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DECOMPOSITION OF LINEAR SINGULARLY PERTURBED FUNCTIONAL DIFFERENTIAL EQUATIONS

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We study a system of linear singularly perturbed functional differential equations by using the method of integral manifolds. We construct a change of variables that decomposes this system into two subsystems: ordinary differential equation on the center manifold and integral equations on the stable manifold.

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1. Let R^n be the n-dimensional Euclidean space, $C = C[-\Delta, 0]$ be the space of continuous function $x : [-\Delta, 0] \to R^n$. Let us denote by x_t an element of the space C, which for any fixed t is defined as the function $x_t = x(t+\theta), -\Delta \le \theta \le 0$.

We denote by $\eta_0(\theta)$ an $n \times n$ matrix whose elements are functions of bounded variation; $\eta_i(t,\theta)$, $i = \overline{1,4}$, are matrixes of dimentions $n \times n$, $n \times m$, $m \times m$, $m \times n$, whose elements are functions of bounded variation in θ for each t and continuous in t uniformly with respect to θ.

Define linear functionals in terms of Stieltjes integrals,

$$L_0\varphi = \int\limits_{-\Delta}^{0} [d\eta_0(\theta)]\varphi(\theta), \quad L_i(t)\varphi = \int\limits_{-\Delta}^{0} [d\eta_i(t,\theta)]\varphi(\theta), \quad i = 1,4,$$
$$L_j(t)\varphi = \int\limits_{-\Delta}^{0} [d\eta_j(t,\theta)]\varphi(\varepsilon\theta), \quad j = 2,3. \tag{1}$$

Consider the system of linear singularly perturbed functional differential equations

$$\frac{dx}{dt} = L_0 x_t + L_1(t)x_t + L_2(t)y_t, \quad x(\theta) = \varphi(\theta), \quad \theta \in [-\Delta, 0],$$
$$\varepsilon\frac{dy}{dt} = L_3(t)y_t + L_4(t)x_t, \quad y(\theta) = \psi(\theta), \quad \theta \in [-\varepsilon\Delta, 0], \tag{2}$$

where $x \in R^n, y \in R^m, \varepsilon > 0$ is a small parameter.

Linear singularly perturbed systems of ordinary differential equations were considered in papers [1, 2], and systems with delay were studied in [3–6] and others. In this paper we will consider application of the method of integral manifolds to study decomposition and stability system (2).

2. We give some results of the theory of functional-differential equations needed later [7, 8]. Define the shift operator corresponding to the autonomous equation

$$\frac{dx}{dt} = L_0 x_t \tag{3}$$

by the relation $T(t)\varphi = x_t(\varphi)$. The family $\{T(t), t \geq 0\}$ is a strongly continuous semigroup, infinitesimal generator A is given by

$$A\varphi = \begin{cases} \dfrac{d\varphi}{d\theta}, & -\Delta \leq \theta < 0, \\ L_0(\varphi), & \theta = 0. \end{cases} \tag{4}$$

The spectrum of A is only a point spectrum and consists of roots of the characteristic equation

$$\det\left(\lambda E - \int_{-\Delta}^{0} e^{\lambda\theta}[d\eta_0(\theta)]\right) = 0. \tag{5}$$

There is only a finite number of roots of the equation (5) in any half plane $\mathrm{Re}\ \lambda \geq \gamma$.

Let $\Lambda = \{\lambda_1, ..., \lambda_l\}$ denote the set of roots of equation (5) such that $|\mathrm{Re}\ \lambda| < \alpha$, and let all the other roots lie in the half plane $\mathrm{Re}\ \lambda < -\alpha$. Denote the subspace in C associated Λ by P and the subspace complementary to P by Q. The subspace P is finite dimensional and its dimension is equal to l.

We give a constructive description of the subspaces P and Q. Consider the equation adjoint to (3),

$$\frac{dy}{dt} = -\int_{-\Delta}^{0} [d\eta_0^T(\theta)] y(t-\theta), \quad t \leq 0. \tag{6}$$

Denote by $\Phi = \Phi(\theta)$, $-\Delta \leq \theta \leq 0$, a basis of P, and $\Psi = \Psi(\theta)$, $0 \leq \theta \leq \Delta$, a basis of the subspace of solutions of (6), P^*, adjoint to P. For elements $\varphi \in C[-\Delta, 0], \psi \in C[0, \Delta]$, we define the scalar product by

$$(\psi, \varphi) = \psi^T(0)\varphi(0) - \int_{-\Delta}^{0}\int_{0}^{\theta} \psi(\xi - \theta) d\eta_0(\theta)\varphi(\xi) d\xi.$$

It is known [7] that the $l \times l$ matrix (Ψ, Φ) is nonsingular and we can take that $(\Psi, \Phi) = E$. Let B denote an $l \times l$ matrix such that $A\Phi = \Phi B$. The set of eigenvalues of the matrix B coincides with the set Λ.

The subspaces P and Q are characterised now by the relation

$$P = \{\varphi \in C[-\Delta, 0] : \varphi = \Phi(\Psi, \varphi)\}, \quad Q = \{\varphi \in C[-\Delta, 0] : (\Psi, \varphi) = 0\}.$$

Every element $x_t \in C$ can be represented in the form

$$x_t = x_t^P + x_t^Q = \Phi u(t) + z_t, \quad u(t) = (\Psi, x_t) \in R^l, \quad z_t \in Q. \tag{7}$$

Define the matrices

$$X_0(\theta) = \begin{cases} 0, & -\Delta \le \theta < 0, \\ E, & \theta = 0, \end{cases} \qquad Y_0(\theta) = \begin{cases} 0, & -\varepsilon\Delta \le \theta < 0, \\ E, & \theta = 0, \end{cases}$$

and the shift operator $V(t, \sigma)$ for the equation

$$\varepsilon \frac{dy}{dt} = L_3(t) y_t. \tag{8}$$

For the projections $X_0(\theta)$ into P and Q, we have the relations

$$X_0^P = \Phi \Psi^T(0), \quad X_0^Q = X_0 - X_0^P = X_0 - \Phi \Psi^T(0).$$

Changing variables (7) in system (2) and using the variation of constant formula [8] we get an equivalent system of differential and integral equations,

$$\frac{du}{dt} = Bu(t) + \Psi^T(0)[L_1(t)(\Phi u(t) + z_t) + L_2(t) y_t],$$

$$z_t = T(t-\sigma) z_\sigma + \int_\sigma^t T(t-s) X_0^Q [L_1(s)(\Phi u(s) + z_s) + L_2(s) y_s] ds, \tag{9}$$

$$y_t = V(t, \sigma) y_\sigma + \frac{1}{\varepsilon} \int_\sigma^t V(t, s) Y_0 L_4(s)(\Phi u(s) + z_s) ds.$$

The integrals in (9), for each θ, are understood as the integral in the Euclidean spaces R^n and R^m.

Suppose that all roots of the characteristic equation for (8),

$$\det \left(\lambda E - \int_{-\Delta}^{0} e^{\lambda\theta} [d\eta_3(t, \theta)] \right) = 0,$$

lie in the half plane $\operatorname{Re} \lambda \le -2\mu < 0$.

This condition and the way the roots of equation (5) are partitioned give the following estimates [7, 8]:

$$
\begin{aligned}
&|e^{Bt}| \le K_1 e^{(\alpha-\beta)|t|},\\
&|T(t)\varphi^P| \le K_2 e^{(-\alpha+\beta)t}|\varphi^P|, \quad t \le 0,\\
&|T(t)\varphi^Q| \le K_2 e^{-(\alpha+\beta)t}|\varphi^Q|, \quad t \ge 0,\\
&|V(t,\sigma)\xi| \le K_3 e^{-\frac{\mu}{\varepsilon}(t-\sigma)}|\xi|, \quad t \ge \sigma,
\end{aligned} \tag{10}
$$

where $K_1, K_2, K_3 > 0, \alpha > \beta > 0, \varphi \in C[-\Delta, 0], \xi \in C[-\varepsilon\Delta, 0]$.

3. Definition. *A set of points $M \subset R \times R^l \times Q \times C[-\varepsilon\Delta, 0]$ is said to be an integral manifold of system (9) if for each $\varepsilon \in (0, \varepsilon_0]$ and any point $(t_0, u_0, z_{t_0}, y_{t_0}) \in M$ it follows that $(t, u(t), z_t, y_t) \in M$ for all $t \ge t_0$, where $(u(t), z_t, y_t)$ is a solution of system (9) with initial values $(t_0, u_0, z_{t_0}, y_{t_0})$.*

Theorem 1. *Let estimates (10) hold and*

$$
|L_1(t)\varphi| \le \nu|\varphi|, \quad |L_2(t)\varphi| \le \nu|\varphi|, \quad |L_4(t)\varphi| \le \nu|\varphi|, \quad \nu > 0. \tag{11}
$$

Then for all sufficiently small ε, ν, system (9) has a central manifold represented in the form

$$
z_t = H_t(\theta)u, \quad y_t = h_t(\theta)u,
$$

where $H_t(\theta) : R^l \to Q$, $h_t(\theta) : R^l \to C[-\varepsilon\Delta, 0]$ are linear bounded operators.

It is not difficult to prove this theorem similarly to [4] using the iteration process

$$
H_t^0 = 0, \quad H_t^{n+1} = \int_{-\infty}^{t} T(t-s)X_0^Q[L_1(s)(\Phi + H_s^n) + L_2(s)h_s^n]U_n(s,t)ds,
$$

$$
h_t^0 = 0, \quad h_t^{n+1} = \frac{1}{\varepsilon}\int_{-\infty}^{t} V(t,s)Y_0L_4(s)(\Phi + H_s^n)U_n(s,t)ds, \quad n = 0, 1, ...,
$$

where $U_n(t,s)$ is the fundamental matrix of the equation

$$
\frac{du}{dt} = [B + \Psi^T(0)(L_1(t)(\Phi + H_t^n) + L_2(t)h_t^n)]u(t).
$$

Let us find differential equations for the functions H_t, h_t. Introduce the following notation: W is a set of continuously differentiable functions $\varphi \in C[-\varepsilon\Delta, 0]$; E_1 is a set of fuctions

$H_t : R^l \to C[-\Delta, 0]$, E_2 is a set of functions $h_t : R^l \to C[-\varepsilon\Delta, 0]$ that are continuously differentiable in t, θ. For $\varphi \in W$, we define the operator

$$D(\varepsilon)\varphi = \begin{cases} \dfrac{d\varphi}{d\theta}, & -\varepsilon\Delta \le \theta < 0, \\ \dfrac{1}{\varepsilon} L_3(t)\varphi, & \theta = 0. \end{cases}$$

Consider the following system for the functions $H_t \in E_1$, $h_t \in E_2$:

$$\begin{aligned} &\frac{\partial H_t}{\partial t} u + H_t[B + \Psi^T(0)(L_1(t)(\Phi + H_t) + L_2(t)h_t)]u \\ &= A(H_t u) + X_0^Q [L_1(t)(\Phi + H_t) + L_2(t)h_t]u, \quad \theta \in [-\Delta, 0], \\ &\frac{\partial h_t}{\partial t} u + h_t[B + \Psi^T(0)(L_1(t)(\Phi + H_t) + L_2(t)h_t)]u \\ &= D(\varepsilon)(h_t u) + \frac{1}{\varepsilon} Y_0 L_4(t)(\Phi + H_t)u, \quad \theta \in [-\varepsilon\Delta, 0], \end{aligned} \tag{12}$$

where $u(t)$ is a solution of the Cauchy problem

$$\begin{aligned} &\frac{du}{dt} = [B + \Psi^T(0)(L_1(t)(\Phi + H_t) + L_2(t)h_t)]u(t), \\ &u(\sigma) = u_0. \end{aligned} \tag{13}$$

Theorem 2. *The functions H_t, h_t determine a central manifold of system (9) if and only if $H_t \in E_1$, $h_t \in E_2$, and they satisfy system (12) for all $t \in R$, $u_0 \in R^l$.*

Proof. Let $u(t)$ be a solution of the Cauchy problem (13). Then the functions $z_t = H_t u$, $y_t = h_t u$ are solutions of the second and third equations of system (9),

$$\begin{aligned} H_t u(t) &= T(t-\sigma)H_\sigma u_0 + \int_\sigma^t T(t-s) X_0^Q [L_1(s)(\Phi + H_s) \\ &\quad + L_2(s)h_s]u(s)ds, \quad \theta \in [-\Delta, 0], \\ h_t u(t) &= V(t,\sigma)h_\sigma u_0 + \frac{1}{\varepsilon}\int_\sigma^t V(t,s) Y_0 L_4(s)(\Phi + H_s)u(s)ds, \quad \theta \in [-\varepsilon\Delta, 0]. \end{aligned} \tag{14}$$

Differentiating the first equation of system (14) on the right with respect to t at $t = \sigma$ for fixed $\theta \in [-\Delta, 0]$. For the derivate of the left-hand side, we obtain

$$\frac{d}{dt}[H_t u(t)]\Big|_{t=\sigma} = \frac{\partial H_\sigma}{\partial t} u_0 + H_\sigma[B + \Psi^T(0)(L_1(\sigma)(\Phi + H_\sigma) + L_2(\sigma)h_\sigma)]u_0. \tag{15}$$

Denote by $X(t)$ the fundamental matrix of solutions of equation (3) such that $X(t) = X_0(t), t \in [-\Delta, 0]$. Let us now make use of the relation $T(t)X_0 = X(t+\theta), -\Delta \le \theta \le 0$. We have that the function $T(t)X_0^Q = T(t)[X_0 - \Phi\Psi^T(0)]$ is continuous for $t \ge 0$. It follows that

$$\lim_{t\to 0}\frac{1}{t}\int_{\sigma}^{\sigma+t} T(t+\sigma-s)X_0^Q[L_1(s)(\Phi+H_s)+L_2(s)h_s]u(s)ds$$

$$= X_0^Q[L_1(\sigma)(\Phi+H_\sigma)+L_2(\sigma)h_\sigma]u_0. \qquad (16)$$

Then, by the definition of the operator $T(t)$, for the right derivative of the first term in right-hand side of the equation, we get

$$\frac{d}{dt}[T(t-\sigma)H_\sigma u_0]\bigg|_{t=\sigma} = A(H_\sigma u_0). \qquad (17)$$

It follows from (15)–(17) that the functions $H_t u$, $h_t u$ satisfy the first equation of system (14) for $t = \sigma$, $u = u_0$ and all $\theta \in [-\Delta, 0]$. Analogously, we proved that these functions satisfy the second equation of system (14) for all $\theta \in [-\varepsilon\Delta, 0]$.

Conversely, let a pair of functions (H_t, h_t) be a solution of system (12). We must prove that the functions $H_t u$, $h_t u$ satisfy the second and the third equations of system (9).

Consider the expression

$$\Omega = H_t u - T(t-\sigma)H_\sigma u_0 - \int_{\sigma}^{t} T(t-s)X_0^Q[L_1(s)(\Phi+H_s)+L_2(s)h_s]u(s)ds. \qquad (18)$$

If $\varphi : [\sigma, \infty) \to C[-\Delta, 0]$ is a continuously differentiable function, then for $t \ge \sigma, \theta \in [-\Delta, 0]$, the following equality holds [9]:

$$\varphi(t) - T(t-\sigma)\varphi(\sigma) = \int_{\sigma}^{t} T(t-s)\left[\frac{d\varphi(s)}{ds} - A\varphi(s)\right]ds. \qquad (19)$$

Using formula (19) with $\varphi(t) = H_t u(t)$ we can write expression (18) in the form

$$\Omega = \int_{\sigma}^{t} T(t-s)\Bigg[\frac{d}{ds}(H_\sigma u(s)) - A(H_s u(s))$$

$$- X_0^Q(L_1(s)(\Phi+H_s)+L_2(s)h_s)u(s)\Bigg]ds.$$

From the first equation of system (12), we have that $\Omega = 0$. Thus, the functions $H_t u$, $h_t u$ satisfy the second equation of system (9). Similarly, it can be proved that these functions satisfy the third equation of system (9). This completes the proof of the Theorem 2.

4. Let us cange the variables in system (9),

$$\bar{z}_t = z_t - H_t u, \quad \bar{y}_t = y_t - h_t u. \tag{20}$$

Using the formulas

$$H_t u - T(t-\sigma)H_\sigma u_0 = \int_\sigma^t T(t-s)\left[\frac{d}{ds}(H_s u(s)) - A(H_s u(s))\right] ds,$$

$$h_t u - V(t,\sigma)h_\sigma u_0 = \int_\sigma^t V(t,s)\left[\frac{d}{ds}(h_s u(s)) - D(\varepsilon)(h_s u(s))\right] ds$$

and equation (12), we get the following system for the new variables,

$$\frac{du}{dt} = [B + \Psi^T(0)(L_1(t)H_t + L_2(t)h_t)]u(t) + \Psi^T(0)[L_1(t)\bar{z}_t + L_2(t)\bar{y}_t],$$

$$\bar{z}_t = T(t-\sigma)\bar{z}_\sigma + \int_\sigma^t T(t-s)[(X_0^Q - H_\sigma \Psi^T(0))(L_1(s)\bar{z}_s + L_2(s)\bar{y}_s)]ds, \tag{21}$$

$$\bar{y}_t = V(t,\sigma)\bar{y}_\sigma + \frac{1}{\varepsilon}\int_\sigma^t V(t,s)[Y_0 L_4(s)\bar{z}_s - \varepsilon h_s \Psi^T(0)(L_1(s)\bar{z}_s + L_2(s)\bar{y}_s)]ds.$$

Theorem 3. *Let estimates (10) and condition (11) of Theorem 1 hold. Then for all sufficiently small ε, ν, system (21) has a stable manifold represented in the form*

$$u(t) = G(t, \bar{z}_t, \bar{y}_t), \tag{22}$$

where $G : R \times Q \times C[-\varepsilon\Delta, 0] \to R^l$ is a linear bounded operator.

To prove this statement we can follow the scheme of [10] used to prove the existence of stable manifold for ordinary differential equations.

Consider the system of integral equations,

$$u(t) = -\int_t^\infty U(t,s)\Psi^T(0)[L_1(s)\bar{z}_s + L_2(s)\bar{y}_s]ds,$$

$$\bar{z}_t = T(t-\sigma)\bar{z}_\sigma + \int_\sigma^t T(t-s)[(X_0^Q - H_s \Psi^T(0))(L_1(s)\bar{z}_s + L_2(s)\bar{y}_s)]ds, \tag{23}$$

$$\bar{y}_t = V(t,\sigma)\bar{y}_\sigma + \frac{1}{\varepsilon}\int_\sigma^t V(t,s)[Y_0 L_4(s)\bar{z}_s - \varepsilon h_s \Psi^T(0)(L_1(s)\bar{z}_s + L_2(s)\bar{y}_s)]ds.$$

It is not difficult to establish the existence of solution of system (23) using the method of successive approximations. Substituting $t = \sigma$ in (23), we obtain a representation of the integral manifold

$$u = G(\sigma, \bar{z}_\sigma, \bar{y}_\sigma) = -\int_\sigma^\infty U(\sigma, s)\Psi^T(0)[L_1(s)\bar{z}_s(\sigma, \bar{z}_\sigma) + L_2(s)\bar{y}_s(\sigma, \bar{y}_\sigma)]ds, \tag{24}$$

where the functions $\bar{z}_t(\sigma, \bar{z}_\sigma), \bar{y}_t(\sigma, \bar{y}_\sigma)$ satisfy the second and the third equations of system (23).

Remark. For any solution $(u(t), \bar{z}_t, \bar{y}_t)$ of system (21) belonging to stable manifold (22) there exist the constants $M > 0, 0 < \delta < \alpha$ such that we have the estimate

$$|u(t)| + |\bar{z}_t| + |\bar{y}_t| \le Me^{-\delta(t-\sigma)}(|u_0| + |\bar{z}_\sigma| + |\bar{y}_\sigma|), \quad t \ge \sigma. \tag{25}$$

Changing variables in (21),

$$\bar{u} = u - G(t, \bar{z}_t, \bar{y}_t). \tag{26}$$

We get the following system:

$$\frac{d\bar{u}}{dt} = [B + \Psi^T(0)(L_1(t)H_t + L_2(t)h_t)]\bar{u}(t),$$

$$\bar{z}_t = T(t-\sigma)\bar{z}_\sigma + \int_0^t T(t-s)[(X_0^Q - H_\sigma\Psi^T(0))(L_1(s)\bar{z}_s + L_2(s)\bar{y}_s)]ds, \tag{27}$$

$$\bar{y}_t = V(t,\sigma)\bar{y}_\sigma + \frac{1}{\varepsilon}\int_0^t V(t,s)[Y_0L_4(s)\bar{z}_s - \varepsilon h_s\Psi^T(0)(L_1(s)\bar{z}_s + L_2(s)\bar{y}_s)]ds.$$

System (27) consists of ordinary differential equation on a central manifold and integral equations on a stable manifold. Relations between systems (9) and (27) are given by the following:

$$u = \bar{u} + G(t, \bar{z}_t, \bar{y}_t), \quad z_t = \bar{z}_t + H_t u, \quad y_t = \bar{y}_t + h_t u. \tag{28}$$

The solution of system (21) on the stable manifold is governed by the functions $(G(t, \bar{z}_t, \bar{y}_t), \bar{z}_t, \bar{y}_t)$ and, for this solution, inequality (25) holds. Then with transformation (28) we obtain a reduction principle for system (9).

Theorem 4. *Let estimates (10) and condition (11) hold. Then for all sufficiently small ε, ν, system (9) by means of tansformation (28) reduces to the form (27). The zero solution of system (9) is stable (asymptotically stable, unstable) if and only if stable (asymptotically stable, unstable) is the zero solution of the system*

$$\frac{d\bar{u}}{dt} = [B + \Psi^T(0)(L_1(t)H_t + L_2(t)h_t)]\bar{u}(t) \tag{29}$$

on the central manifold.

Theorem 4 allows to reduce the study of stability of the initial singularly perturbed differential-functional system (9) to an analysis of system of ordinary differential equations (29) which is regular and has no lag in the argument.

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