $\Gamma(t)$ is presented as

$$\Gamma(t) = S(t) \cup \Sigma(t), \quad S(t) \cap \Sigma(t) = l(t),$$

where $\Sigma(t)$ is the free surface to be found; $l(t)$ is the intersection curve of the considered pieces of the boundary. The unknown surface $\Sigma(t)$ can change its configuration and location relative to the vessel. We suppose that the surface $\Sigma(t)$ is described by the equation

$$z = \zeta(x, y, t).$$

In the boundary condition (2) the function h_1 is known and defined by the formula

$$h_1(x, y, z, t, n) = [v(t) + \omega(t) \times r] \cdot n(x, y, z, t), \tag{7}$$

where $v(t)$, $\omega(t)$: $[t_0, t_1] \to R^3$ are the known vector-valued functions; $r = (x, y, z)$. The symbols "$\cdot$" and "$\times$" denote the scalar and vector product, respectively. (In function (7) the normal $n(x, y, z, t)$ is taken in the points of the surface $S(t)$.)

The functions h_2 and Φ from the boundary conditions (3), (4) are of the following form:

$$h_2(x, y, z, t, n) = h_1(x, y, z, t, n) + (1 + \zeta_x^2 + \zeta_y^2)^{-\frac{1}{2}} \zeta_t(x, y, t), \tag{8}$$

$$\Phi(x, y, z, t, \nabla u, u_t) = u_t + \frac{1}{2}(\nabla u)^2 - \nabla u \cdot (v(t) + \omega(t) \times r) - g(t) \cdot r + \frac{p_0}{\rho} - F(t). \tag{9}$$

Here $g(t)$: $[t_0, t_1] \to R^3$ is the known vector-valued function; $p_0, \rho > 0$ are certain real constant parameters; $F(t)$ is an arbitrary function of time. (In contrast to (7), the normal $n(x, y, z, t)$ of the function h_1 in (8) is taken in the points of the free surface $\Sigma(t)$.)

The sought for quantity of the problem (1) – (6) is the pair of functions (u, ζ). The function $F(t)$ is unknown too, but as we shall see below it is determined from additional considerations.

The formulated problem is known from dynamics of the bounded volume of an inviscid incompressible fluid under the vessel's arbitrary motion in gravity field [1]. In equations (1) – (8) $u(r, t)$ is a potential of the fluid's absolute velocity (∇u gives the projections of the fluid's absolute velocity on the axes of the *moving* coordinate system $Oxyz$); equation (2) is the impermeability condition of the vessel's rigid walls for fluid; the nonlinear relations (3), (4) are the kinematic and dynamic conditions on the fluid free surface, respectively; equation (5) is the condition of constancy of the fluid volume; equations (6) are the initial conditions. The function $F(t)$ is an arbitrary function from the Lagrange – Cauchy integral. The domain Π is nothing else but a cavity of the vessel and $\Omega(t)$ is the fluid volume.

In a partial case, when $\Omega(t)$ coincides with Π, the function $\zeta(x, y, t) \equiv \zeta(x, y)$, which is of practical interest too (this problem constitutes the well-known Zhukovsky problem).

In general, the problem (1) – (9) is very difficult. An existence of its solutions has not been proved hitherto. The obstacles are connected with nonlinearity of the boundary conditions, variability of the solutions' domain and indeterminacy of the domain's boundaries. There exists another less evident difficulty, namely, the functions h_1 on $S(t)$ and h_2 on $\Sigma(t)$ are irregular as a consequence of their dependence on the normal vector n which, in the general case of the unsmooth surfaces $\Gamma(t)$, must be considered as a distribution (generalized function). Under this circumstances, as far as we know, even the solution's notion has not been yet formulated accurately.

On the other hand, there exist many papers which are devoted to the development of approximate methods of solving the considered problem in different particular cases (of the vessel's shapes and the laws of the vessel's motion). However, each of such methods must not contradict to a priori mathematical properties of an exact solution about which there exist little information in the case under consideration. As a pay for not meeting this requirement, there may be either an instability of the numerical method for solving the problem or an absence of the convergence of the approximate solution to the exact solution.

In connection with the indicated circumstances, the purpose of this paper is formulated as follows: to clarify the concept of the solution of the problem (1) – (9), to prove its existence under certain conditions and to show the example of such a solution.

One uses the S. L. Sobolev's [2] and J. Nečas' [3] results which are most suitable regarding the equation, the boundary functions, and unsmoothness of the boundary surfaces (see, for example, [4 – 8]).

2. The Region $\Omega(\mathbf{t})$

The properties of space region in which the initial boundary-value problem is considered play an important role in the solvability questions of this problem. In linear hydrodynamics the case of the Lipschitzian region is the most general [9, p. 31]. Let us extent this approach to the nonlinear case.

We shall assume that the boundary $\partial\Pi$ of the region Π consists of a finite number M of the surface pieces S_k, $k = 1, \ldots, M$, each of which is described by the following relations

$$y_3^{(k)} = f^{(k)}(y_1^{(k)}, y_2^{(k)}), \quad (y_1^{(k)}, y_2^{(k)}) \in D^{(k)}, \quad k = 1, \ldots, M, \tag{10}$$

where $(y_1^{(k)}, y_2^{(k)}, y_3^{(k)})$ is the local Cartesian coordinate system; $f^{(k)}(y_1^{(k)}, y_2^{(k)})$ is a continuous function in the square ($\alpha > 0$ is some constant)

$$D^{(k)} = \{(y_1^{(k)}, y_2^{(k)}) : \quad | \, y_i^{(k)} \, |< \alpha, \quad i = 1, 2\}$$

which satisfies the Lipschitz condition

$$| \, f^{(k)}(\xi) - f^{(k)}(\eta) \, |\leq Ld(\xi, \eta) \quad \forall \xi, \eta \in D^{(k)} \subset R^2 \quad (L > 0). \tag{11}$$

Here $d(\cdot, \cdot)$ denotes the Euclidean distance between the vectors.

In addition, we assume that the surface $\partial\Pi$ has not any singular points, i.e., there exists $\beta > 0$ such that the points $(y_1^{(k)}, y_2^{(k)}, y_3^{(k)})$ for which $| \, y_i^{(k)} \, |< \alpha$, $i = 1, 2$, and the coordinate $y_3^{(k)}$ satisfies one of the conditions

$$\begin{aligned} f^{(k)}(y_1^{(k)}, y_2^{(k)}) - \beta < y_3^{(k)} &< f^{(k)}(y_1^{(k)}, y_2^{(k)}), \\ f^{(k)}(y_1^{(k)}, y_2^{(k)}) < y_3^{(k)} &< f^{(k)}(y_1^{(k)}, y_2^{(k)}) + \beta \end{aligned} \tag{12}$$

lie inside the open region Π or outside the closed region $\bar{\Pi} = \Pi \cup \partial\Pi$, respectively.

The totality of relations (10) – (12) defines the Lipschitzian surface in the sense of definition of paper [3, p. 14, 15].

Next, let us fix the origin O of the coordinate system $Oxyz$ at any point inside Π. Then considering $\Sigma(t)$ as a family of surfaces which depend on t and are situated inside Π, we shall define each of these surfaces as

$$z = \zeta(x, y, t), \quad (x, y) \in D(\zeta), \quad t \in [t_0, t_1], \tag{13}$$

where $D(\zeta)$ is the orthogonal projection of the surface $\Sigma(t)$ to the plane Oxy; the function $\zeta(x, y, t)$ is continuous with respect to x, y in the region $D(\zeta)$ and satisfies the Lipschitz condition

$$| \, \zeta(\xi, t) - \zeta(\eta, t) \, |\leq Ld(\xi, \eta) \quad \forall \xi, \eta \in D(\zeta), \quad t \in [t_0, t_1]. \tag{14}$$

We note that for all t the relation $D(\zeta) \subseteq D_0$ holds, where D_0 is the orthogonal projection of the region Π to the plane Oxy.

From the two subsets into which the surface $\Sigma(t)$ divides the region Π, the one we select for $\Omega(t)$ is that which is situated, entirely or partially, under the surface $\Sigma(t)$ (along the axis Oz). For example, if the fluid filling a rolling rectangular vessel touches the vessel's lid, $\Omega(t)$ is situated partially under $\Sigma(t)$ only. If it does not, it is situated under $\Sigma(t)$ entirely.

Remark 1. In connection with the suggested parametrization of the boundary surfaces in the considered hydrodynamic problem it is appropriate to note the following. Since the points of the surface pieces S_k are determined by the local coordinates $(y_1^{(k)}, y_2^{(k)}, y_3^{(k)})$, the projections x, y, z of the corresponding vector r to the axis x, y, z will be functions of these local coordinates if formula (7) is considered together with the boundary conditions (2). This also refers to the vector r in the expressions (8), (9) if they are considered together with the boundary conditions

(3), (4), but here the local coordinates for the free surface $\Sigma(t)$ coincide with x, y, z. We should also add that in all cases the normal n to the considering surfaces will be taken in projections to the axes of the "support" coordinate system $Oxyz$, because the input vectors $v_0(t), \omega(t)$ are usually given in this coordinate system.

Besides the condition (14), one also supposes that a condition on the construction of the β-neighbourhood of the surface $z = \zeta(x, y, t)$, which is analogous to (12), is fulfilled, namely: the points (x, y, z) for which $\mid x \mid< \alpha, \mid y \mid< \alpha$ and the coordinate z satisfies one of the conditions

$$\zeta(x,y,t) - \beta < z < \zeta(x,y,t), \quad \zeta(x,y,t) < z < \zeta(x,y,t) + \beta \tag{15}$$

are situated inside the open region $\Omega(t)$ or outside the closed region $\bar{\Omega}\ (t) = \Omega(t) \cup \Gamma(t)$, respectively.

To ensure the Lipschitz condition for the entire surface $\Gamma(t)$, we also introduce the following condition (in short, the l-condition): at the points of the line $l(t)$ in which the tangent planes to both the surface $S(t)$ and the surface $\Sigma(t)$ exist, the angles between this tangent planes are not equal to zero.

In general, the l-condition may be not fulfilled. Indeed, the points of the line $l(t)$ are situated on the boundaries of three media (the fluid, the gas, and the rigid body). As is known, the angles between the contact surfaces of this media at the mentioned points are functions of the corresponding coefficients of the surface tension [10, p. 32]. Theoretically, these coefficients can be such that the mentioned angles will be equal to zero. In our case, however, the surface tension is neglected and therefore one can say nothing about the values of these angles. But as we shall see below, there exists an example for which the l-condition holds.

The class of the surfaces $\Sigma(t)$ which are defined by the explicit equations (13) is narrower than the class of the surfaces which are defined both in the implicit form $f(x, y, z, t) = 0$ and in the parametric form $x = x(\tau_1, \tau_2)$, $y = y(\tau_1, \tau_2)$, $z = z(\tau_1, \tau_2)$ (τ_1, τ_2 are the parameters). (The surfaces (13) make up a subset both of the set of the implicitly defined surfaces when $f(x, y, z, t) = z - \zeta(x, y, t)$) and of the set of the surfaces which are defined parametrically when $\tau_1 = x$, $\tau_2 = y$, $x(\tau_1, \tau_2) = x$, $y(\tau_1, \tau_2) = y$, $z(\tau_1, \tau_2) = \zeta(x, y, t)$.) At the same time this class of surfaces contains a sufficiently large number of practical cases. In connection with this, it is advisable to extract the corresponding regions $\Omega(t)$ as a separate class.

Definition. *A changing region $\Omega(t)$ is called a ζ-region if for all t the fixed boundary surface $S(t)$ satisfies the conditions (10) – (12), the free boundary surface $\Sigma(t)$ is set by the explicit equation (13) and satisfies the conditions (14), (15), and at the line of intersection of the surfaces $S(t)$ and $\Sigma(t)$ the l-condition is fulfilled.*

By the classification of paper [6], the ζ-regions belong to the Hölder class $C_{l,\lambda}$, where $l = 0$, $\lambda = 1$ (since one requires only a continuity of the functions which define the pieces of the boundary (the derivatives' continuity is not obligatory) and the Lipschitz conditions coincide with the corresponding Hölder conditions for which the exponent λ is one).

Let us note some peculiarity of the ζ-regions which is important in what follows and refers to the connection between structure and properties of its boundary.

Proposition 1. *The ζ-region is a locally star-shaped region at any time t.*

Proof. Indeed, this property for the Lipschitz regions is proved in [11, p. 308] but the Lipschitz property of the ζ-region at any t follows from Definition.

The locally star-shaped structure of the ζ-regions will be used below when applying Sobolev's results to the problem (1) – (9).

Remark 2. The assumptions (13), (14) and the l-condition concern the unknown mathematical objects (the sought for surface $\Sigma(t)$ and the line $l(t)$ of its intersection with $S(t)$). Therefore, there exists a certain risk to make a mistake if we endow these objects with the noted properties beforehand. An exception is the condition (13) which is fulfilled unconditionally, because it is present in the relation (3) implicitly. As for the rest of the conditions, the example constructed below shows that the initial problem (1) – (9) can have a solution (u, ζ) component ζ of which satisfies these conditions and, consequently, the corresponding $\Omega(t)$ is a ζ-region.

3. The Class of Functions for Possible Solutions

In this section one clarifies an important question regarding the classes of functions in which the solutions of the considered initial boundary-value problem must be sought.

A priory information about the solutions of the problem (1) – (9) can be obtained if one notes that any its solution (u^0, ζ^0) (if it exists) necessarily is a solution of the shortened problem (1) – (3), (5). After the substitution of $\zeta^0(x, y, t)$ for $\zeta(x, y, t)$ this shortened problem turns into the Neyman problem in a changing region. (We note that the shortened problem (1) – (3), (5) is of an independent practical interest too. In hydrodynamics it corresponds to the case when at any time t the free surface is known and it is necessary to find the potential and the pressure in the fluid.)

When t is fixed the problem (1) – (3) is a classical Neyman problem for the Laplace equation. For the class of regions with the so-called "simple" boundary (or else, with the piecewise smooth boundary) its solution is given by S.L. Sobolev [2, p. 124]. In this case, the conditions on the boundary functions are formulated in the form of their belonging to the space $L_2(\Gamma)$. O.A. Ladyzhenskaya and N.N. Uraltzeva [6] considered the regions with a piecewise smooth boundary too, but they supposed that the boundary functions are defined at once in the entire domain $\Omega + \partial\Gamma$ (p. 25), which cannot be said about the functions h_1 and h_2. The cases of irregular boundary functions from the space $W_2^{-1/2}(\Gamma) \supset L_2(\Gamma)$ are investigated by J.-L. Lions and E. Magenes [7] but they restricted themselves to boundaries Γ from the class C^∞ of infinitely-differentiated surfaces [p. 48]. The Neyman problem for the Laplace equation (1) – (3) with "hydrodynamic" specific character may be investigated if one combines Sobolev's [2] and Nečas' [3] results; this is carried out below.

Before solving the question regarding the class of functions for the solutions, we consider properties of the vector-valued function $n(x, y, z, t)$ which is included in the boundary conditions (2), (3) and makes them irregular.

Let $n = (n_1, n_2, n_3)$ be the vector-valued function n in projections to the axes of the local coordinate systems, where

$$n_i = \begin{cases} n_i(y_1^{(k)}, y_2^{(k)}, y_3^{(k)}, t), & \text{if} \quad (x^{(k)}, y^{(k)}, z^{(k)}) \in S_k, \\ n_i(x, y, z, t), & \text{if} \quad (x, y, z) \in \Sigma(t), \end{cases}$$

$$i = 1, 2, 3, \quad k = 1, \ldots, M.$$

In the first row of this formula, the coordinates $x^{(k)}, y^{(k)}, z^{(k)}$ of the points of the surface S_k in the coordinate system $Oxyz$ are connected with the local coordinates $y_1^{(k)}, y_2^{(k)}, y_3^{(k)}$ of these points by the following relations:

$$\begin{pmatrix} x^{(k)} \\ y^{(k)} \\ z^{(k)} \end{pmatrix} = \begin{pmatrix} x_0^{(k)} \\ y_0^{(k)} \\ z_0^{(k)} \end{pmatrix} + A^{(k)} \begin{pmatrix} y_1^{(k)} \\ y_2^{(k)} \\ y_3^{(k)} \end{pmatrix},$$

where $x_0^{(k)}, y_0^{(k)}, z_0^{(k)}$ are coordinates of the origin of the k-th local coordinate system in the axes x, y, z; $A^{(k)}$ is the matrix consisting of cosines of the angles between the axes x, y, z and $y_1^{(k)}, y_2^{(k)}, y_3^{(k)}$.

For every piece S_k of the surface $\Gamma(t)$, the matrix $A^{(k)}$ has its own form and is constant, i.e., independent of the space variables and time. For the piece $\Sigma(t)$ of the surface $\Gamma(t)$, an analogous relation holds, where one must assume $x_0^{(k)} = y_0^{(k)} = z_0^{(k)} = 0$ and $A^{(k)}$ is the identity matrix.

In addition, let $n = (n_x^{(k)}, n_y^{(k)}, n_z^{(k)})$ be the unit normal vector to the surface S_k in projections to the axes x, y, z (see Remark 1). It is evident that

$$\begin{pmatrix} n_x^{(k)} \\ n_y^{(k)} \\ n_z^{(k)} \end{pmatrix} = A^{(k)} \begin{pmatrix} n_1(y_1^{(k)}, y_2^{(k)}, y_3^{(k)}) \\ n_2(y_1^{(k)}, y_2^{(k)}, y_3^{(k)}) \\ n_3(y_1^{(k)}, y_2^{(k)}, y_3^{(k)}) \end{pmatrix}. \tag{16}$$

Proposition 2. *Let the region $\Omega(t)$ be a ζ-region for all t. Then at any time t there exists a vector $n(x, y, z, t)$ normal to its boundary $\Gamma(t)$ almost everywhere on $\Gamma(t)$ and its components n_x, n_y, n_z, in the coordinate system Oxyz, are measurable and bounded functions of x, y, z defined almost everywhere on $\Gamma(t)$.*

Proof. For the Lipschtzian region which does not depend on time, existence almost everywhere, measurability, and boundedness of the projections n_1, n_2, n_3 follow from Lemma 4.2 of Paper [3, p. 88]. (Here and later, the measurability is regarded in the sense of Lebesgue.) For the boundary $\Gamma(t)$, this remains valid because $\Gamma(t)$ is Lipschitzian for all t in virtue of Definition. Since n_x, n_y, n_z are linear combinations of n_1, n_2, n_3 with piecewise continuous coefficients on $\Gamma(t)$ (in virtue of (16)), the projections of the normal $n(x, y, z, t)$ to the axes of the coordinate system $Oxyz$ also exist almost everywhere and are measurable, bounded functions on $\Gamma(t)$, which is what was to be proved.

The measurability and boundedness of the function $n(x, y, z, t)$ will be used to determine the class of functions for the functions h_1, h_2 from the boundary conditions (2), (3).

To prove Theorem of the present paper, it will also be necessary to use the following known result of Sobolev concerning solution of the Neyman problem [2, p. 124] and formulated in terms of belonging the boundary functions to the space $L_2(\Gamma)$.

Proposition 3 [8, p. 323]. *The Neyman problem for the Laplace equation in some finite region Ω of the Euclidean space R^n has a unique weak solution from the space $W_2^1(\Omega)$ if Ω is* *locally star-shaped and the boundary function $h(x)$ ($x \in R^n$) satisfies the following conditions*

$$h \in L_{q^*}(\Gamma), \quad \frac{1}{q} + \frac{1}{q^*} = 1, \quad q < \frac{2(n-1)}{n-2}, \tag{17}$$

$$\int\limits_{\Gamma} h(x) d\Gamma = 0, \tag{18}$$

where Γ is the boundary of the region Ω.

Now, taking into account the concrete form of the functions h_1, h_2, we can prove the following statement.

Theorem. *Let $\Omega(t)$ be a ζ-region, the functions $h_1(x,y,z,t)$, $h_2(x,y,z,t)$ be defined by the formulae (7), (8), and the derivative $\zeta_t^0(x,y,t)$ satisfy the condition*

$$\forall t \in [t_0, t_1] \quad \textit{the function} \quad (x,y) \mapsto \zeta_t^0(x,y,t) \quad \textit{is measurable and bounded on} \quad D(\zeta). \tag{19}$$

Then there exists a unique weak solution $u^0(x,y,z,t)$ of the problem (1) – (3), (5) which, for any t, belongs to the Sobolev space $W_2^1(\Omega(t))$.

Proof. Let us fix t, suppose $n = 3$ in the relation (17) and replace Γ with $\Gamma(t)$ in (18). We also note that in the considered case,

$$h = h_1(x,y,z,t) \quad \text{for} \quad (x,y,z) \in S(t), \qquad h = h_2(x,y,z,t) \quad \text{for} \quad (x,y,z) \in \Sigma(t).$$

Then, first of all, according to Proposition 1 the region $\Omega(t)$ is locally star-shaped.

Let us apply Proposition 3 and prove that the conditions (17), (18) are fulfilled for the considered shortened hydrodynamic problem. Indeed, let $q = 2$. Then we should prove that

$$h \in L_2(\Gamma(t)).$$

However, the function h_1 is a scalar product of the vector-valued functions $[v(t) + \omega(t) \times r]$ and n with the first of them being continuous with respect to the space variables x, y, z in virtue of Remark 1 and definition of the Lipschitzian surface. At the same time, it follows from Proposition 2 that the components of the vector n are measurable and bounded functions of x, y, z. Hence, h_1 is measurable and bounded, too, and $h_1 \in L_2(S(t))$.

In turn, h_2 is the sum of h_1 and

$$\frac{\zeta_t^0}{\sqrt{1 + (\zeta_x^0)^2 + (\zeta_y^0)^2}}.$$

However, the function h_1 on $\Sigma(t)$ is an element of the space $L_2(\Sigma(t))$ in virtue of the above argument. The derivative ζ_t^0 is measurable and bounded in virtue of the condition (19) of Theorem. The denominator $\sqrt{1 + (\zeta_x^0)^2 + (\zeta_y^0)^2}$ is measurable, bounded, and nonzero because, in the coordinate system $Oxyz$ (which is local for the surface $\Sigma(t)$), the normal n to the free surface $\Sigma(t)$ is of the form

$$n(x,y,z,t) = \frac{(-\zeta_x^0, -\zeta_y^0, 1)}{\sqrt{1+(\zeta_x^0)^2+(\zeta_y^0)^2}},$$

and is measurable and bounded in virtue of Proposition 2. Hence, $h_2 \in L_2(\Sigma(t))$.

At last, since $S(t)$ and $\Sigma(t)$ intersect themselves at most along the line $l(t)$, one can write

$$\int\limits_{\Gamma(t)} h^2 d\Gamma(t) = \int\limits_{S(t)} h_1^2 dS(t) + \int\limits_{\Sigma(t)} h_2^2 d\Sigma(t).$$

Remark 3. The integrals over the surface $\Gamma(t)$ everywhere are understood in the Lebesgue sense and defined with the help of partition of unity. The latter is associated with the description of S_k (in the general case) in different local coordinate systems.

It follows from this that $h \in L_2(\Gamma(t))$, and hence the conditions (17) are satisfied.

Let us consider the condition (18). We have the relation

$$\int\limits_{\Gamma^0(t)} h d\Gamma^0(t) = \int\limits_{S^0(t)} [v(t)+\omega(t)\times r]\cdot n dS^0(t)$$

$$+ \int\limits_{\Sigma^0(t)} \Big\{[v(t)+\omega(t)\times r]\cdot n + [1+(\zeta_x^0)^2+(\zeta_y^0)^2]^{-\frac{1}{2}}\zeta_t^0\Big\} d\Sigma^0(t). \tag{20}$$

Here $\Gamma^0(t)$, $S^0(t)$, and $\Sigma^0(t)$ correspond to the surface $z = \zeta^0(x,y,t)$. By the divergence theorem (which is true in the case of the Lipschitzian regions too), we can find that

$$\int\limits_{\Gamma^0(t)} [v(t)+\omega(t)\times r]\cdot n d\Gamma^0(t) = 0. \tag{21}$$

Then we shall use the additional integral condition (5). By differentiating (5) with respect to t (such differentiation is possible for the regions which are bounded by piecewise smooth surfaces), we obtain the following relations:

$$\frac{d}{dt}\int\limits_{\Omega(t)} d\Omega(t) = \int\limits_{\Gamma(t)} V_n d\Gamma(t) = \int\limits_{S(t)} V_n dS(t) + \int\limits_{\Sigma(t)} V_n d\Sigma(t) \equiv 0, \tag{22}$$

where $V_n = V_n(x,y,z,t)$ is the normal component of the velocity of the point $\Gamma(t)$ as it moves relative to the coordinate system $Oxyz$. However, it follows from equations (10) and (13) for the parts of the surface $\Gamma(t)$ that

$$V_n \equiv 0 \quad \text{on} \quad S^0(t), \quad V_n = [1+(\zeta_x^0)^2+(\zeta_y^0)^2]^{-\frac{1}{2}}\zeta_t^0 \quad \text{on} \quad \Sigma^0(t). \tag{23}$$

From (22) using (23) we find that

$$\int\limits_{\Sigma^0(t)} [1 + (\zeta_x^0)^2 + (\zeta_y^0)^2]^{-\frac{1}{2}} \zeta_t^0 d\Sigma^0(t) \equiv 0. \tag{24}$$

Substituting (21) and (24) into (20) we obtain the equality (18).

Up to now we supposed that t was fixed. Since it was arbitrary, the statement of the theorem is true for any $t \in [t_0, t_1]$. Theorem is proved.

Thus, for the input problem (1) – (9) there exists a reason to look for the solutions (u, ζ) among the functions that satisfy the following conditions (the arguments that are not specified in the following formulae vary on the sets indicated in the notations of the corresponding function space):

$$\Omega(t) \quad \text{is a} \quad \zeta\text{-region},$$

$$u(r,t) : \Omega(t) \times [t_0, t_1] \to R, \quad \zeta(x,y,t) : D(\zeta) \times [t_0, t_1] \to R;$$

the properties with respect to the space variables are

$\forall\ t \in [t_0, t_1]$:
$u(\cdot, t), u_t(\cdot, t) \in W_2^1(\Omega(t))$,
$u_x(\cdot, t), u_y(\cdot, t), u_z(\cdot, t) \in L_2(\Omega(t))$,
$\zeta(\cdot, \cdot, t), \zeta_t(\cdot, \cdot, t) \in C(D(\zeta))$ and is Lipschitzian,
$\zeta_x(\cdot, \cdot, t), \zeta_y(\cdot, \cdot, t)$ exist almost everywhere and are measurable and bounded in $D(\zeta)$;

the properties with respect to time t are

$\forall\ r \in \Omega(t)$:
$u(r, \cdot), u_x(r, \cdot), u_y(r, \cdot), u_z(r, \cdot) \in C([t_0, t_1])$,
$u_t(r, \cdot)$ is piecewise continuous in $[t_0, t_1]$ and has a finite number of discontinuities of the first kind,

$\forall\ (x, y) \in D(\zeta)$:
$\zeta(x, y, \cdot), \zeta_x(x, y, \cdot), \zeta_y(x, y, \cdot) \in C^1([t_0, t_1])$,
$\zeta_t(x, y, \cdot) \in C([t_0, t_1])$.

One can verify that the properties of u, ζ and their derivatives as functions of t agree with the properties of the given functions $v(t), \omega(t), g(t)$ imposed by physical reasons and consist in the following:

$$v(t), \omega(t), g(t) : [t_0, t_1] \subset R \to R^3,$$

$v(t), \omega(t), g(t) \in C([t_0, t_1])$,
$\dot{v}(t)$, $\dot{\omega}(t)$ are piecewise continuous and have a finite number of discontinuities of the first kind.

This assumptions correspond to the translational and angular motions of the vessel with jumps (discontinuities) of acceleration. Such motions occur, for example, during an acceleration, retardation, instantaneous stopping, pushes of the vessel, and so on. The "ordinary" motions with continuously changing acceleration are also included here.

In the formulated requirements on smoothness of the sought for functions, the conditions on $\zeta_x(\cdot,t), \zeta_y(\cdot,t)$ and $u(\cdot,t), u_x(\cdot,t), u_y(\cdot,t), u_z(\cdot,t), u_t(\cdot,t)$ follow from Proposition 2 and Theorem, respectively. The condition on derivative $\zeta_t(\cdot,t)$, evidently, agrees with the requirement (19) of Theorem on ζ_t. All the rest of the properties either follow from physical reasons or are their corollaries.

4. Example

Let us show that the set of the pairs of functions (u^0, ζ^0) mentioned in Theorem is not empty. To this end, we shall construct one exact solution of the input problem (1) – (9).

Let the region Π be a cylinder the axis of which coincides with the coordinate axis Ox, i.e.,

$$\Pi = \left\{(x,y,z): \quad | x |< a, \quad y^2+z^2 < R^2\right\},$$

$$\partial\Pi = \left\{(x,y,z): \quad x = \pm a, \quad y^2+z^2 < R^2\right\} \cup \left\{(x,y,z): \quad | x |< a, \quad y^2+z^2 = R^2\right\},$$

where R, a is the radius and a half of length of the cylinder, respectively. The origin O of the coordinate system $Oxyz$ is placed in the cylinder center, the axis Oz is directed in an opposite way to the vector g (the location of the Oy axis is determined, in this case, uniquely).

Let us consider the following family of surfaces $\Sigma(t)$ (13) depending on parameter t:

$$\zeta(x,y,t) = h(t) + k(t)y, \quad (x,y) \in D(\zeta) \subset D_0, \quad t \in [t_0, t_1].$$

Here $h(t), k(t)$ are certain functions of time and the sets $D(\zeta)$, D_0 are rectangles in the plane Oxy which are determined by the relations

$$D(\zeta) = \{(x,y): \quad | x |< a, \quad y_1(t) < y < y_2(t)\}, \quad D_0 = \{(x,y): \quad | x |< a, \quad | y |< R\}.$$

In the penultimate formula, the functions $y_1(t), y_2(t)$ are solutions for y of the following system of equations:

$$y^2 + z^2 = R^2, \quad z - h(t) - k(t)y = 0,$$

and are of the form

$$y_{1,2}(t) = \frac{\{-hk \pm [(1+k^2)R^2 - h^2]^{\frac{1}{2}}\}}{1+k^2}.$$

Here the minus and plus sings correspond to the functions $y_1(t)$ and $y_2(t)$, respectively. Further we suppose $h(t)$, $k(t)$, R to be such that

$$y_1(t) < 0, \quad y_2(t) > 0.$$

Thus, the function $\zeta(x,y,t)$ is independent of the space variable x and, as a function of y, is determined in the changing region $D(\zeta)$ which is bounded by abscissas $y_1(t), y_2(t)$ of the intersection points of the circle and the straight line.

The region $\Omega(t)$, in this particular case, can be presented in the following form:

$$\Omega(t) = \Omega_1(t) \cup \Omega_2(t),$$

where

$$\Omega_1(t) = \{(x,y,z): \quad |x| < a, \quad -R < z < z_1(t), \quad |y| < \sqrt{R^2 - z^2}\},$$

$$\Omega_2(t) = \left\{(x,y,z): \quad |x| < a, \quad z_1(t) < z < z_2(t), \quad \frac{z - h(t)}{k(t)} < y < \sqrt{R^2 - z^2}\right\}.$$

Here the functions $z_1(t), z_2(t)$ are the ordinates of the above-mentioned intersection points of the circle and straight line, correspond to the abscissas $y_1(t), y_2(t)$, and are determined by the formulae

$$z_{1,2}(t) = \{h \pm k[(1+k^2)R^2 - h^2]^{\frac{1}{2}}\}/(1+k^2).$$

The minus and the plus sings correspond to the functions $z_1(t)$, $z_2(t)$, respectively. If $k(t) > 0$, then $z_1(t) < z_2(t)$, which is supposed further.

The surface $S(t)$ consists of two plane sets and the cylindrical surface so that we can write

$$S(t) = \bigcup_{j=1}^{3} S_j(t),$$

where

$$S_{1,2}(t) = \{(x,y,z): \quad x = \pm a, \quad y^2 + z^2 < R^2, \quad -R < z < h(t) + k(t)y\},$$

$$S_3(t) = \{(x,y,z): \quad |x| < a, \quad y^2 + z^2 = R^2, \quad -R < z < h(t) + k(t)y\}.$$

The surface $S(t)$ is evidently piecewise smooth for any t.

The line $l(t)$ of intersection of the surfaces $S(t)$ and $\Sigma(t)$ is a set of the following form:

$$l(t) = \bigcup_{j=1}^{4} l_j(t),$$

where

$$l_{1,2}(t) = \{(x,y,z): \quad |x| < a, \quad y = y_{1,2}(t), \quad z = z_{1,2}(t)\},$$

$$l_{3,4}(t) = \{(x,y,z): \quad x = \pm a, \quad y_1(t) < y < y_2(t), \quad z = h(t) + k(t)y\}.$$

The l-condition on the line $l(t)$ can be violated only if $h(t) \equiv \pm R$ when $k(t) \equiv 0$ (in this case the l-line degenerates into segments coinciding with the generator of the cylindrical surface). Further we shall suppose that $\mid h(t) \mid< R$. With this inequality, the l-condition is always fulfilled.

Thus, in the considered example the region $\Omega(t)$ is a ζ-region.

In respect to the functions $v(t), \omega(t)$ in (7), we suppose that in projections to the axes x, y, z, they are of the following form:

$$v(t) = (0, v_y(t), v_z(t)),$$

$$v_y(t) = (C_1 + C_2 t)\cos\alpha(t) + (C_3 + C_4 t)\sin\alpha(t),$$

$$v_z(t) = -(C_1 + C_2 t)\sin\alpha(t) + (C_3 + C_4 t)\cos\alpha(t),$$

$$\omega(t) = \left(\frac{d\alpha}{dt}, 0, 0\right),$$

where $\alpha(t) \in C^1([t_0, t_1])$ is an arbitrary function and also $-\pi/2 < \alpha(t) < \pi/2$; $C_j, j = 1, 2, 3, 4$, is a certain real constant.

Under this conditions, we can give a solution of the shortened problem (1) – (3), (5) which simultaneously will be a solution of the initial problem (1) – (9). Indeed, one can directly verify that for the function

$$\zeta^0(x, y, t) = h_0\sqrt{1 + k^2(t)} + k(t)y,$$

where

$$k(t) = \frac{C_2\cos\alpha(t) + (g + C_4)\sin\alpha(t)}{C_2\sin\alpha(t) - (g + C_4)\cos\alpha(t)},$$

and h_0 is a certain real constant, the function

$$u^0(x, y, z, t) = [(C_1 + C_2 t)\cos\alpha(t) + C_3 + C_4 t)\sin\alpha(t)]y$$

$$+ \; [(C_3 + C_4 t)\cos\alpha(t) - (C_1 + C_2 t)\sin\alpha(t)]z,$$

is a solution of the shortened problem (1) – (3), (5). The constant h_0 is determined from condition (5).

Supposing (we use arbitrariness of $F(t)$ from the Lagrange – Cauchy integral)

$$F(t) = \frac{p_0}{\rho} + \frac{1}{2}\left[(C_1 + C_2 t)^2 + C_3 + C_4 t)^2\right] + [C_2\sin\alpha(t) - (g + C_4)\cos\alpha(t)]h_0\sqrt{1 + k^2(t)}$$

and substituting u^0 and ζ^0 in (4), one can verify that a pair of the constructed functions (u^0, ζ^0) is in fact a solution of the initial nonlinear problem (1) – (9) in this special case.

It is obvious that the condition (15) is satisfied. Besides, it is clear that the constructed functions u^0, ζ^0 satisfy all the formulated smoothness conditions. Thus the set of a pairs of the functions (u^0, ζ^0) which figure in Theorem is indeed nonempty.

We note that the constructed solution is a solution in the usual sense although Theorem guarantees existence of a more general solution only (more precisely, a weak solution for the function u^0). It should also be added that the important question of the parameters ($v_0(t), \omega(t)$, $g(t)$, and so on) for which the conditions of Theorem are realized remains to be solved.

5. Conclusion

The general nonlinear initial boundary-value problem of the potential absolute motion of an inviscid incompressible fluid partially filling a Lipschitzian cavity of a certain moving vessel has a solution (u, ζ) for which the potential $u(r, t)$ is a function of both the Sobolev space with respect to r and the space of continuous functions with respect to t and the surface $\zeta(x, y, t)$ is a function of both the Lipschitzian class with respect to x, y and the class of continuously differentiable functions with respect to t.

The constructed example of an exact solution may be used for testing and graduating the corresponding computing methods.

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