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ON THE MINIMAL SPEED OF TRAVELING WAVES FOR A NON-LOCAL DELAYED REACTION-DIFFUSION EQUATION*

ПРО МІНІМАЛЬНУ ШВИДКІСТЬ ПЕРЕСУВАЮЧИХ ХВИЛЬ ДЛЯ НЕЛОКАЛЬНОГО РЕАКЦІЙНО-ДИФУЗІЙНОГО РІВНЯННЯ З ЗАПІЗНЕННЯМ

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In this note, we give constructive upper and lower bounds for the minimal speed of propagation of traveling waves for non-local delayed reaction-diffusion equation.

Наведено конструктивні верхня і нижня межі поширення пересуваючих хвиль для нелокального реакційно-дифузійного рівняння з запізненням.

1. Introduction and the main results. In this note, we estimate the minimal speed of propagation of positive traveling wave solutions for the non-local delayed reaction-diffusion equation

$$u_t(t,x) = u_{xx}(t,x) - u(t,x) + \int_{\mathbb{R}} K(x-s)g(u(t-h,s))ds, \quad u \geq 0, \quad x \in \mathbb{R}, \tag{1.1}$$

which is widely used in applications, e.g. see [1–5] and references wherein. It is assumed that the birth function g is of the monostable type, $p := g'(0) > 1$ and $h \geq 0$. The non-negative kernel K is such that $K(s) = K(-s)$ for $s \in \mathbb{R}$, $\int_{\mathbb{R}} K(s)ds = 1$ and $\int_{\mathbb{R}} K(s)\exp(\lambda s)ds$ is finite for all $\lambda \in \mathbb{R}$. Consider

$$\psi(z,\varepsilon) = \varepsilon z^2 - z - 1 + p\exp(-zh)\int_{\mathbb{R}} K(s)\exp(-\sqrt{\varepsilon}zs)ds, \tag{1.2}$$

which determines the eigenvalues of Eq. (1.1) at the trivial steady state. From [4, 6], we know that there is $\varepsilon_0 = \varepsilon_0(h) > 0$ such that $\psi(z,\varepsilon_0) = 0$ has a unique multiple positive root $z_0 = z_0(h)$. Furthermore, if $g(s) \leq g'(0)s$ for $s \geq 0$, then the minimal speed c_* is equal to $c_* = 1/\sqrt{\varepsilon_0}$. Note that z_0 and ε_0 are the unique solutions of the system

$$\psi(z,\varepsilon) = 0, \quad \psi_z(z,\varepsilon) = 0. \tag{1.3}$$

It is known [5] that for various systems modeled by equation (1.1), the minimal wave speed c_* coincides with the spreading speed. Therefore, it is important to study the effects caused by the delay and other parameters (depending on specific models) on c_*, cf. [2, 3, 5, 7, 8]. Another aspect of the problem concerns easily calculable upper and lower bounds for c_*. In particular, in

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the recent work [9], Wu et al. give several nice estimations for c_* when $K_\alpha(s) = \dfrac{1}{\sqrt{4\pi\alpha}} e^{-s^2/4\alpha}$ and $\alpha \le h$. However, the approach of [9] depends heavily on the condition $\alpha \le h$ and on the special form of K which is the fundamental solution of the heat equation. In the present work, we use a completely different idea to estimate the minimal speed for general kernels and without imposing any restriction on h.

Let us state our main result. Set

$$k_1 = 2\sqrt{\frac{p-1}{1+\frac{p}{2}\int_{\mathbb{R}} s^2K(s)ds}} - p\int\limits_{\mathbb{R}} sK(s)\exp\left(-s\sqrt{\frac{p-1}{1+\frac{p}{2}\int_{\mathbb{R}} s^2K(s)ds}}\right)ds,$$

$$k_2 = \frac{1}{\sqrt{\ln p}}\ln\left(p\int\limits_{\mathbb{R}} K(s)\exp(-\sqrt{\ln p}s)ds\right).$$

It is clear that $k_2 > 0$ and bellow we will show that k_1 is positive.

Theorem 1.1. *Assume that $K(s) \ge 0$ is such that $K(s) = K(-s)$ for $s \in \mathbb{R}$, $\int_{\mathbb{R}} K(s)ds = 1$ and $\int_{\mathbb{R}} K(s)\exp(\lambda s)ds$ is finite for all $\lambda \in \mathbb{R}$. Then $c_* = c_*(h) = 1/\sqrt{\varepsilon_0(h)}$ is a C^∞-smooth decreasing function of variable $h \in \mathbb{R}_+$. Moreover,*

1) $\max\left\{2\sqrt{\dfrac{p-1}{p(2h+h^2)+1}}, \dfrac{2\sqrt{\ln p}}{1+h}\right\} < c_* < \min\left\{\dfrac{k_1}{1+h}, \dfrac{k_2}{h}\right\}, h \in [0,1],$

2) $\max\left\{2\sqrt{\dfrac{p-1}{p(2h+h^2)+1}}, \dfrac{\sqrt{\ln p}}{h}\right\} < c_* < \min\left\{\dfrac{k_1}{2}, \dfrac{k_2}{\sqrt{h}}\right\}, h \in [1,+\infty).$

Furthermore, $\dfrac{C_1}{h} \le c_*(h) \le \dfrac{C_2}{h}$, $h \ge 1$, for some positive $C_1 < C_2$.

Observe that Theorem 1.1 implies that $c_*(h) = O(h^{-1})$, $h \to +\infty$, in this way we improve the estimation $c_*(h) = O(h^{-1/2})$, $h \to +\infty$, proved in [8, 9].

Proof. It follows from [4, 6] that the functions $z_0 = z_0(h)$ and $\varepsilon_0 = \varepsilon_0(h)$ are well defined for all $h \ge 0$. Set $F(h,z,\varepsilon) = (\psi(z,\varepsilon), \psi_z(z,\varepsilon))$. It is easy to see that $F \in C^\infty(\mathbb{R}_+\times\mathbb{R}\times(0,\infty), \mathbb{R}^2)$, $F(h,z_0,\varepsilon_0) = 0$, and

$$\left|\frac{\partial F(h,z_0,\varepsilon_0)}{\partial(z_0,\varepsilon_0)}\right| = \psi_{zz}(z_0,\varepsilon_0)\psi_\varepsilon(z_0,\varepsilon_0) =$$

$$= \left(2\varepsilon_0 + p\int\limits_{\mathbb{R}} K(s)\exp(-z_0(h+\sqrt{\varepsilon_0}s))(h+\sqrt{\varepsilon_0}s)^2ds\right) \times$$

$$\times \frac{z_0}{2\varepsilon_0}\left(1 + hp\int\limits_{\mathbb{R}} K(s)\exp(-z_0(h+\sqrt{\varepsilon_0}s))ds\right) > 0.$$

Applying the Implicit Function Theorem we find that $z_0, \varepsilon_0 \in C^\infty(0,+\infty)$.

On the other hand, after introducing a new variable $w = \sqrt{\varepsilon}z$ we find that system (1.3) takes the following form:

$$\left(1 + \frac{w}{\sqrt{\varepsilon}} - w^2\right) \exp\left(\frac{wh}{\sqrt{\varepsilon}}\right) = p \int\limits_{\mathbb{R}} K(s) \exp(-ws) ds, \tag{1.4}$$

$$\left(\frac{h}{\sqrt{\varepsilon}} w^2 + \left(2 - \frac{h}{\varepsilon}\right) w - \frac{1+h}{\sqrt{\varepsilon}}\right) \exp\left(\frac{wh}{\sqrt{\varepsilon}}\right) = p \int\limits_{\mathbb{R}} sK(s) \exp(-ws) ds. \tag{1.5}$$

Let

$$G(w) = \left(1 + \frac{w}{\sqrt{\varepsilon_0}} - w^2\right), \quad H(w) = \left(1 + \frac{w}{\sqrt{\varepsilon_0}} - w^2\right) \exp\left(\frac{wh}{\sqrt{\varepsilon_0}}\right)$$

and

$$R(w) = p \int\limits_{\mathbb{R}} K(s) \exp(-ws) ds.$$

Set also $w_0 = w_0(h) = \sqrt{\varepsilon_0(h)} z_0(h)$. First, note that $G(w_0) = \exp\left(\frac{-wh}{\sqrt{\varepsilon_0}}\right) R(w_0) > 0$ and $G(w) \geq 1$ when $0 \leq w \leq 1/\sqrt{\varepsilon_0}$. As can be checked directly, H has a unique positive local extremum (maximum) at some $\bar{w}$. Since $K(s) = K(-s)$, $s \in \mathbb{R}$, it is easy to see that R increases on $\mathbb{R}_+$.

Differentiating equation (1.4) with respect to h and using (1.5) we get the following differential equation:

$$\varepsilon_0'(h) = \frac{2\varepsilon_0(h) G(w_0(h))}{1 + hG(w_0(h))} > 0. \tag{1.6}$$

The remainder of the proof will be divided in several steps.

Step I. If $h \in [0,1]$, then $H'(1/\sqrt{\varepsilon_0}) = \left(\frac{h-1}{\sqrt{\varepsilon_0}}\right) e^{h/\varepsilon_0} \leq 0$. Hence, $\bar{w} \leq 1/\sqrt{\varepsilon_0}$. In addition, if $w \in (0, \bar{w})$ then $H'(w) > 0$. As $R'(w) > 0$ for $w > 0$, we have $w_0 < \bar{w} \leq 1/\sqrt{\varepsilon_0}$. Thus, we get $G(w_0) \geq 1$. In this way, $\varepsilon_0'(h) \geq 2\varepsilon_0(h)/(1+h)$ for $h \in [0,1]$ that yields $(1+h)^2 \varepsilon_0(0) \leq \leq \varepsilon_0(h) \leq (1+h)^2 \varepsilon_0(1)/4$ (equivalently, $2c_*(1)/(1+h) \leq c_*(h) \leq c_*(0)/(1+h)$, for $h \in [0,1]$). Next, taking $h = 0$ in equations (1.4) and (1.5) we obtain that

$$\frac{1}{\sqrt{\varepsilon_0(0)}} = 2w_0(0) - p \int\limits_{\mathbb{R}} sK(s) \exp\left(-w_0(0)s\right) ds, \tag{1.7}$$

$$1 + w_0^2(0) = p \int\limits_{\mathbb{R}} K(s)(1 + w_0(0)s) \exp(-w_0(0)s) ds =$$

$$= p\left(1 - \frac{\int_{\mathbb{R}} s^2 K(s) ds}{2} w_0^2(0) - \frac{\int_{\mathbb{R}} s^4 K(s) ds}{8} w_0^4(0) - \ldots\right).$$

As a consequence of the latter formula, we get

$$w_0(0) < \sqrt{\frac{p-1}{1+\frac{p}{2}\int_{\mathbb{R}} s^2K(s)ds}}.$$

Then (1.7) implies that $c_*(0) < k_1$ so that $c_*(h) < k_1/(1+h)$ for $h \le 1$. Note that $k_1 > 0$ since R is increasing for $w > 0$. Finally, since $c_*(h)$ is decreasing, we have that $c_*(h) < k_1/2$ for $h \ge 1$.

Step II. If $h \ge 1$, then $\bar{w} \ge 1/\sqrt{\varepsilon_0}$. As a consequence, $G(\bar{w}) \le 1 = G(1/\sqrt{\varepsilon_0})$ so that $G(w) \ge G(\bar{w})$ for all $w \in [0, \bar{w}]$ (see Fig. 1). Additionally, $G(\bar{w}) = (2\bar{w}\sqrt{\varepsilon_0} - 1)\dfrac{1}{h} \ge \dfrac{1}{h}$, therefore we conclude that $G(w_0) \ge 1/h$. Hence, we have $\varepsilon_0'(h) \ge \varepsilon_0(h)/h$, so that $\varepsilon(h) \ge \varepsilon(1)h$ (equivalently, $c_*(h) \le c_*(1)/\sqrt{h}$) for $h \ge 1$. Now, if $h = 1$ we have $\bar{w} = 1/\sqrt{\varepsilon_0(1)}$. Thus, taking $h = 1$ and $w = \bar{w}$ in (1.4) we get $\exp(1/\varepsilon_0(1)) = R(\bar{w}) > R(0) = p$ that yields $\sqrt{\ln p} < 1/\sqrt{\varepsilon_0(1)} = c_*(1)$. On the other hand, for all $0 \le w < 1/\sqrt{\varepsilon_0}$, we have

$$\exp\left(\frac{wh}{\sqrt{\varepsilon_0}}\right) < \left(1 + \frac{w}{\sqrt{\varepsilon_0}} - w^2\right)\exp\left(\frac{wh}{\sqrt{\varepsilon_0}}\right) \le p\int_{\mathbb{R}} K(s)\exp(-ws)ds. \tag{1.8}$$

In particular, taking $h = 1$ and $w = \sqrt{\ln p}$ in (1.8) we conclude that $c_*(1) < k_2$ so that $c_*(h) < k_2/\sqrt{h}$ for $h \ge 1$. Additionally, using $c_*(h) \ge 2c_*(1)/(1+h)$ obtained in step I, we also concluded that $c_*(h) > 2\sqrt{\ln p}/(1+h)$, for $h \in [0,1]$.

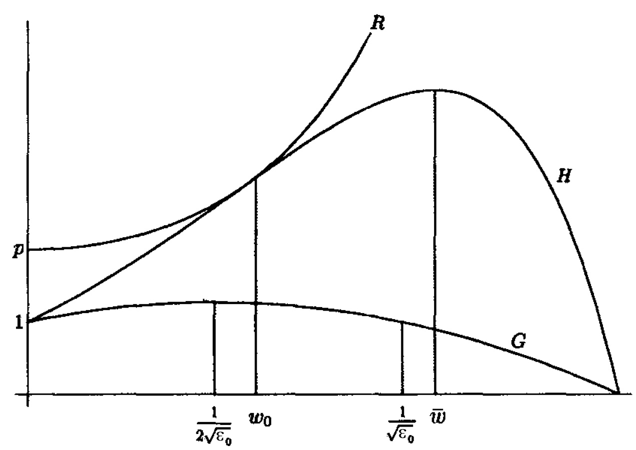

Fig. 1. G, H and R for $h > 1$.

Step III. For $h > 0$, it is evident that $\varepsilon_0'(h) \le 2\varepsilon_0(h)/h$. Integrating the latter inequality on $[h,1]$ we obtain $\varepsilon(h) \ge \varepsilon(1)h^2$ (equivalently, $c_*(h) \le c_*(1)/h$), for $0 < h \le 1$ so that $c_*(h) < k_2/h$, for $h \in (0,1]$. Analogously, by integrating $\varepsilon_0'(h) \le 2\varepsilon_0(h)/h$ on $[1,h]$ we have $\varepsilon_0(h) \le \varepsilon_0(1)h^2$ (equivalently, $c_*(h) \ge c_*(1)/h$), for $h \ge 1$. Thus, we obtain $c_*(h) > \sqrt{\ln p}/h$, $h \ge 1$.

On the other hand, for all $h \ge 0$, we have $G(w_0) \le 1 + 1/(4\varepsilon_0)$. As a consequence, $\varepsilon_0'(h) \le (4\varepsilon_0(h)+1)/(2(1+h))$ for all $h \ge 0$ so that $\varepsilon_0(h) \le ((4\varepsilon_0(0)+1)(1+h)^2 - 1)/4$. Taking

$h = 0$ in (1.4), we get $1 + 1/(4\varepsilon_0(0)) > G(w_0(0)) = R(w_0(0)) > p$ so that $c_*(0) > 2\sqrt{p-1}$. In consequence,

$$c_*(h) > 2\sqrt{\frac{p-1}{p(2h+h^2)+1}}, \quad h \geq 0. \tag{1.9}$$

Step IV. Setting $w = r$, $r \in (0,1)$, in the second inequality of (1.8) we obtain

$$\left(1-r^2\right)\exp\left(\frac{rh}{\sqrt{\varepsilon_0(h)}}\right) < p\int\limits_{\mathbb{R}} K(s)\exp(-rs)ds,$$

from which we get that

$$\frac{1}{\sqrt{\varepsilon_0(h)}} < \frac{1}{hr}\ln\left(\frac{p}{1-r^2}\int\limits_{\mathbb{R}} K(s)\exp(-rs)ds\right), \quad h > 0. \tag{1.10}$$

Considering (1.9) and (1.10) we get $\dfrac{C_1}{h} \leq c_*(h) \leq \dfrac{C_2}{h}$ for $h \geq 1$. This completes the proof.

2. Example. Consider the heat kernel $K_\alpha(s) = (4\pi\alpha)^{-1/2}\exp\left(-s^2/(4\alpha)\right)$. Then Theorem 1.1 applies with

$$k_1 = 2\sqrt{p-1}\left(\frac{1+\alpha p\exp\left(\frac{\alpha(p-1)}{1+\alpha p}\right)}{\sqrt{1+\alpha p}}\right), \quad k_2 = (1+\alpha)\sqrt{\ln p}.$$

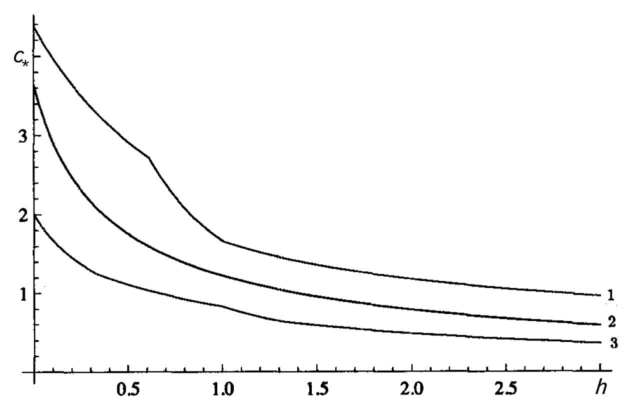

Fig. 2. The minimal speed (curve *2*) and its upper (curve *1*) and lower (curve *3*) bounds ($p = 2$ and $\alpha = 1$).

In fact, in this case we can plot graphs of c_* against h using standard numerical methods to solve some appropriately chosen initial value problem $\varepsilon_0(h_0) = \rho_0$ for differential equation (1.6). For example, if we take $h_0 = \alpha$ then ρ_0 coincides with a positive solution of the equation

$1+\dfrac{1}{4\rho}=p\exp\Big(-\dfrac{\alpha}{4\rho}\Big)$. Next, we can explicitly find $G(w_0)$ in (1.6) by using Cardano's formulas to solve the cubic equation $(w_0^2-w_0/\sqrt{\varepsilon_0}-1)(2\sqrt{\varepsilon_0}\alpha w_0-h)+1-2\sqrt{\varepsilon_0}w_0=0$. It is easy to see that this equation has three real roots for all $h\geq 0$ and $\alpha>0$, and that w_0 is the leftmost positive root.

Fig. 2 shows the minimal speed c_* and its estimations when $p=2$ and $\alpha=1$. Remark that we do not need the restriction $\alpha\leq h$ required in [9].

Finally, note that letting $\alpha\to 0^+$ in (1.1) and (1.2) we recover the characteristic equation for the delayed reaction-diffusion equation

$$u_t(t,x)=u_{xx}(t,x)-u(t,x)+g(u(t-h,x)),$$

which was studied by various authors (e.g. see [8, 10] and references therein). In this case, our results complete and partially improve the estimations of [8].

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