COMPUTATION OF THE WEAKLY RELATIVISTIC PLASMA DISPERSION FUNCTIONS USING SUPERASYMPTOTIC AND HYPERASYMPTOTIC EXPANSIONS

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A new method to compute the weakly relativistic plasma dispersion functions for complex argument is presented. It is demonstrated that Jacobi fractions represent those functions asymptotically for $z \to \infty$ in the upper semi-plane.
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1. INTRODUCTION

The evaluation of the weakly relativistic plasma dispersion functions (PDFs) also known as the Shkarofsky functions [1] is a ground of EC wave analysis in the laboratory thermonuclear plasmas. As a rule, in numerical applications these functions are calculated massively, therefore the efficiency of the involved computational algorithm is of primary importance.

The theory of continued fractions of Jacobi has been proved to provide a basis for such fast calculation methods for the case of the complex error function $w(z) = \exp(-z^2)\,\mathrm{erfc}(-iz)$, related to the nonrelativistic PDF by $Z(z) = i\sqrt{\pi}w(z)$. The large-$|z|$ asymptotic expansion in the region $\mathrm{Im}(z) \ge 0$ for this function,

$$w(z) \sim \frac{i}{\pi}\sum_{k=0}^{\infty}\frac{\mu_k}{z^{k+1}}, \quad \mu_k = \begin{cases} 0, & k \text{ odd} \\ \Gamma\left(\frac{k+1}{2}\right), & k \text{ even} \end{cases}$$

is a divergent series, so it is undesirable to use this expansion directly for the computation of $w(z)$. Gautschi indicated [2] that changing the series to the continued J-fraction

$$\sum_{k=0}^{\infty}\frac{\mu_k}{z^{k+1}} = \frac{|\sqrt{\pi}}{z-|} + \frac{|1/2}{z-|} + \frac{|1}{z-|} + \frac{|3/2}{z-|} + \ldots = \cfrac{\sqrt{\pi}}{z - \cfrac{1/2}{z - \cfrac{1}{z - \cfrac{3/2}{z - \ldots}}}}$$

can eliminate these difficulties and speed up the convergence in the region, where $|w(z)|$ is of monotonic type. In this way, the evaluation of $w(z)$ requires some few first terms even if $|z|$ is not very large. He has also shown empirically that for moderate and small $|z|$ values one can use the continued fractions technique applied for computations at the outer boundary of this domain in combination with the Taylor expansion in the descending direction of the imaginary axis towards the target points.

One can apply this method for fast computation of the weakly relativistic PDFs, as follows. The two lowest-order PDFs can be expressed in terms of $w(z)$ [3] and computed using the above method, and then those of higher orders are sequentially evaluated by employing the 2nd-order recursion relation between them. However, this technique lacks stability when the argument of $w(z)$ becomes large [4].

The main purpose of the present work is to develop another, direct (without use of the recurrent calculations) computational algorithm for the weakly relativistic PDFs, on the basis of superasymptotic and hyperasymptotic similar to that of Ref. [2].

2. MATHEMATICAL GROUNDS

From several equivalent definitions of the function $w(z)$ we shall exploit the Cauchy integral

$$w(z) = \frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{\exp(-t^2)dt}{t-z}, \tag{1}$$

where the contour of integration passes below the pole. According to the theory of Cauchy integral, the function $w(z)$ can be analytically continued to the whole complex plane. Taking the derivative of (1) with respect to z, we will obtain another definition of $w(z)$

$$w'(z) = \frac{2i}{\sqrt{\pi}} - 2zw(z) \tag{2}$$

with the condition $w(0) = 1$. Substituting the Taylor expansion $w(z) = \sum_{n=0}^{+\infty} a_n z^n$ in (2) will bring us to the power series definition

$$w(z) = \sum_{n=0}^{+\infty}\frac{(iz)^n}{\Gamma\left(1+\frac{n}{2}\right)}. \tag{3}$$

This series has infinite radius of convergence, therefore $w(z)$ belongs to the class of entire functions, which is the extension of a polynomial class and hence it is the class of simplest, after polynomials, analytical functions.

The most basic properties of a given analytical function are defined via its singularities (poles, essential singularities, branching points). In this regard the theory of entire functions can be named the theory of essential singularity because any entire function excepting a polynomial has an essential singularity at infinity. The entire functions, as more simple, were the first studied (see e.g. [5]). Particularly, according to Picard's theorem, in the vicinity of an infinite point the modulus of an entire function attains every finite positive value. In the view of asymptotic expansion theory, both the concept of modulus of an analytical function and its behavior near the essential singularity at infinity are very important. Indeed, asymptotic properties of analytical functions are associated not only with the principle of modulus maximum but also with that of modulus minimum, which is valid except for the vicinities of the function's zeroes, generally located at the Stokes lines [6]. According to these principles, the modulus of an entire function has equal number of regions of maximum and minimum in any vicinity of infinity, with the Stokes lines lying between them. If the modulus between two adjacent Stokes lines tends to zero (or constant) in some region at infinity, the asymptotic properties of the function depend on the value of its decrement. In this regard $w(z)$ is the simplest case of a single maximum and minimum, which are separated by two Stokes lines corresponding to the bisectors of the 3rd and 4th quadrant and pass through the zeroes of $w(z)$. In the more wide upper sector, which corresponds to the minimum, the function $w(z)$ has a unique asymptotic expansion in the form of divergent series because otherwise (convergent series) there would be no essential singularity at infinity. This series converges in asymptotic sense and the speedy of this convergence has the exponential character. If it is necessary the further convergence acceleration can be provided on the base the continued fractions or Pade approximants techniques. Nevertheless, such common reasoning certainly needs a clear proof.

Substituting the expansion at infinity $w(z) = \sum_{n=0}^{+\infty} a_n / z^n$ in (2) and using the condition $w(\infty) = 0$, which is again the result of (1), we obtain the real-axis asymptotic expansion:

$$w(z) \approx \frac{i}{\pi} \sum_{n=0}^{+\infty} \Gamma\left(n + \frac{1}{2}\right) \Big/ z^{2n+1} \quad . \tag{4}$$

Firstly, let us show that (4) is valid not only for the real valued z, but at least for the whole upper half-plane. As it is known from the theory of Cauchy integral, analytical continuation of the real-axis integral (1) to the domain of $\operatorname{Im} z > 0$ can be made by corresponding analytical continuation of the subintegral function $1/(t-z)$. We shall use this fact in the following proof.

It is also known that the functions $u(z) \equiv \operatorname{Re} w(z) = \exp(-z^2)$, $v(z) \equiv \operatorname{Im} w(z) = \frac{2}{\sqrt{\pi}} D(z)$, where $D(z) = \exp(-z^2) \int_0^z \exp(t^2) dt$ is the Dawson integral, are connected through the Hilbert transform formulas:

$$iv(z) = -\frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{u(\tau) d\tau}{\tau - z}, \qquad u(z) = -\frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{v(\tau) d\tau}{\tau - z} \tag{5}$$

(P denotes the Cauchy principal value of the integral). Relations (5) are valid as far as $u(z)$ and $v(z)$ satisfy both the Holder condition on the real axis and the conditions $u(z) \to 0$, $v(z) \to 0$ $(z \to \pm\infty)$. One more representation of $w(z)$ can be stated using (1) and (5):

$$w(z) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{D(t) dt}{t - z} \quad . \tag{6}$$

Now it follows from (1) and (6) that

$$w(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{w(t) dt}{t - z} \quad . \tag{7}$$

On the other hand, $w(z)$ is an analytical function at the domain of $\operatorname{Im} z \geq 0$, therefore one can apply the Cauchy integral formula to any internal finite contour. Let this contour be a semicircle of radius R traversed counterclockwise, then the following representation is valid inside it:

$$w(z) = \frac{1}{2\pi i} \int_{-R}^{+R} \frac{w(t) dt}{t - z} + \frac{1}{2\pi i} \int_{arc} \frac{w(t) dt}{t - z} \quad . \tag{8}$$

Comparison of (7) and (8) in the limit $R \to \infty$ yields

$$\int_{arc} \frac{w(t) dt}{t - z} \to 0$$

for any contour with $R \to \infty$. From this result it follows that $w(z) \to 0$ at $z \to \infty$ in the upper half-plane.

It is known from the theory that an analytical function, which is given by the Cauchy integral defined on the real axis, can be continued from the upper half-plane to the lower one by the following relation:

$$w(z^*) = w^*(z) + 2\pi i \exp(-z^2) \ .$$

Therefore, the asymptotic behavior of $w(z)$ at infinity in the lower half-plane is the same as in the upper one only in the sectors $-\pi/4 < \arg z \leq 0$ and $\pi \leq \arg z < 5\pi/4$, where $\exp(-z^2) \to 0$ at $z \to \infty$.

It is clear from the above shown proof that at a fixed x and $y \to +\infty$ in the upper half-plane $\operatorname{Re} w(x+iy) \to \exp(\sqrt{x^2+y^2})$, $i \operatorname{Im} w(x+iy) \to -\exp(\sqrt{x^2+y^2})$, eliminating each other exactly and providing $w(z) \to 0$. As a consequence, either integration or Taylor expansion of the function $w(z)$ for its calculation in the region of small and moderate $|z|$ values has to be directed downward the imaginary axis in order to preserve the accuracy, as it was done in the algorithm of Gautschi.

With a few modifications, this proof can be used for wide class of functions capable of being represented as a Cauchy-type integral defined on the real axis, provided that the density of the corresponding integral vanishes at infinity. This is the case of the weakly relativistic PDFs. Really, the weakly relativistic PDF of index $q+3/2$ can be written in the following form [7]:

$$F_{q+3/2}(z,a) = \frac{e^{-a}}{(\sqrt{a})^{q+1/2}} \int_0^\infty du \frac{(\sqrt{u})^{q+1/2} e^{-u} I_{q+1/2}(2\sqrt{au})}{u + z - a}, \tag{9}$$

where $I_{q+1/2}$ is the modified Bessel function and the integration contour passes above the pole.

The functions $F_{q+3/2}$ defined by expressions (9) have the real branching points at a. In such cases the integration contour has to be added with the semicircle of radius ε passing around the branching point in the upper half-plane. Therefore the respective elaboration of the formula (8) will contain an extra term in the right hand side, which is vanishing at $\varepsilon \to 0$ (due to continuity of $F_{q+3/2}$ at a point), and hence has no effect onto the further reasoning.

Note that the Stokes lines of the weakly relativistic PDFs are not coincided with that of $w(z)$, as far as the densities of integral (9) determine them. But evidently, these two lines are still located in the 3rd and 4th quadrants, and this important fact will be of further use.

3. EVALUATION OF THE WEAKLY RELATIVISTIC PDFs

We'll assume $z = x + iy$ to lie in the region $y \ge 0$ of the plane. This is no restriction of generality, since

$$F_q(z^*, a) = \begin{cases} F_q(z,a)^* + 2\pi i f(a - z, a), & \operatorname{Re} z < a \\ F_q(z,a)^*, & \operatorname{Re} z > a, \end{cases}$$

where $f(u,a) = e^{-a-u}(u/a)^{(q-1)/2} I_{q-1}(2\sqrt{au})$, can be used to continue $F_q(z,a)$ into the remaining half-plane.

Computation of $F_q(z,a)$ in the region Q of large $|z-a|$ values i.e. superasymptotic part of calculations, was performed on the basis of asymptotic expansion

$$F_q(z,a) = -\sum_{j=0}^{+\infty} \frac{C_{q,j}}{(a-z)^{j+1}}, \qquad C_{q,0} = 1,$$

$$C_{q,j} = (j + q - 1)C_{q,j-1} + aC_{q+1,j-1}$$

and use of Jacobi continued fractions in total correspondence with [8] and the chain length $n = 9$ to provide 10 correct decimal digits after the decimal point.

In the region S of small $|z-a|$ values we use the method of [3] since in this region it is stable and most fast.

In the region R of moderate $|z-a|$ there simultaneously used both the J-continued fraction expansion with higher the values of n and the Taylor expansion of $F_q(z,a)$ in the direction of negative imaginary semi-axis

$$F_q(x_0 + iy, a) = F_q(x_0 + iy_0, a) + F_q'(x_0 + iy, a)i(y - y_0) + \frac{1}{2}F_q''(x_0 + iy, a)[i(y - y_0)]^2 + ...,$$

which is most favorable as was shown in the previous section. In this expansion the derivatives of $F_q(z,a)$ were defined from the equation

$$(a - z)F_q^{(n)} - [2(a - z) - q + n]F_q^{(n-1)}[z + q - 2(n-1)]F_q^{(n-2)} - (n-2)F_q^{(n-3)} = 0.$$

with the starting values $F_q(x_0 + iy_0, a)$ and $F_q'(x_0 + iy_0, a) = F_q(x_0 + iy_0, a) - F_{q-1}(x_0 + iy_0, a)$ at the boundary between regions R and Q obtained from continued fraction expansions.

Calculations show that the application of this method gives an average gain in time in comparison with the technique of Cauchy-type integrals approximately of a factor 20. In comparison with the method [3] this technique gives an average loss in time with a factor 5 but nevertheless it allows to avoid instability of computation $F_q(z,a)$ for high enough q.

4. CONCLUSIONS

1. It is demonstrated that for the weakly relativistic PDFs asymptotic expansions in the vicinity of infinity totally define those PDFs in the whole complex plane.
2. On this basis a new method to compute the weakly relativistic plasmas dispersion functions for complex argument is presented.
3. It is shown that the method of Gautschi for fast evaluation of non-relativistic plasma dispersion function can be generalized to the weakly relativistic case directly, without usage of the unstable, for large $|z|$ values, recursive equation for the weakly relativistic functions.

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ВЫЧИСЛЕНИЕ СЛАБОРЕЛЯТИВИСТСКИХ ПЛАЗМЕННЫХ ДИСПЕРСИОННЫХ ФУНКЦИЙ НА ОСНОВЕ СУПЕРАСИМПТОТИЧЕСКИХ И ГИПЕРАСИМПТОТИЧЕСКИХ РАЗЛОЖЕНИЙ

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Предлагается новый метод вычисления слаборелятивистских плазменных дисперсионных функций комплексного аргумента.

ОБЧИСЛЕННЯ СЛАБКОРЕЛЯТИВІСТСЬКИХ ПЛАЗМОВИХ ДІСПЕРСІЙНИХ ФУНКЦІЙ НА ОСНОВІ СУПЕРАСИМПТОТИЧНИХ ТА ГІПЕРАСИМПТОТИЧНИХ РОЗКЛАДАНЬ

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Пропонується новий метод обчислення слабкорелятивістських плазмових дисперсійних функцій комплексного аргументу.