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# ON λ-STABILITY OF ONE ESSENTIALLY NONLINEAR NONAUTONOMOUS SECOND ORDER EQUATION

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*There are obtained sufficient conditions for $\lambda$-stability of the trivial solution of a some essentially nonlinear differential second order equation.*

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## 1. Setting of the Problem

The subject of investigation in the theory of wave propagation [1] is the behavior of solutions of the second order differential equation (d.e.) with respect to degrees of the coefficient at the first degree of the unknown function in the theory of a diffusion of waves. Therefore in what follows we investigate asymptotic local and global $\lambda$-stability of the d.e. of the type

$$y'' + \lambda(t)y = F(t, y, y'), \tag{1}$$

where $t \in \mathbf{\Delta}$, $\mathbf{\Delta} \equiv [t_0, \omega[$ or $\mathbf{\Delta} \equiv ]\omega, t_0]$, $|\omega| \leq +\infty$, $\lambda : \mathbf{\Delta} \mapsto \mathbf{R}_+$, $F : \mathbf{\Delta} \times \mathbf{R^2} \mapsto \mathbf{R}$, $\mathbf{R} \equiv ]-\infty, +\infty[$, $\mathbf{R}_- \equiv ]-\infty, 0[$, $\mathbf{R}_+ \equiv ]0, +\infty[$, $\mathbf{R^2} \equiv \mathbf{R} \times \mathbf{R}$, $\lambda \in \mathbf{C^1_\Delta}$, $F \equiv \sum\limits_{s+k=2}^{m} f_{s,k} y^s (y')^k + F^*$, $f_{sk} \in \mathbf{C^1_\Delta}$, $s + k = \overline{2, m}$, $m \in \{3, 4, \ldots\}$, $|F^*| \leq L\left(|y| + |y'|\right)^{m+\alpha}$, $L \in \mathbf{C_\Delta}$, $L : \mathbf{\Delta} \mapsto \mathbf{R}_+$, $\alpha \in \mathbf{R}_+$. The results given below are applied to the d.e. (1) with slowly variable coefficients (their derivates are smalls with respect to the coefficients themselves as $t \uparrow \omega$). Let us denote

$$S(t, \lambda, a, b) \equiv \frac{|y(t; t_0, y_0, y_0')|}{\lambda^a} + \frac{|y'(t; t_0, y_0, y_0')|}{\lambda^b},$$

where $y = y(t; y_0, y_0')$ is a solution of the d.e. (1), $a, b \in \mathbf{R}$;

$$B_{sk}(t; a, b) \equiv 0,5 \sum_{p=0}^{k+p-q} \sum_{q=0}^{s-p+q} (-1)^p i^{k+p-q} \lambda^{(a+b)(s+k-1) - \frac{1}{2}(k+p-q)} C^p_{k+p-q} C^q_{s-p+q} f_{k+p-q, s-p+q},$$

$$i^2 = -1;$$

$$\Lambda \equiv \text{maxi}\, \{f_s : \mathbf{\Delta} \mapsto \mathbf{R};\ s = \overline{1, n}\},$$

if $\Lambda : \mathbf{\Delta} \mapsto \mathbf{R}_+$, $\Lambda^{-1} f_s = c_s + o_s(1)$, $t \uparrow \omega$, $c_s \in \mathbf{R}$, $s = \overline{1, n}$, $|c_1| + \ldots + |c_n| > 0$;

$$L^*(t; a, b) \equiv 2^{m+\alpha-1} \lambda^{(a+b-\frac{1}{4})(m+\alpha)} (\lambda^{-\frac{1}{4}} + \lambda^{\frac{1}{4}})^{m+\alpha} L;$$

$$G(t,a,b,x) \equiv \sum_{\substack{s+k=2\\ s\neq k+1}}^{m} \Lambda^{-1}(t;a,b)|B_{s,k}(t;a,b)|\phi^{s+k-1}(t;a,b)x^{s+k}$$
$$+\sum_{s=1}^{m_0}\Big|\Lambda^{-1}(t;a,b)\operatorname{Re}B_{s+1,s}(t;a,b)\phi^{2s}(t;a,b)-B_{2s+1}(a,b)\Big|x^{2s+1}$$
$$+\phi^{m+\alpha-1}(t;a,b)\Lambda^{-1}(t;a,b)L^*(t;a,b)x^{m+\alpha};\;\; m_0=\frac{m}{2}-\frac{3+(-1)^m}{4}.$$

**Definition 1.** *The d.e. (1) has the property $St_\lambda$ as $t\uparrow\omega$ if there are constants $a,\, b\in\mathbf{R}$ such that for any arbitrarily small $\varepsilon\in\mathbf{R}_+$ there are $T_\varepsilon\in\mathbf{\Delta}$, $\delta_\varepsilon\in]0,\varepsilon[$, such that any solution $y=y(t;T_\varepsilon,y_0,y_0')$ with an initial condition $S(T_\varepsilon,\lambda,a,b)\le\delta_\varepsilon$ satisfies the inequality $S(t,\lambda,a,b)<\varepsilon$ for all $t\in[T_\varepsilon,\omega[$ (the local $\lambda$-stability).*

**Definition 2.** *The d.e. (1) has the property $AsSt_\lambda$ as $t\uparrow\omega$ if Definition 1 takes place and $S(T_\varepsilon,\lambda,a,b)=o(1)$, $t\uparrow\omega$ (the local asymptotic $\lambda$-stability).*

**Definition 3.** *The d.e. (1) has the property $G_{\mathbf{\Delta}}St_\lambda$ as $t\uparrow\omega$ if there are constants $a,\, b\in\mathbf{R}$ such that for any arbitrarily small $\varepsilon\in\mathbf{R}_+$ there is $\delta_\varepsilon\in]0,\varepsilon[$ such that any solution $y=y(t;t_0,y_0,y_0')$ of the d.e. with an initial condition $S(t_0,\lambda,a,b)\le\delta_\varepsilon$ satifies the inequality $S(t,\lambda,a,b)<\varepsilon$ for all $t\in\mathbf{\Delta}$ (the global $\lambda$-stability).*

**Definition 4.** *The d.e. (1) has the property $G_{\mathbf{\Delta}}AsSt_\lambda$ as $t\uparrow\omega$ if Definition 3 takes place and $S(t,\lambda,a,b)=o(1)$, $t\uparrow\omega$ (the global asymptotic $\lambda$-stability).*

**Definition 5.** *The d.e. (1) has the property $UnSt_\lambda$ as $t\uparrow\omega$, if the Definition 1 doesn't take place.*

## 2. Auxiliary Results

Let us give tranfotmations that help to obtain an estimate of the value $S(t,\lambda,a,b)$.

**Lemma 1.** *The transformation*

$$y=-i\lambda^{a+b-0,5}(x-\overline{x}),\;\; y'=\lambda^{a+b}(x+\overline{x}), \tag{2}$$

*reduces the d.e. (1) to the d.e. of the type*

$$x'=\Phi(t,x), \tag{3}$$

*where*

$$\Phi(t,x)\equiv\Big[i\lambda^{0,5}-(a+b-0,25)\lambda'\lambda^{-1}\Big]x-0,25\lambda'\lambda^{-1}\overline{x}+\sum_{s+k=2}^{m}B_{s,k}(t;a,b)x^s\overline{x}^k+\Phi^*(t,x),$$

$$\Phi^*(t,x)\equiv\frac{1}{2}\lambda^{-(a+b)}F^*[t,-i\lambda^{a+b-0,5}(x-\overline{x}),\lambda^{a+b}(x+\overline{x})],\;\; |\Phi^*|\le L^*(t;a,b)|x|^{m+\alpha}.$$

***Proof*** is obvious.

**Lemma 2.** *The substitution*

$$x = \rho\exp(i\theta),\ \rho = |x|,\ \theta \in \mathbf{R}, \tag{4}$$

*reduces the d.e. (3) to the differential system (d.s.) of the type*

$$\begin{aligned}\rho' &= -(a+b-0,5\cos^2\theta)\lambda'\lambda^{-1}\rho + \sum_{s+k=2}^{m} \operatorname{Re}\Big[B_{s,k}(t;a,b)\exp(i\theta(s-k-1))\Big]\rho^{s+k} + \Phi_1,\\ \theta' &= \lambda^{0,25} + 0,25\lambda'\lambda^{-1}\sin 2\theta + \sum_{s+k=2}^{m} \operatorname{Im}\Big[B_{s,k}(t;a,b)\exp(i\theta(s-k-1))\Big]\rho^{s+k-1} + \rho^{-1}\Phi_2,\end{aligned} \tag{5}$$

*where* $\Phi_1 \equiv \operatorname{Re}\Big[\Phi^*\exp(-i\theta)\Big]$, $\Phi_2 \equiv \operatorname{Im}\Big[\Phi^*\exp(-i\theta)\Big]$, $|\Phi_s| \le L^*(t;a,b)\rho^{m+\alpha}$, $s = 1,2$.

***Proof*** is obvious.

***Remark 1.*** The following estimate

$$\lambda^{-0,25}\min\{\lambda^{a+0,25},\lambda^{b-0,25}\}\rho \le S(t,\lambda,a,b) \le 2\lambda^{-0,25}\Big(\lambda^{a+0,25}+\lambda^{b-0,25}\Big)\rho \tag{6}$$

is implied by transformations (2), (4).

## 3. Main Results

It should be noted that the behavior of the value $S(t,\lambda,a,b)$ depends on solutions of the first d.e. of the d.s. (5), as implied by the estimate (6). To understand the behavior of these solutions we shall note that there is a group of terms which don't depend of $\theta$. This allows to obtain the from the differential system (5) a d.e. of the type

$$\rho' = \sum_{s=1}^{m_0} \operatorname{Re} B_{s+1,s}(t;a,b)\rho^{2s+1}, \tag{7}$$

the behavior of all solutions of which can by investigated by using the method [2].

**Theorem 1.** *Let for the d.e. (1) there are* $a, b \in \mathbf{R}$ *such that*

*1) among the functions* $\Big[-2s\int\limits_{t_0}^{t}\operatorname{Re} B_{s+1,s}(\tau;a,b)d\tau\Big]^{-\frac{1}{2s}}$,

$\Big[-\operatorname{Re} B_{s+1,s}(t;a,b)\operatorname{Re}^{-1} B_{k+1,k}(t;a,b)\Big]^{\frac{1}{2(k-s)}}$, $\quad s \ne k$, $s,k = \overline{1,m_0}$,

*there is* $\phi(t;a,b) : \mathbf{\Delta} \mapsto \mathbf{R}_+$ *such that there exists*

$$\Lambda(t;a,b) \equiv \operatorname{maxi}\{(a+b)\lambda'\lambda^{-1} + \phi'(t;a,b)\phi^{-1}(t;a,b) - 0,5|\lambda'|\lambda^{-1},$$
$$\operatorname{Re} B_{s+1,s}(t;a,b)\phi^{2s}(t;a,b);\ s = \overline{1,m_0}\},$$

*and*

$$[-(a+b)\lambda'\lambda^{-1} - \phi'(t;a,b)\phi^{-1}(t;a,b) + 0,5|\lambda'|\lambda^{-1}]x + \sum_{s=1}^{m_0} \operatorname{Re} B_{s+1,s}(t;a,b)\phi^{2s}(t;a,b)x^{2s+1}$$

$$\equiv \Lambda(t;a,b)\sum_{s=0}^{m_0}\Big[B_{2s+1}(a,b) + o_{2s+1}(1)\Big]x^{2s+1}, \ \ B_{2s+1}(a,b) \in \mathbf{R}, \ s = \overline{0,m_0};$$

*2) there is* $c_0 \in \mathbf{R}_+$ *such that* $\sum_{s=0}^{m_0} B_{2s+1}(a,b)c_0^{2s+1} \in \mathbf{R}_-$*, and*

$$\phi^{-1}(t,a,b)G[t,a,b,\phi(t,a,b)c_0] = o(1), \ t \uparrow \omega$$

$$\left(\sup_{t\in\mathbf{\Delta}} \phi^{-1}(t,a,b)G[t,a,b,\phi(t,a,b)c_0] \le -\sum_{s=0}^{m_0} B_{2s+1}(a,b)c_0^{2s+1}\right);$$

*3)* $\lambda^{-0,25}(\lambda^{a+0,25} + \lambda^{b-0,25})\phi(t;a,b) = o(1)$*,* $t \uparrow \omega$*.*
*Then it has the property* $AsSt_\lambda$ *(*$G_{\mathbf{\Delta}}AsSt_\lambda$*) as* $t \uparrow \omega$*.*

***Proof.*** For the first d.e. of the d.s. (5) we make the transformation

$$\rho = \phi x \tag{8}$$

which, under the condition 1), reduces it to a d.e. of the type

$$x' = \Lambda(t;a,b)\left\{\sum_{s=0}^{m_0} B_{2s+1}(a,b)x^{2s+1} + \phi^{-1}(t,a,b)G^*[t;a,b,\theta,\phi(t,a,b)x]\right\}, \tag{9}$$

where we have the estimate $|G^*[t;a,b,\theta,\phi(t,a,b)x]| \le G[t,a,b,\phi(t,a,b)x]$ for all $\theta \in \mathbf{R}$, $t \in \mathbf{\Delta}$. Then, under the condition 2), there is $T_0 \in \mathbf{\Delta}$ such that if $x = c_0$ for all $t \in [T_0,\omega]$, then we have the inequality $x' < 0$. It means that any solution $x = x(t;t_0^*,\theta_0,x_0)$ of the d.e. (9) with a sufficiently small initial value $x_0$ is bounded for all $\theta \in \mathbf{R}$ and $t \in [t_0^*,\omega[$, $t_0^* \in [T_0,\omega]$.

Using condition 3) and substitution (8) we see that smallness of the value $x$ corresponds to smallness of the value $S(t,\lambda,a,b)$.

***Remark 2.*** It follows from condition 2) of Theorem 1 that the coefficients $\operatorname{Re} B_{s,k}(t,a,b)$, $s \ne k+1$, $s,k = \overline{1,m}$, of the first d.e. of the d.s. (5) must be small with respect to the coefficients $\operatorname{Re} B_{s+1,s}(t,a,b)$, $s = \overline{1,s_0}$, of this d.e. If this condition doesn't hold, then the order of growth of these coefficients can be decreased if the method of nonlinear "frozen" transformations [3] is applied to the d.e. (4).

**Lemma 3.** *Transformations (2) and*

$$x = z + H(t,z,\overline{z}) \equiv z + \sum_{s+k}^{m} h_{sk}z^s\overline{z}^k, \ \ z = \rho\exp(i\theta), \tag{10}$$

*where $\rho$, $\theta$ are polar coordinates reduce the d.e. (1) to the d.s. of the type*

$$\begin{aligned} \rho' &= -(a+b-0,5\cos^2\theta)\lambda'\lambda^{-1}\rho + \sum_{s=1}^{m_0} A_{2s+1}(t,a,b)\rho^{2s+1} + \Phi_1^*(t,a,b,\theta,\rho), \\ \theta' &= \lambda^{0,5} + \sum_{s=0}^{m_0} B_{2s+1}^*(t,a,b)\rho^{2s} + 0,25\lambda'\lambda^{-1}\sin 2\theta + \Phi_2^*(t,a,b,\theta,\rho), \end{aligned} \tag{11}$$

*for which $A_{2s+1}(t,a,b)$, $B_{sk+1}^*(t,a,b)$ are known real functions of t only, $s = \overline{1,m_0}$, $\Phi_k^*(t,\theta,\rho)$, $k = 1,2$, are known real $2\pi$-periodic in $\theta$ functions, $\Phi_k^*(t,\theta,0) \equiv 0$, $k = 1,2$.*

***Proof.*** First let us apply the transformation (2) to the d.e. (1). And next, the transformation (10) is applied to the obtained d.e. (1), where $h_{rl}$, $r+l = \overline{2,m}$, are defined such that the d.e. in $z$ in the autonomous case will have the type $z' = i\lambda^{\frac{1}{2}} z$.

For the forms $H_k \equiv \sum\limits_{r+l=k} h_{rl} z^r \overline{z}^l$, $k = \overline{2,m}$, we have the d.e.

$$\begin{aligned} &-\frac{\partial H_k}{\partial t} + H_k - z\frac{\partial H_k}{\partial z} + \overline{z}\frac{\partial H_k}{\partial \overline{z}} \\ &= i\lambda_1^{-\frac{1}{2}}\Bigg[F_k(t,z,\overline{z}) + \sum_{s=2}^{k-1} F_s\Big(t, z + \sum_{j=2}^{m} H_j, \overline{z} + \sum_{j=2}^{m}\overline{G_j}\Big) \\ &\quad - \sum_{j=2}^{k-1}\Big(F_{k-j+1}\frac{\partial H_j}{\partial z} - \overline{F}_{k-j+1}\frac{\partial H_j}{\partial \overline{z}}\Big)\Bigg] \\ &\quad + \sum_{j=2}^{k-1}\Big(H_{k-j+1}\frac{\partial H_j}{\partial x} - \overline{H_{k-j+1}}\frac{\partial H_j}{\partial \overline{z}}\Big), \ k = \overline{2,m}, \end{aligned} \tag{12}$$

where $F_k(t,z,\overline{z}) \equiv \sum\limits_{r+l=k} B_{r,l} z^r \overline{z}^l$, $k = \overline{2,m}$.

We can't solve the d.e. (12) exactly. Therefore we use the method of "frozen" $t$ and replace this d.e. by an algebraic equation of the type

$$\begin{aligned} H_k - z\frac{\partial H_k}{\partial z} + \overline{z}\frac{\partial H_k}{\partial \overline{z}} &= i\lambda_1^{-\frac{1}{2}}\Bigg[F_k(t,z,\overline{z}) + \sum_{s=2}^{k-1} F_s\Big(t, z + \sum_{j=2}^{m} H_j, \overline{z} + \sum_{j=2}^{m}\overline{H_j}\Big) \\ &- \sum_{j=2}^{k-1}\Big(F_{k-j+1}\frac{\partial H_j}{\partial x} - \overline{F}_{k-j+1}\frac{\partial H_j}{\partial \overline{x}}\Big)\Bigg] + \sum_{j=2}^{k-1}\Big(H_{k-j+1}\frac{\partial H_j}{\partial x} - \overline{H_{k-j+1}}\frac{\partial H_j}{\partial \overline{x}}\Big) \\ &+ z\sum_{s=2}^{k-1}\Big(\frac{\partial H_s}{\partial x}\frac{\partial H_{k-s+2}}{\partial \overline{z}} - \frac{\partial H_s}{\partial \overline{z}}\frac{\partial \overline{H}_{k-s+2}}{\partial z}\Big), \ k = \overline{2,m}. \end{aligned} \tag{13}$$

If in the right-hand side of the d.e. (13) we shall denote by $A_{rl}^{**}$ the coefficient at $z^r\overline{z}^l$, then for $h_{rl}$, $r+l = k$, $k = \overline{2,m}$, we shall obtain the d.e. of the type

$$(1-r+l)h_{rl} = A_{rl}^{**}, \ r+l = k, \ k = \overline{2,m}. \tag{14}$$

It's easy to understand that the d.e. (14) does not have a solution only if $r = l + 1$. In this case we define $h_{l+1,l} \equiv 0$.

As the result of applying transformations (2), (10) to the d.e. (1), the d.s. (10) will have a polynomial in $\rho$ the coefficients of which depend only on $t$. This allows to select the d.e. of the type

$$\rho' = \sum_{s=1}^{m_0} A_{2s+1}(t,a,b)\rho^{2s+1}, \tag{15}$$

the behavior of regular solutions of which can determine $\lambda$-stability of the d.e. (1).

Since the substitution (10) has $\theta$ under the signs of $\sin$ and $\cos$, to get conditions for properties $St_\lambda$, $AsSt_\lambda$ to hold for the d.e. (1), it is sufficient to investigate the properties of solutions of the first d.e. (11) relatively to $\rho$ for any variation of $\theta \in \mathbf{R}$.

**Theorem 2.** *Let for the d.e. (1) there are $a, b \in \mathbf{R}$ such that*
*1) for the d.s. (11) there exists a solution of the Cauchy problem;*
*2) among the functions*

$$\Big[-2s\int_{t_0}^{t} A_{2s+1}(\tau,a,b)d\tau\Big]^{-\frac{1}{2s}},\ \big[-A_{2s+1}^{-1}(t,a,b)A_{2k+1}(t,a,b)\big]^{\frac{1}{2(s-r)}},\ s,k = \overline{1,m_0},\ s \neq k,$$

*there exists $\phi(t,a,b) : \mathbf{\Lambda} \mapsto \mathbf{R}_+$ such that there is*

$$\Lambda(t;a,b) = \text{maxi}\{(a+b)\lambda'\lambda^{-1} + \phi'(t;a,b)\phi^{-1}(t;a,b) - 0,5|\lambda'|\lambda^{-1},$$
$$A_{2s+1}(t;a,b)\phi^{2s}(t;a,b);\ s = \overline{1,m_0}\}$$

*and*

$$-(a+b)\lambda'\lambda^{-1} - \phi'(t;a,b)\phi^{-1}(t;a,b) + 0,5|\lambda'|\lambda^{-1}]z + \sum_{s=1}^{m_0} A_{2s+1}(t;a,b)\phi^{2s}(t;a,b)z^{2s+1}$$
$$\equiv \Lambda(t;a,b)\sum_{s=0}^{m_0}\Big[A_{2s+1}^*(a,b) + o_{2s+1}(1)\Big]x^{2s+1},\ \ A_{2s+1}^*(a,b) \in \mathbf{R},\ s = \overline{0,m_0};$$

*3) there exists a constant $c_0 \in \mathbf{R}_+$ such that $\sum_{s=0}^{m_0} A_{2s+1}^*(a,b)c_0^{2s+1} \in \mathbf{R}_-$ and*

$$\Lambda^{-1}(t,a,b)\phi^{-1}(t,a,b)\Phi_1^*[t,a,b,\theta,\phi(t,a,b)c_0)] = o(1),\ t \uparrow \omega,\ \textit{for all}\ \theta \in \mathbf{R}$$

$$\Bigg(\sup_{t\in\mathbf{\Delta},\theta\in\mathbf{R}}\Big|\Lambda^{-1}(t,a,b)\Phi_1^{**}(t,a,b,\theta) - \sum_{s=0}^{m_0} A_{2s+1}^* c_0^{2s+1}\Big| < -\sum_{s=0}^{m_0} A_{2s+1}^* c_0^{2s+1},$$

$$\Phi_1^{**}(t,a,b,\theta) \equiv -[(a+b-0,5\cos^2\theta)\lambda'\lambda^{-1}+\phi'(t,a,b)\phi^{-1}(t,a,b)]c_0 + \sum_{s=1}^{m_0} A_{2s+1}(t,a,b)\phi^{2s}(t,a,b)c_0^{2s+1} + \phi^{-1}(t,a,b)\Phi_1^*[t,a,b,\theta,\phi(t,a,b)c_0]\Big);$$

*4)* $\lambda^{-0,25}(\lambda^{a+0,25}+\lambda^{b-0,25})\phi(t;a,b) = o(1)$, $h_{sk}(t,a,b)\phi^{s+k-1}(t,a,b) = O(1)$, $t\uparrow\omega$, $s+k=\overline{2,m}$.

*Then it has the property* $AsSt_\lambda$ *(*$G_{\mathbf{\Delta}}AsSt_\lambda$*) as* $t\uparrow\omega$.

***Proof.*** For the first d.e. of the d.e. (11) we apply the substitution $z=\phi(t,a,b)v$. And to the obtained d.e. for $v$ we apply Theorem 1.

For the sake of clarity of the obtained results, let us consider the d.e. (1) in a special case. Let, for the d.e. (1),

$$\omega=+\infty,\ \ m=3,\ \ \int^{+\infty}\lambda^{0,5}dt=+\infty, \tag{16}$$

and there are $a,b\in\mathbf{R}$ such that there exist finite limits

$$\lim_{t\to+\infty}\lambda^{(k-1)(a+b)-0,5(s+1)}f_{s,k-s},\ \ 0\le s\le k,\ \ k=2,3. \tag{17}$$

Then the substitution (2) transforms it into d.e. (3) for which

$$B_{20}(t,a,b)\equiv\overline{B}_{02}(t,a,b)\equiv 0,5\lambda^{a+b}\left(\lambda^{-1}f_{20}-i\lambda^{-0,5}f_{11}+f_{02}\right),$$

$$B_{11}(t,a,b)\equiv-\lambda^{a+b}\left(\lambda^{-1}f_{20}-f_{02}\right),$$

$$B_{30}(t,a,b)\equiv\overline{B}_{03}(t,a,b)\equiv 0,5\lambda^{2(a+b)}\left(i\lambda^{-1,5}f_{30}-\lambda^{-1}f_{21}-i\lambda^{-0,5}f_{12}+f_{03}\right),$$

$$B_{21}(t,a,b)\equiv\overline{B}_{12}(t,a,b)\equiv-0,5\lambda^{2(a+b)}\left(3i\lambda^{-1,5}f_{30}-\lambda^{-1}f_{21}+i\lambda^{-0,5}f_{12}-3f_{03}\right).$$

Let us apply the transformation

$$x=z+H,\ H\equiv H_2+H_3,\ \ H_k\equiv\sum_{s=0}^{k}h_{k-s,s}(t,a,b)z^{k-s}\overline{z}^s,\ \ k=2,3, \tag{18}$$

to d.e. (3) where the forms $H_2$, $H_3$ are defined so that, in the autonomous case, the d.e. for $z$ has the form $z'=i\lambda^{0,5}z$. Under this condition, the forms $H_2$, $H_3$ satisfy a d.e. of the form

$$-\frac{\partial H_2}{\partial t}+i\lambda^{\frac12}\left(H_2-z\frac{\partial H_2}{\partial z}+\overline{z}\frac{\partial H_2}{\partial\overline{z}}\right)+\sum_{s=0}^{2}B_{2-s,s}z^{2-s}\overline{x}^s=0,$$

$$-\frac{\partial H_3}{\partial t} + i\lambda^{0,5}\left(H_3 - z\frac{\partial H_3}{\partial z} + \overline{z}\frac{\partial H_3}{\partial \overline{z}}\right) + \sum_{s=0}^{3} B_{3-s,s} z^{3-s}\overline{z}^{s}$$

$$+ [\overline{B}_{02}h_{11} + B_{11}\overline{h}_{02} - i\lambda^{0,5}(2h_{20}^2 - h_{11}\overline{h}_{02})]z^3$$

$$+ [B_{20}h_{11} + B_{11}\overline{h}_{11} - B_{11}h_{20} + 2B_{02}\overline{h}_{02} + \overline{B}_{11}h_{11} + 2\overline{B}_{02}h_{02}$$

$$- i\lambda^{0,5}(2h_{11}h_{02} + h_{20}h_{11} - |h_{11}|^2) - 2|h_{02}|^2]z^2\overline{z}$$

$$+ [2B_{20}h_{02} + B_{11}\overline{h}_{20} + 2B_{02}\overline{h}_{11} - 2B_{02}h_{20} + \overline{B}_{20}h_{11} + 2\overline{B}_{11}h_{02}$$

$$- i\lambda^{0,5}(h_{11}^2 + 2h_{20}h_{02} - \overline{h}_{20}h_{11} - 2\overline{h}_{11}h_{02})]z\overline{z}^2$$

$$+ [B_{11}h_{02} + 2B_{02}\overline{h}_{20} - B_{02}h_{11} + 2\overline{B}_{20}h_{02} - i\lambda^{0,5}(h_{11}h_{02} - 2\overline{h}_{20}h_{02})]\overline{z}^3 = 0.$$

Applying the method of the “frozen” $t$ to the obtained d.e. we find the coefficients $h_{rl}$, $r + l = 2, 3$, i.e.,

$$h_{20}(t,a,b) \equiv -0,5i\lambda^{a+b-0,5}\left(\lambda^{-1}f_{20} - i\lambda^{-0,5}f_{11} + f_{02}\right),$$

$$h_{11}(t,a,b) \equiv -\lambda^{a+b-0,5}\left(\lambda^{-1}f_{20} - f_{02}\right), \quad h_{02}(t,a,b) \equiv \frac{1}{3}\overline{h}_{20}(t,a,b),$$

$$h_{30}(t,a,b) \equiv -0,25i\lambda^{2(a+b)-0,5}\left(i\lambda^{-1,5}f_{30} - \lambda^{-1}f_{21} - i\lambda^{-0,5}f_{12} + f_{03}\right)$$

$$- \frac{1}{12}\lambda^{2(a+b)-1}\left(\lambda^{-1}f_{20} - i\lambda^{-0,25}f_{11} + f_{02}\right)\left(4\lambda^{-1}f_{20} - 3i\lambda^{-0,5}f_{11} + 2f_{02}\right),$$

$$h_{21}(t,a,b) \equiv 0, \tag{19}$$

$$h_{12}(t,a,b) = -0,25i\lambda^{2(a+b)-0,5}\left(3i\lambda^{-1,5}f_{30} - \lambda^{-1}f_{21} + i\lambda^{-0,5}f_{12} - 3f_{30}\right)$$

$$- \frac{1}{6}\lambda^{2(a+b)-1}\left(\lambda^{-1}f_{20} + i\lambda^{-0,5}f_{11} + f_{02}\right)\left(8\lambda^{-1}f_{20} + 5i\lambda^{-0,5}f_{11} - 12f_{20}\right),$$

$$h_{03}(t,a,b) = 0,25i\lambda^{2(a+b)-0,5}\left(i\lambda^{-1,5}f_{03} + \lambda^{-1}f_{21} - i\lambda^{-0,5}f_{12} - f_{03}\right)$$

$$- \frac{1}{12}\lambda^{2(a+b)-1}\left(\lambda^{-1}f_{20} + i\lambda^{-0,5}f_{11} + f_{02}\right)^2.$$

It is easy to see that, under the condition (17), the coefficients of the transformation (18) have the property: there exist finite limits $\lim\limits_{t\to+\infty} h_{sk} = h_{sk}^*$, $s + k = 2, 3$, where $h_{sk} - h_{sk}^* \equiv \omega_{sk} = o(1)$, $t \to +\infty$, $s + k = 2, 3$.

It means that there exists $z_0 \in \mathbf{R}_+$ such that

$$\sup_{t\in\boldsymbol{\Delta}, |z|\le z_0}\left|\frac{\partial H}{\partial z} + \frac{\partial \overline{H}}{\partial \overline{z}} + \frac{\partial H}{\partial z}\frac{\partial \overline{H}}{\partial \overline{z}} - \frac{\partial H}{\partial \overline{z}}\frac{\partial \overline{H}}{\partial z}\right| < 1,$$

i.e., the transformation (18) exists and it is nonsingular.

Then the d.e. (15) has the form

$$\rho' = \lambda^{0,5}A_3(t,a,b)\rho^3,$$

where

$$A_3(t,a,b) \equiv 0,5\lambda^{2(a+b)-0,5}(\lambda^{-1}f_{21}+3f_{03}) - \frac{1}{6}\lambda^{2(a+b)-1,5}f_{11}(\lambda^{-1}f_{20}-f_{02}),$$

moreover under the condition (17) there exists the finite limit $\lim_{t\to+\infty} A_3^* = A_3^0$. Therefore under the condition $A_3^0 < 0$, the function $\phi(t,a,b)$ from Theorem 2 has the form

$$\phi(t,a,b) \equiv \psi(t) \equiv \left(1 - 2A_3^0 \int_{t_0}^{t} \lambda^{0,5}d\tau\right)^{-0,5}.$$

From the condition (16) it follows that $\phi(t) = o(1),\ t \to +\infty$.

The function $\Phi_1^*(t,a,b,\theta,\rho)$ from d.s. (11) has the form

$$\Phi_1^*(t,a,b,\theta,\rho) \equiv \operatorname{Re}\left[\exp(i\theta)\Psi(t,a,b,\theta,\rho)\right] - A_3^0\lambda^{0,5}\rho^3 + \Big(a+b-\frac{1}{2}\cos^2\theta\Big)\lambda'\lambda^{-1}\rho,$$

where

$$\Psi(t,a,b,\theta,\rho) \equiv \left(1+\frac{\partial H}{\partial z}+\frac{\partial \overline{H}}{\partial \overline{z}}+\frac{\partial H}{\partial z}\frac{\partial \overline{H}}{\partial \overline{z}}-\frac{\partial H}{\partial \overline{z}}\frac{\partial \overline{H}}{\partial z}\right)^{-1}\Bigg|_{z=\rho\exp(i\theta)}$$
$$\times\left\{\left[\Psi^*(t,a,b,z+H)-\frac{\partial H}{\partial t}\right]\left(1+\frac{\partial \overline{H}}{\partial \overline{z}}\right) - \left[\overline{\Psi^*}(t,a,b,z+H)-\frac{\partial \overline{H}}{\partial t}\right]\frac{\partial H}{\partial \overline{z}}\right\}\Bigg|_{z=\rho\exp(i\theta)},$$

$$\Psi^*(t,a,b,x) \equiv [i\lambda^{0,25} - (a+b-0,25)\lambda'\lambda^{-1}]x - 0,25\lambda'\lambda^{-1}\overline{x} + \sum_{s+k=2}^{3} B_{sk}(t;a,b)x^s\overline{x}^k.$$

Let us assume that

$$\Lambda(t,a,b) \equiv \Lambda(t) = -A_3^0\lambda^{0,5}\left(2A_3^0\int_{t_0}^{t}\lambda^{0,5}d\tau - 1\right)^{-1}$$
$$\equiv \max\Bigg\{\Big[(a+b)\lambda' - 0,5|\lambda'|\Big]\lambda^{-1} - A_3^0\lambda^{0,5}\Big(1-2A_3^0\int_{t_0}^{t}\lambda^{0,5}d\tau\Big)^{-1},$$
$$A_3^0\lambda^{0,5}\Big(1-2A_3^0\int_{t_0}^{t}\lambda^{0,5}d\tau\Big)^{-1}\Bigg\}.$$

Then under the condition of existence of the finite limit

$$\lim_{t\to+\infty}\lambda'\lambda^{-1,5}\int_{t_0}^{t}\lambda^{0,5}d\tau = B_0 \geq 0,$$

making the transformation $\rho = \phi(t)x$ we obtain a differential inequality for $x$ in the form

$$x' \leq \frac{A_3^0}{2A_3^0 \int\limits_{t_0}^{t} \lambda^{0,5} d\tau - 1} \left\{ \left[ -(a+b)\frac{B_0}{A_3^0} + 0,5\frac{|B_0|}{A_3^0} + 1 \right] x - x^3 + \Psi^{**}(t,a,b,x) \right\}. \tag{20}$$

Note that if, for example, $\lambda = Bt^{\sigma}$, $B = \text{const} > 0$, then $B_0 = \sqrt{B} > 0$. If, for example, $\lambda = B \ln t$, then $B_0 = 0$.

The positive constant $c_0$ can be found from the inequality

$$c_0^2 > 1 - (a+b)\frac{B^0}{A_3^0} + 0,5\frac{|B_0|}{A_3^0}.$$

For condition 4) of Theorem 2 to hold, we must require that $\Psi^{**}(t,a,b,c_0) = o(1), t \to +\infty$ in (20). The latter condition is fullfilled if

$$(\omega'_{2-s,s})^2 \lambda^{-1} \int\limits_{t_0}^{t} \lambda^{0,5} d\tau = o(1), \;\; \omega'_{3-k,k}\lambda^{-0,5} = o(1), \;\; t \to +\infty, \;\; s = 0,1,2, \;\; k = 0,2,3,$$

$$\lambda^{(a+b-0,5-1)(m+\alpha)-1}(\lambda^{-0,5} + \lambda^{0,5})^{m+\alpha} \left( \int\limits_{t_0}^{t} \lambda^{0,5} d\tau \right)^{-0,5\alpha} L = o(1), \quad t \to +\infty.$$

If we consider the behavior of the so-called tail, i.e., the terms powers of which in $x$ equal $4, \ldots, 9$ and which are contained in the function $\Psi^{**}$, then their coefficients with the factor $\Lambda^{-1}(t)$ disappear for $t \to +\infty$ and $x$ fixed. For example, at $x^4$ we have

$$\begin{aligned} &B_{20}\phi^3(t)\Lambda^{-1} \\ &\quad = 0,5\lambda^{a+b-0,5}(\lambda^{-1}f_{20} - i\lambda^{-0,5}f_{11} + f_{02})\lambda^{0,5}\left(1 - 2A_3^0 \int\limits_{t_0}^{t} \lambda^{0,5} d\tau\right)^{-1,5} \\ &\qquad \times \left(1 - 2A_3^0 \int\limits_{t_0}^{t} \lambda^{0,5} d\tau\right)(A_3^0)^{-1}\lambda^{-0,5} \\ &\quad = i(A_3^0)^{-1}h_{20}\left(1 - 2A_3^0 \int\limits_{t_0}^{t} \lambda^{0,5} d\tau\right)^{-0,5} = o(1), \;\; t \to +\infty, \end{aligned}$$

$$B_{30}\phi^3(t)\Lambda^{-1}$$

$$= 0,5\lambda^{2(a+b)-0,5}(i\lambda^{-1,5}f_{30} - \lambda^{-1}f_{21} - i\lambda^{-0,5}f_{12} + f_{03})\lambda^{0,5}\Big(1 - 2A_3^0\int_{t_0}^{t}\lambda^{0,5}d\tau\Big)^{-1,5} \times \left(1 - 2A_3^0\int_{t_0}^{t}\lambda^{0,5}d\tau\right)(A_3^0)^{-1}\lambda^{-0,5}$$

$$= 2i(A_3^0)^{-1}h_{30}\left(1 - 2A_3^0\int_{t_0}^{t}\lambda^{0,5}d\tau\right)^{-0,5} = o(1), \;\; t \to +\infty,$$

$$(\omega'_{2-s,s})^2\lambda^{-1}\int_{t_0}^{t}\lambda^{0,5}d\tau = o(1), \;\; \omega'_{3-k,k}\lambda^{-0,5} = o(1), \;\; t \to +\infty, \;\; s = 0,1,2, \;\; k = 0,2,3,$$

$$\lambda^{(a+b-0,5)(m+\alpha)}\lambda^{0,5}L\left(\int_{t_0}^{t}\lambda^{0,5}d\tau\right)^{-0,5\alpha} = o(1), \;\; t \to +\infty.$$

We formulate the obtained results as the following theorem.

**Theorem 3.** *Let the d.e. (1) be such that*

*1)* $\omega = +\infty$, $m = 3$, $\int^{+\infty}\lambda^{0,5}dt = +\infty$, *the limit* $\lim_{t\to+\infty}\lambda'\lambda^{-1,5}\int_{t_0}^{t}\lambda^{0,5}d\tau < +\infty$ *exists, and it's possiple to find* $a, b \in \mathbf{R}$ *such that there exist finite limits*

$$\lim_{t\to+\infty}\lambda^{(k-1)(a+b)-0,5(s+1)}f_{s,k-s}, \;\; 0 \le s \le k, \;\; k = 2,3,$$

*and*

$$\lim_{t\to+\infty}[\lambda^{2(a+b)-0,5}(\lambda^{-1}f_{21} + 3f_{03}) - \frac{1}{6}\lambda^{2(a+b)-1,5}f_{11}(\lambda^{-1}f_{20} - f_{02})] \in \mathbf{R}_-,$$

$$(h'_{2-s,s})^2\lambda^{-1}\int_{t_0}^{t}\lambda^{0,5}d\tau = o(1), \;\; h'_{3-k,k}\lambda^{-0,5} = o(1), \;\; t \to +\infty, \;\; s = 0,1,2, \;\; k = 0,2,3$$

*(*$h_{sk}$, $s + k = 2,3$, *are defined by formulas (19)),*

$$\lambda^{(a+b-0,5)(m+\alpha-1)-1}(\lambda^{-0,5} + \lambda^{0,5})^{m+\alpha}\left(\int_{t_0}^{t}\lambda^{0,5}d\tau\right)^{-0,5\alpha} L = o(1), \;\; t \to +\infty;$$

*2)* $\lambda^{-0,5}(\lambda^{a+0,5} + \lambda^{b-0,5})\left(\int_{t_0}^{t}\lambda^{0,5}d\tau\right)^{-0,5} = o(1), \;\; t \to +\infty.$

*Then it has the property* $AsSt_\lambda$ *as* $t \to +\infty$.

**Theorem 4.** *Let for the d.e. (1) there exist $a, b \in \mathbf{R}$ such that*
*1) for the d.s. (11) there is a solution of the Cauchy problem;*

*2) among the functions*

$$\Big[-2s\int\limits_{t_0}^{t} A_{2s+1}(\tau,a,b)d\tau\Big]^{-\frac{1}{2s}},\ \big[-A_{2s+1}^{-1}(t,a,b)A_{2k+1}(t,a,b)\big]^{\frac{1}{2(s-r)}},\ s,k=\overline{1,m_0},\ s\neq k,$$

*there is $\phi(t,a,b):\mathbf{\Lambda}\mapsto\mathbf{R}_+$ such that there exists*

$$\Lambda(t;a,b)=\mathrm{maxi}\Big\{(a+b)\lambda'\lambda^{-1}+\phi'(t;a,b)\phi^{-1}(t;a,b)+0,5|\lambda'|\lambda^{-1},$$
$$A_{2s+1}(t;a,b)\phi^{2s}(t;a,b);\ s=\overline{1,m_0}\Big\}$$

*and*

$$-[(a+b)\lambda'\lambda^{-1}+\phi'(t;a,b)\phi^{-1}(t;a,b)+0,5|\lambda'|\lambda^{-1}\Big]z$$
$$+\sum_{s=1}^{m_0}A_{2s+1}(t;a,b)\phi^{2s}(t;a,b)z^{2s+1}$$
$$\equiv\Lambda(t;a,b)\sum_{s=0}^{m_0}\Big[A_{2s+1}^*(a,b)+o_{2s+1}(1)\Big]x^{2s+1},$$
$$\int^{\omega}\Lambda dt=+\infty,\quad A_{2s+1}^*(a,b)\in\mathbf{R},\ s=\overline{0,m_0};$$

*3) there exists a constant $c_0\in]0,1]$ such that for all $t\in\mathbf{\Delta}$, $\theta\in\mathbf{R}$, $z\in]0,c_0]$,*

$$\sum_{s=0}^{m_0}A_{2s+1}^*(a,b)z^{2s+1}>0,$$
$$\left|\Lambda^{-1}(t,a,b)\Phi_1^{**}[t,a,b,\theta,z)]-\sum_{s=0}^{m_0}A_{2s+1}^*(a,b)z^{2s+1}\right|<\sum_{s=0}^{m_0}A_{2s+1}^*(a,b)z^{2s+1},$$

$$\Phi_1^{**}(t,a,b,\theta,z)\equiv-[(a+b)\lambda'\lambda^{-1}+\phi'(t,a,b)\phi^{-1}(t,a,b)+0,5|\lambda'|\lambda^{-1}]z$$
$$+\sum_{s=1}^{m_0}A_{2s+1}(t,a,b)\phi^{2s}(t,a,b)z^{2s+1}$$
$$+\phi^{-1}(t,a,b)\Phi_1^*[t,a,b,\theta,\phi(t,a,b)z];$$

*4) for any* $\alpha_0 \in ]0, c_0]$ *there is a constant* $l(\alpha_0) \in \mathbf{R}_+$ *such that*

$$\inf_{t\in\mathbf{\Delta},\ \theta\in\mathbf{R},\ z\in[\alpha_0,c_0]} \left| z\exp(i\theta) + \sum_{s+k=2}^{m} h_{sk}(t,a,b)\phi^{s+k-1}(t,a,b)\exp[i\theta(s-k)]z^{s+k} \right| \geq l(\alpha_0),$$

$$\sup_{t\in\mathbf{\Delta},\ \theta\in\mathbf{R},\ z\in[\alpha_0,c_0]} \Lambda^{-1}(t,a,b)|\Phi_1^{**}[t,a,b,\theta,z]| < \inf_{z\in[\alpha_0,c_0]} \sum_{s=0}^{m_0} A_{2s+1}(a,b)z^{2s+1};$$

*5)* $\lambda^{0,25}(\lambda^{a+0,25} + \lambda^{b-0,25})^{-1}\phi^{-1}(t;a,b) = o(1)$.
*Then it has the property* $UnSt_\lambda$ *as* $t \uparrow \omega$.

***Proof.*** Let us use the left-hand side of inequality (6). In the first d.e. of d.s. (11) we use the substitution $z = \phi(t,a,b)v$. Then we apply Chetaev's theorem [4, p. 199] to the obtained d.e. in $v$.

To clarity the obtained results we sive, for example, the following theorem.

**Theorem 5.** *Let for the d.e. (1) the condition 1) of Theorem 3 be fulfilled and*

$$\lambda^{0,5}(\lambda^{a+0,5} + \lambda^{b-0,5})^{-1}\left(\int_{t_0}^{t} \lambda^{0,5} d\tau\right)^{0,5} = o(1), \quad t \to +\infty.$$

*Then it has the property* $UnSt_\lambda$ *as* $t \to +\infty$.

***Proof.*** is reduced to a check of conditions of Theorem 4.

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