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Optimum Plans of Step-Stress Life Tests Using Failure-Censored Data Form Burr Type-XII Distribution

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Оптимальное планирование частично ускоренных ресурсных испытаний с цензурированием по времени для распределения Бурра XII типа

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Выполнено оптимальное планирование частично ускоренных ресурсных испытаний при пошаговом изменении напряжений с использованием цензурированных по времени данных из распределения Бурра XII типа. В рамках концепции максимальной вероятности выполнена оценка коэффициента ускорения испытаний и параметров распределения. В качестве критерия оптимизации планируемых частично ускоренных ресурсных испытаний используется минимизация обобщенной асимптотической дисперсии для показателей максимальной вероятности параметров распределения. Эффективность предложенного метода демонстрируется на примере численных расчетов.

Ключевые слова: частично ускоренные ресурсные испытания, максимальная вероятность, распределение Бурра XII типа, оптимальное планирование испытаний, цензурирование II типа.

Introduction. Recently, the focus of inhabitant communities, manufacturing organizations and governments on the reliability issue is being increased. In fact, the majority of manufacturers increase their efforts to enhance the performance of their products in order to improve the trust and demand of their customers. However, information concerning the lifetime of materials of high reliability cannot be easily obtained when normal testing conditions are considered. Therefore, severe conditions (stresses) must be utilized in combination with normal (use) ones in order to obtain information about the lifetime of such materials in a shortest time. When such testing models are carried out under stresses, they are then known as partially accelerated life tests (PALT) or accelerated life tests (ALT). In the ALTs, it is supposed that the mathematical model used to explore the relation between the stress and the unit lifetime is either predefined or can be assumed. When that model is not applicable, the PALTs come to be a good alternative to the ALTs in order to investigate and examine specimens or materials of high reliability.

Nelson [1] indicated that the stresses can be categorized into common two types, namely; constant-stress and step-stress. The constant-stress test method runs all items only under either accelerated conditions or use ones until the experiment is ended, see for example, Ismail and Al Tamimi [2]. The second type is step-stress test method which in turn can be categorized into two types, namely; time-step-stress (TSS) and failure-step stress (FSS) tests. In the TSS test, all test units are run under use condition until a pre-specified time τ. If a unit does not fail until that time, it is run under accelerated condition until it fails or the experiment is terminated. In the FSS test, the test units are run under use condition until a pre-assigned number of failures occurs and the unfailed units after that time are run under accelerated condition until the test is finished.

For an overview of the literature about the estimation and optimal design problems of step-stress PALTs, readers can be referred, for example, to Goel [3], Bai and Chung [4], Ismail [5–7], Ismail and Al-Habardi [8], among others.

The remaining of this article is organized as follows. Section 1 presents a lifetime model, namely, two-parameter Burr type-XII distribution. Section 2 presents the maximum likelihood estimations (MLEs) of the distribution parameters and acceleration factor. Also, the asymptotic confidence intervals of the model parameters are considered in this section. Section 3 considers the optimal design of step-stress PALTs. In Section 4 simulation studies are presented to illustrate the theoretical results. Section Six concludes the article.

1. **The Model Description and Its Assumptions**. This section describes the used model and presents its assumptions.

1.1. ***Burr Type-XII Distribution as a Lifetime Model***. The Burr type-XII distribution, which was originally derived by Burr [9] received more attention by the researchers due to its broad applications in different fields, mainly in the modeling of failure time and reliability.

The two parameters Burr type-XII distribution has the following density function:

$$f(t) = ckt^{(c-1)}(1+t^c)^{-(k+1)}, \qquad t>0,\ c,\ k>0. \tag{1}$$

Its reliability function is given by

$$R(t) = (1+t^c)^{-(k)}, \qquad t>0. \tag{2}$$

The associated hazard function can be expressed by

$$h(t) = \frac{ckt^{(c-1)}}{1+t^c}. \tag{3}$$

This section presents the procedure of step-stress PALTs and its main assumptions.

1.2. ***Test Procedure***.

(1) Putting the test units under normal (use) condition for a pre-specified stress change-time τ.

(2) After that time τ, each unit still alive is then tested under accelerated condition until it fails or the experimented is terminated.

1.3. ***Test Assumptions***.

(1) The total lifetime Y of an item can be expressed by

$$Y = \begin{cases} T & \text{if } \ T \le \tau, \\ \tau + \beta^{-1}(T-\tau) & \text{if } \ T > \tau, \end{cases} \tag{4}$$

where T is the lifetime at use condition and β is the acceleration factor.

(2) The Burr type-XII distribution is assumed at both use and accelerated conditions.

(3) The failure times y_i, $i=1,\dots,n$ are independent and identically distributed random variables (i.i.d.) random variables (r.v.).

2. **Maximum Likelihood Estimation**. The PDF of the total lifetime Y for a specific item in step-stress PALTs is set as follows:

$$f(y)=\begin{cases}0 & \text{if} \quad y\le 0,\\ f_1(y) & \text{if} \quad 0<y\le\tau,\\ f_2(y) & \text{if} \quad y>\tau,\end{cases}$$

where $f_1(y)=cky^{(c-1)}(1+y^c)^{-(k+1)}$, $c,k>0$ which is given in Eq. (1), and $f_2(y)=$ $=\beta ck[\tau+\beta(y-\tau)]^{(c-1)}[1+\{\tau+\beta(y-\tau)\}^c]^{-(k+1)}$, $c,k>0$, $\beta>1$ is derived by the transformation variable technique using the equations given in (1) and (4).

When the pre-specified number of failures (r) is reached, the test ends immediately.

The observed lifetimes are expressed as $y_{(1)}\le\ldots\le y_{(n_u)}\le\tau\le y_{(n_u+1)}\le\ldots\le y_{(r)}$, where $r=n_u+n_a$ is the total number of failures. Two indictor functions can be then considered, namely, δ_{1i}, δ_{2i}, as

$$\delta_{1i}=\begin{cases}1 & y_{(i)}\le\tau\\ 0 & \text{otherwise}\end{cases} \qquad i=1,2,\dots,n$$

and

$$\delta_{2i}=\begin{cases}1 & y_{(i)}<\tau\le y_{(r)}\\ 0 & \text{otherwise}\end{cases} \qquad i=1,2,\dots,n.$$

The likelihood function of the lifetimes $y_1,\dots,y_n$ of n items under step-stress PALTs can be written as

$$L(y;\beta,c,k)=\prod_{i=1}^{n}\{f_1(y_i)\}^{\delta_{1i}}\{f_2(y_i)\}^{\delta_{2i}}\{R(y_{(r)})\}^{\bar\delta_{1i}\bar\delta_{2i}}=$$

$$=\prod_{i=1}^{n}\{cky^{c-1}(1+y_i^c)^{-(k+1)})\}^{\delta_{1i}}\times$$

$$\times\{\beta ck[\tau+\beta(y_i-\tau)]^{c-1}[1+(\tau+\beta(y_i-\tau))^c]^{-(k+1)}\}^{\delta_{2i}}\{1+[\tau+\beta(y_{(r)}-\tau)]^c\}^{-k\bar\delta_{1i}\bar\delta_{2i}},$$

where $\bar\delta_{1i}=1-\delta_{1i}$ and $\bar\delta_{2i}=1-\delta_{2i}$.

Practically, the maximization of the natural logarithm of the likelihood function is much simpler than that of the likelihood function itself. The natural logarithm of the likelihood function can be defined as

$$\ln L=n_0\ln c+n_0\ln k+(c-1)\left\{\sum_{i=1}^{n}\delta_{1i}\ln y_i+\sum_{i=1}^{n}\delta_{2i}\ln A\right\}+$$

$$+n_a\ln\beta-(k+1)\left\{\sum_{i=1}^{n}\delta_{1i}\ln(1+y_i^c)+\sum_{i=1}^{n}\delta_{2i}\ln(1+A^c)\right\}-k(n-n_0)\ln(1+D^c),$$

where

$$A = \tau + \beta(y_i - \tau), \qquad D = \tau + \beta(y_{(r)} - \tau), \qquad \sum_{i=1}^{n} \delta_{1i} = n_u,$$

$$\sum_{i=1}^{n} \delta_{2i} = n_a, \qquad \sum_{i=1}^{n} \bar{\delta}_{1i}\bar{\delta}_{2i} = n - n_u - n_a, \qquad n_0 = n_u + n_a.$$

The first partial derivatives of $\ln L$ with respect to (w.r.t.) β, c, and k are given by

$$\frac{\partial \ln L}{\partial \beta} = \frac{n_a}{\beta} - (c-1)\sum_{i=1}^{n} \delta_{2i}(y_i - \tau)A^{-1} - kc(n - n_0)D^{c-1}(y_{(r)} - \tau)(1 + D^c)^{-1} -$$

$$-(k+1)c\sum_{i=1}^{n} \delta_{2i} A^{c-1}(y_i - \tau)(1 + A^c)^{-1}, \tag{5}$$

$$\frac{\partial \ln L}{\partial c} = \frac{n_0}{c} + \sum_{i=1}^{n} \delta_{1i} \ln y_i + \sum_{i=1}^{n} \delta_{2i} \ln A - k(n - n_0)D^c \ln D(1 + D^c)^{-1} -$$

$$-(k+1)\left\{\sum_{i=1}^{n} \delta_{1i} y_i^c \ln y_i (1 + y_i^c)^{-1} + \sum_{i=1}^{n} \delta_{2i} A^c \ln A(1 + A^c)^{-1}\right\}, \tag{6}$$

and

$$\frac{\partial \ln L}{\partial k} = \frac{n_0}{k} - \sum_{i=1}^{n} \delta_{1i} \ln(1 + y_i^c) - \sum_{i=1}^{n} \delta_{2i} \ln(1 + A^c) - (n - n_0)\ln(1 + D^c).$$

Set $\frac{\partial \ln L}{\partial k} = 0$ we get

$$\frac{n_0}{k} = \sum_{i=1}^{n} \delta_{1i} \ln(1 + y_i^c) + \sum_{i=1}^{n} \delta_{2i} \ln(1 + A^c) + (n - n_0)\ln(1 + D^c).$$

The MLE of k is then given by

$$\hat{k} = \frac{n_0}{a_1},$$

where

$$a_1 = \sum_{i=1}^{n} \delta_{1i} \ln(1 + y_i^c) + \sum_{i=1}^{n} \delta_{2i} \ln(1 + A^c) + (n - n_0)\ln(1 + D^c).$$

Substitute $\hat{k}$ into Eqs. (5) and (6), then we have the following two nonlinear equations:

$$\frac{n_a}{\hat{\beta}} + (\hat{c} - 1)\sum_{i=1}^{n} \delta_{2i}(y_i - \tau)A^{-1} - \left(\frac{n_0}{a_1} + 1\right) a_4 - \frac{n_0}{a_1} a_5 = 0 \ldots \tag{7}$$

and

$$\frac{n_0}{\hat{c}} + \sum_{i=1}^{n} \delta_{1i} \ln y_i + \sum_{i=1}^{n} \delta_{2i} \ln A - \left(\frac{n_0}{a_1} + 1\right) a_2 - \frac{n_0}{a_1} a_3 = 0, \ldots \tag{8}$$

where

$$a_2 = \Sigma_{i=1}^{n} \delta_{1i} y_i^{\hat{c}} \ln y_i (1+y_i^{\hat{c}})^{-1} + \Sigma_{i=1}^{n} \delta_{2i} A^{\hat{c}} \ln A (1+A^{\hat{c}})^{-1},$$

$$a_3 = (n-n_0) D^{\hat{c}} \ln D (1+D^{\hat{c}})^{-1}, \qquad a_4 = \hat{c} \sum_{i=1}^{n} \delta_{2i} A^{\hat{c}-1} (y_i - \tau)(1+A^{\hat{c}})^{-1},$$

$$a_5 = (n-n_0) \hat{c} D^{\hat{c}-1} (y_{(r)} - \tau)(1+D^{\hat{c}})^{-1}.$$

It is shown that the MLEs of the model parameters cannot be obtained in a closed form. Hence, some iterative methods such as Newton–Raphson method are necessary.

Regarding the asymptotic Fisher information matrix, the elements can be found as the negative of the second partial derivatives of the natural logarithm of the likelihood function w.r.t. the parameters of the model as indicated below.

$$F = \begin{bmatrix} -\frac{\partial^2 \ln L}{\partial^2 \beta^2} & -\frac{\partial^2 \ln L}{\partial \beta \partial c} & -\frac{\partial^2 \ln L}{\partial \beta \partial k} \\ -\frac{\partial^2 \ln L}{\partial c \partial \beta} & -\frac{\partial^2 \ln L}{\partial^2 c^2} & -\frac{\partial^2 \ln L}{\partial c \partial k} \\ -\frac{\partial^2 \ln L}{\partial k \partial \beta} & -\frac{\partial^2 \ln L}{\partial k \partial c} & -\frac{\partial^2 \ln L}{\partial^2 k^2} \end{bmatrix}.$$

The elements of the above matrix are defined and expressed in more details in Appendix A.

The approximate $100(1-\gamma)\%$ confidence intervals for the model parameters β, c, and k can be, respectively, presented by

$$\hat{\beta} \pm Z_{\gamma/2} \sqrt{F_{11}^{-1}}, \qquad \hat{c} \pm Z_{\gamma/2} \sqrt{F_{22}^{-1}}, \qquad \hat{k} \pm Z_{\gamma/2} \sqrt{F_{33}^{-1}},$$

where $Z_{\gamma/2}$ is the upper $(\gamma/2)$ percentile of the standard normal distribution.

3. **Optimum Test Plan**. This section presents the optimal plans of step-stress PALTs. The optimal plans provide the most accurate estimates of life at use condition. Based on a certain optimality criterion, we can choose the best or optimal stress-change time. The D-optimality criterion is adopted in this paper. It is a more general optimality criterion. It takes into account the overall parameter space. It minimizes the generalized asymptotic variance (GAV) of MLEs of the model parameters.

The GAV of the MLEs of the model parameters is defined as the reciprocal of the determinant of the Fisher information matrix (F), see Bai et al. [10]. That is

$$GAV(\hat{\beta}, \hat{c}, \hat{k}) = \frac{1}{|F|}.$$

Now, the optimum test plan for products having Burr-XII distribution is to find the optimum stress-change time τ such that the GAV of the MLE of the model parameters at use condition is minimized. The minimization of the GAV over τ can be achieved by solving the following equation:

$$\frac{\partial GAV}{\partial \tau} = 0.$$

It can be then reduced to the equation below:

$$\frac{\partial |F|}{\partial \tau} = 0, \tag{9}$$

where $|F|$ and its derivative w.r.t. τ are defined in Appendix B.

The above equation can be solved by using an iterative method such as Newton–Raphson to obtain the optimal value of τ. Accordingly, the expected optimal number of failed items under use and accelerated conditions are expressed, respectively, as

$$nP_u = n(1-(1+(\tau^*)^{\hat{c}})^{-(\hat{k})},$$

$$nP_a = n(1+(\tau^*)^{\hat{c}})^{-(\hat{k})}(1-(1+(\hat{\beta}(y_{(r)}-\tau^*))^{\hat{c}})^{-(\hat{k})}),$$

where P_u is the probability that a tested item under use condition fails by $y_{(r)}$ and P_a is the probability that a tested item under accelerated condition fails by $y_{(r)}$.

4. **Simulation Studies**. Monte Carlo simulation studies are conducted to discuss the performance of the MLEs in terms of their mean square errors (MSE) for different choices of n, c, β, and k values based on failure censored data in step-stress PALTs. Also, the variance of the MLEs is computed and 95% approximate confidence intervals (ACI) of the model parameters are constructed. In addition, the optimal design results of the life test are discussed.

Average values of the MLEs with MSEs and 95% ACIs are obtained based on samples generated from Burr type-XII distribution of various combinations of true parameter values of c, β, and k. The used combinations are (0.5, 2, 1), (1.5, 2, 0.5), (1, 2, 0.5) and (0.4, 2, 0.5) with sample sizes set at 20, 25, 30, 40, 50, 75, and 100. The number of replications used for each sample size is 20,000.

Tables 1–4 show a summary of the estimation results when $\tau = 3$ and $r = 0.75n$. The estimates, MSEs, variance, lower and upper limits for c, β, and k are presented for different sample sizes with different combinations of the parameter values. As indicated from the results that the MSEs of the estimates become smaller as the sample size increases. Also, the confidence intervals become narrower as the sample size increases. That is, we obtain good estimates.

T a b l e 1

Average Values of the Estimates, MSEs, Variances, and Confidence Limits for $(c, \beta, k, \tau, r) = (0.5, 2, 1, 3, 0.75n)$

n	Parameters	Estimates	MSEs	Variance	Lower bound	Upper bound
1	2	3	4	5	6	7
20	c	0.5558331	0.02000864	0.0169	0.1673	0.6768
	β	1.4039627	0.79268474	0.4374	0.7038	3.2965
	k	1.0638589	0.07783380	0.0738	0.3924	1.4570
25	c	0.5510322	0.01529511	0.0127	0.3092	0.7508
	β	1.4554021	0.68603244	0.3895	0.7787	3.2251
	k	1.0533883	0.06258916	0.0597	0.5507	1.5089
30	c	0.5488667	0.01308809	0.0107	0.4173	0.8228
	β	1.4934548	0.60259759	0.3460	0.8471	3.1530
	k	1.0471862	0.05159331	0.0494	0.8183	1.6893

Continued Table 1

1	2	3	4	5	6	7
40	c	0.5443688	0.01093166	0.0090	0.4185	0.7898
	β	1.5471787	0.50230063	0.2973	0.7464	2.8837
	k	1.0387819	0.03989735	0.0384	0.3957	1.1638
50	c	0.5255432	0.007638101	0.0070	0.3532	0.6809
	β	1.5833846	0.431768628	0.2582	0.7769	2.7688
	k	1.0352212	0.030679060	0.0294	0.5132	1.1858
75	c	0.5130436	0.00311434	0.0030	0.3805	0.5973
	β	1.6434398	0.32589565	0.1988	1.1263	2.8740
	k	1.0204070	0.01569335	0.0153	0.7370	1.2215
100	c	0.5143938	0.002584185	0.0024	0.4208	0.6119
	β	1.6831689	0.263691937	0.1633	1.2079	2.7920
	k	1.0189076	0.012451442	0.0121	0.9435	1.3746

Table 2

Average Values of the Estimates, MSEs, Variances, and Confidence Limits for $(c, \beta, k, \tau, r) = (1.5, 2, 0.5, 3, 0.75n)$

n	Parameters	Estimates	MSEs	Variance	Lower bound	Upper bound
20	c	1.7570995	1.02382632	0.9578	0.1057	3.9421
	β	1.6075434	0.54472141	0.3907	0.8744	3.3247
	k	0.5183056	0.03163381	0.0313	0.0761	0.7697
25	c	1.6987884	0.47179510	0.4323	0.7509	3.3282
	β	1.6235957	0.47828527	0.3366	0.6884	2.9627
	k	0.5118737	0.02101283	0.0209	0.2081	0.7744
30	c	1.6563729	0.19159083	0.1671	1.1031	2.7057
	β	1.6501155	0.41601437	0.2936	1.0774	3.2015
	k	0.5126854	0.01617681	0.0160	0.1669	0.6630
40	c	1.6093054	0.11626641	0.1043	1.0507	2.3169
	β	1.6839737	0.33755963	0.2377	0.6734	2.5845
	k	0.5188141	0.01339067	0.0130	0.1798	0.6274
50	c	1.5847601	0.08622207	0.0790	1.4492	2.5513
	β	1.7033682	0.29343447	0.2055	0.7305	2.5073
	k	0.5228587	0.01213649	0.0116	0.2419	0.6644
75	c	1.5613185	0.051841188	0.0481	1.3211	2.1816
	β	1.7432533	0.217272645	0.1514	0.7280	2.2531
	k	0.5023715	0.007077984	0.0071	0.4288	0.7584
100	c	1.553086	0.038601260	0.0358	0.8031	1.5446
	β	1.771735	0.174691394	0.1226	1.1030	2.4755
	k	0.501235	0.004560504	0.0046	0.4978	0.7625

T a b l e 3

Average Values of the Estimates, MSEs, Variances, and Confidence Limits for $(c, \beta, k, \tau, r) = (1, 2, 0.5, 3, 0.75n)$

n	Parameters	Estimates	MSEs	Variance	Lower bound	Upper bound
20	c	1.143961	0.19859550	0.1829	0.8465	2.2847
	β	1.473192	0.65951500	0.3858	0.0975	2.5789
	k	0.547272	0.02754216	0.0255	0.1604	0.7871
25	c	1.132550	0.18771716	0.1702	0.4688	2.0858
	β	1.542232	0.55121442	0.3417	0.0643	2.3557
	k	0.517588	0.02098195	0.0207	0.0942	0.6578
30	c	1.1065001	0.11910894	0.1078	0.2643	1.5512
	β	1.5766455	0.48121523	0.3020	1.0132	3.1675
	k	0.5202578	0.01753665	0.0171	0.1536	0.6666
40	c	1.0854660	0.06071411	0.0534	0.6080	1.5131
	β	1.6165245	0.39292298	0.2459	1.0526	2.9964
	k	0.5209746	0.01438465	0.0139	0.3931	0.8567
50	c	1.0690090	0.04405074	0.0393	0.7131	1.4901
	β	1.6501520	0.3303979	0.2107	0.9175	2.7167
	k	0.5230048	0.01316781	0.0126	0.2061	0.6468
75	c	1.0414909	0.02231290	0.0206	0.7892	1.3517
	β	1.7027006	0.24449222	0.1561	1.2837	2.8325
	k	0.5022402	0.00602722	0.0060	0.2177	0.5219
100	c	1.0323358	0.015548694	0.0145	0.8172	1.2893
	β	1.7315798	0.202133419	0.1301	1.2973	2.7112
	k	0.5053709	0.004267453	0.0042	0.3273	0.5825

T a b l e 4

Average Values of the Estimates, MSEs, Variances, and Confidence Limits for $(c, \beta, k, \tau, r) = (0.4, 2, 0.5, 3, 0.75n)$

n	Parameters	Estimates	MSEs	Variance	Lower bound	Upper bound
1	2	3	4	5	6	7
20	c	0.4984639	0.25801551	0.2483	0.1562	0.9751
	β	1.3282733	0.94312057	0.4919	0.6108	3.9751
	k	0.5223291	0.03438470	0.0309	0.0210	0.6521
25	c	0.4688156	0.07526569	0.0705	0.0548	0.5304
	β	1.3756230	0.83089715	0.4411	0.8209	3.1791
	k	0.5187950	0.02588938	0.0255	0.5274	1.1364
30	c	0.4403062	0.02141981	0.0198	0.1047	0.5926
	β	1.4244297	0.72943502	0.3982	0.2477	2.6291
	k	0.5130699	0.01897816	0.0178	0.4190	0.9841

Continued Table 4

1	2	3	4	5	6	7
40	c	0.4259220	0.008332961	0.0077	0.2067	0.5810
	β	1.4821521	0.612243463	0.3441	0.6356	2.9481
	k	0.5098138	0.011472813	0.0114	0.2354	0.6897
50	c	0.4242359	0.006346346	0.0058	0.1401	0.4407
	β	1.5267159	0.527254038	0.3033	0.6688	2.8088
	k	0.5082884	0.009183374	0.0091	0.4831	0.8933
75	c	0.4220691	0.004551284	0.0041	0.3563	0.6062
	β	1.6037429	0.386799434	0.2298	1.0614	2.9405
	k	0.5083173	0.007267912	0.0072	0.2343	0.5669
100	c	0.4173501	0.004153889	0.0039	0.3669	0.6102
	β	1.6418385	0.323044804	0.1948	1.3501	2.8650
	k	0.5051016	0.006595488	0.0066	0.2836	0.6013

T a b l e 5

The Results of Optimal Design of Step-Stress PALTs for Different Sized Samples Based on the Results Presented in Table 1

n	τ^*	$n_u^* = nP_u$	$n_a^* = nP_a$	Optimal GAV
20	4.095819	11	4	0.00110
25	4.033014	13	6	0.00049
30	3.824391	16	7	0.00017
40	3.652942	20	10	0.00013
50	3.578992	26	12	0.00004
75	3.345305	37	19	0.000002
100	3.299687	49	26	0.000001

T a b l e 6

The Results of Optimal Design of Step-Stress PALTs for Different Sized Samples Based on the Results Presented in Table 2

n	τ^*	$n_u^* = nP_u$	$n_a^* = nP_a$	Optimal GAV
20	3.566034	11	4	0.00112
25	3.722411	13	6	0.00064
30	3.601075	16	7	0.00046
40	3.585876	20	10	0.00014
50	3.371908	25	13	0.00010
75	3.156042	35	21	0.00004
100	3.133222	46	29	0.00001

Moreover, Tables 5–8 show the results of the optimal design of the life test. The optimal stress-change time τ^* is determined to minimize the GAV of the MLEs of the model parameters. Also, the expected optimal numbers of failed items under use and accelerated conditions are obtained for each sample size. Moreover, the optimal GAV of the MLEs of the model parameters are optimally determined for each sample size. As shown from the results, the optimal GAV decreases as the sample size increases.

Table 7

The Results of Optimal Design of Step-Stress PALTs for Different Sized Samples Based on the Results Presented in Table 3

n	τ^*	$n_u^* = nP_u$	$n_a^* = nP_a$	Optimal GAV
20	3.98608	9	6	0.00037
25	3.68533	11	8	0.00035
30	3.56103	13	10	0.00023
40	3.42640	17	13	0.00012
50	3.55931	21	17	0.00005
75	3.31212	30	26	0.00002
100	3.21435	39	36	0.00001

Table 8

The Results of Optimal Design of Step-Stress PALTs for Different Sized Samples Based on the Results Presented in Table 4

n	τ^*	$n_u^* = nP_u$	$n_a^* = nP_a$	Optimal GAV
20	3.7044	6	9	0.002891
25	3.8967	8	11	0.000125
30	4.3621	10	13	0.000034
40	3.9035	13	17	0.000012
50	3.8494	15	22	0.000007
75	3.3324	22	34	0.000006
100	3.3432	29	46	0.000003

Conclusions. This paper considers the step-stress PALTs with failure-censored data from Burr type-XII distribution. Average values of the MLEs of the parameters and acceleration factor with MSEs and 95% ACIs are obtained based on samples generated from Burr type-XII distribution of various combinations of true parameter values. Moreover, optimum test plans were developed such that the GAV of the MLEs of the model parameters is minimized. These plans determine the optimal value of the stress change-point τ^* that minimizes the GAV. Accordingly, the quality of the statistical inference is improved. That is, more efficient MLEs of the model parameters are obtained.

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Appendix A.

Derivation of the second-order partial derivatives:

$$\frac{\partial^2 \ln L}{\partial \beta^2} = -\frac{n_a}{\beta^2} - (c-1)\sum_{i=1}^{n} \delta_{2i}(y_i - \tau)\frac{(y_i - \tau)}{(\tau + \beta(y_i - \tau))^2} -$$

$$-(k+1)c\sum_{i=1}^{n}(y_i - \tau)\delta_{2i}[(y_i - \tau)(c-1)A^{c-2}(1+A^c)^{-1} -$$

$$-(y_i - \tau)cA^{2(c-1)}(1+A^c)^{-2}] - kc(n-n_0)(y_{(r)} - \tau)\times$$

$$\times[(c-1)(y_{(r)} - \tau)D^{c-2}(1+D^c)^{-1} - (y_{(r)} - \tau)cD^{2(c-1)}(1+D^c)^{-2}] =$$

$$= -\frac{n_a}{\beta^2} - (c-1)\sum_{i=1}^{n} \delta_{2i}(y_i - \tau)^2 A^{-2} -$$

$$-(k+1)c\sum_{i=1}^{n}(y_i - \tau)^2 \delta_{2i}[(c-1)A^{c-2}(1+A^c)^{-1} - cA^{2(c-1)}(1+A^c)^{-2} -$$

$$-kc(n-n_0)(y_{(r)} - \tau)^2[(c-1)D^{c-2}(1+D^c)^{-1} - cD^{2(c-1)}(1+D^c)^{-2}],$$

$$\frac{\partial^2 \ln L}{\partial \beta \partial c} = \sum_{i=1}^{n} \delta_{2i}(y_i - \tau)A^{-1} - k(y_{(r)} - \tau)(n-n_0)\times$$

$$\times[D^{c-1}(1+D^c)^{-1} + c(1+D^c)^{-1}D^{c-1}\ln D - cD^{c-1}(1+D^c)^{-2}D^c \ln D] -$$

$$-(k+1)\sum_{i=1}^{n} \delta_{2i}(y_i - \tau)[A^{c-1}(1+A^c)^{-1} + c(1+A^c)^{-1}A^{c-1}\ln A -$$

$$-cA^{c-1}(1+A^c)^{-2}A^c \ln A],$$

$$\frac{\partial^2 \ln L}{\partial \beta \partial k} = -c\sum_{i=1}^{n} \delta_{2i}(y_i - \tau)A^{c-1}(1+A^c)^{-1} - (y_{(r)} - \tau)(n-n_0)cD^{c-1}(1+Dc)^{-1},$$

$$\frac{\partial^2 \ln L}{\partial c^2} = -\frac{n_0}{c^2} - k(n-n_0)\ln D[(1+D^c)^{-1}D^c \ln D - (1+D^c)^{-2}D^{2c}\ln D] -$$

$$-(k+1)\left[\sum_{i=1}^{n} \delta_{1i} \ln y_i \{(1+y_i^c)^{-1} y_i^c \ln y_i - y_i^{2c}(1+y_i^c)^{-2}\ln y_i\}\right] -$$

$$-(k+1)\left[\sum_{i=1}^{n} \delta_{2i} \ln A\{(1+A^c)^{-1}A^c \ln A - A^{2c}(1+A^c)^{-2}\ln A\}\right],$$

$$\frac{\partial^2 \ln L}{\partial c \partial k} = -\sum_{i=1}^{n} \delta_{1i} \ln y_i \{(1+y_i^c)^{-1} y_i^c \ln y_i\} - \sum_{i=1}^{n} \delta_{2i} \ln A (1+A^c)^{-1} A^c \ln A -$$

$$-(n-n_0)(1+D^c)^{-1} D^c \ln D,$$

and

$$\frac{\partial^2 \ln L}{\partial k^2} = -\frac{n_0}{k^2}.$$

Appendix B.

The determinant of F and its partial derivative w.r.t. τ. The determinant of F is given by

$$|F| = f_{11}(f_{22} f_{33} - f_{23}^2) - f_{12}(f_{12} f_{33} - f_{13} f_{23}) + f_{13}(f_{12} f_{23} - f_{13} f_{22}).$$

Its partial derivative w.r.t. τ is obtained as

$$\frac{\partial |F|}{\partial \tau} = f_{11}(f'_{22} f_{33} + f_{22} f'_{33} - 2 f_{23} f'_{23}) + f'_{11}(f_{22} f_{33} - f_{23}^2) -$$

$$-f_{12}(f'_{12} f_{33} + f_{12} f'_{33} - f'_{13} f_{23} - f_{13} f'_{23}) - f'_{12}(f_{12} f_{33} - f_{13} f_{23}) +$$

$$+f_{13}(f'_{12} f_{23} + f_{12} f'_{23} - f'_{13} f_{22} - f_{13} f'_{22}) + f'_{13}(f_{12} f_{23} - f_{13} f_{22}).$$

where

$$f'_{11} = (c-1)\sum_{i=1}^{n} \delta_{2i}[-2(y_i - \tau)A^{-2} - 2(y_i - \tau)^2 A^{-3}(1-\beta)] +$$

$$+(k+1)c\sum_{i=1}^{n}(c-1)\delta_{2i}[-2(y_i - \tau)A^{c-2}(1+A^c)^{-1} + (y_i - \tau)^2 \times$$

$$\times(1-\beta)((c-2)A^{c-3}(1+A^c)^{-1} - cA^{2c-3}(1+A^c)^{-2})] -$$

$$-(k+1)c\sum_{i=1}^{n} c\delta_{2i}[-2(y_i - \tau)A^{2(c-1)}(1+A^c)^{-2} + (y_i - \tau)^2 \times$$

$$\times 2(1-\beta)((c-1)A^{2c-3}(1+A^c)^{-2} - cA^{3(c-1)}(1+A^c)^{-3})] +$$

$$+kc(n-n_0)(c-1)[-2(y_{(r)} - \tau)D^{c-2}(1+D^c)^{-1} + (1-\beta)(Y_c - \tau)^2 \times$$

$$\times((c-2)D^{c-3}(1+D^c)^{-1} - cD^{2c-3}(1+D^c)^{-2})] -$$

$$-kc^2(n-n_0)[-2(y_{(r)} - \tau)D^{2(c-1)}(1+D^c)^{-2} + 2(y_{(r)} - \tau)^2(1-\beta) \times$$

$$\times((c-1)D^{2c-3}(1+D^c)^{-2} - D^{3(c-1)}c(1+D^c)^{-3})],$$

$$f'_{22} = k(n-n_0)(1-\beta)[2D^{c-1}(1+D^c)^{-1} \ln D - c(\ln D)^2 \times$$

$$\times(D^{2c-1}(1+D^c)^{-2} + D^{c-1}(1+D^c)^{-1})] -$$

$$-k(n-n_0)(1-\beta)[D^{2c-1}(1+D^c)^{-2}2\ln D+2c(1-\beta)(\ln D)^2\times$$

$$\times\{-D^{3c-1}(1+D^c)^{-3}+D^{2c-1}(1+D^c)^{-2}\}]+$$

$$+(k+1)(1-\beta)\sum_{i=1}^{n}\delta_{2i}[2A^{c-1}(1+A^c)^{-1}\ln A+c(\ln A)^2\times$$

$$\times\{-(1+A^c)^{-2}A^{2c-1}+(1+A^c)^{-1}A^{c-1}\}]-$$

$$-(k+1)(1-\beta)\sum_{i=1}^{n}\delta_{2i}[2A^{2c-1}(1+A^c)^{-2}\ln A+2c(\ln A)^2\times$$

$$\times\{(1+A^c)^{-2}A^{2c-1}-(1+A^c)^{-3}A^{3c-1}\},$$

$$f'_{33}=0,$$

$$f'_{23}=\sum_{i=1}^{n}\delta_{2i}(1-\beta)[2A^{c-1}(1+A^c)^{-1}\ln A+c(\ln A)^2\times$$

$$\times(1+A^c)^{-1}A^{c-1}-(1+A^c)^{-1}A^{2c-1}]+$$

$$+(n-n_0)(1-\beta)[-D^{2c-1}(1+D^c)^{-2}c\ln D+(1+D^c)^{-1}D^{c-1}\{c\ln D+1\}],$$

$$f'_{12}=\sum_{i=1}^{n}\delta_{2i}[A^{-1}+(1-\beta)(y_i-\tau)A^{-2}]+k(n-n_0)\{(1-\beta)(Y_c-\tau)\times$$

$$\times[D^{c-2}(c-1)(1+D^c)^{-1}-c(1+D^c)^{-2}D^{2(c-1)}]-D^{c-1}(1+D^c)^{-1}-$$

$$-c(1+D^c)^{-1}D^{c-1}\ln D+c(Y_c-\tau)(1-\beta)\{-D^{2(c-1)}c(1+D^c)^{-2}\ln D+$$

$$+D^{c-2}(1+D^c)^{-1}[1+(c-1)\ln D]\}+cD^{2c-1}(1+D^c)^{-2}\ln D-c(1-\beta)(Y_c-\tau)\times$$

$$\times[D^{2c-2}(1+D^c)^{-2}-2c(1+D^c)^{-3}D^{3c-2}\ln D+(2c-1)(1+D^c)^{-2}D^{2c-2}\ln D]\}+$$

$$+(k+1)\sum_{i=1}^{n}\delta_{2i}(y_i-\tau)\{(1-\beta)\times$$

$$\times[(c-1)A^{c-2}(1+A^c)^{-1}-cA^{2(c-1)}(1+A^c)^{-2}-c^2A^{2(c-1)}(1+A^c)^{-2}\ln A+$$

$$+cA^{c-2}(1+A^c)^{-1}(1+(c-1)\ln A)-cA^{2(c-1)}(1+A^c)^{-2}+$$

$$+2c^2A^{3c-2}(1+A^c)^{-3}\ln A-c(2c-1)A^{2c-2}(1+A^c)^{-2}\ln A\}-$$

$$-[A^{c-1}(1+A^c)^{-1}+c(1+A^c)^{-1}A^{c-1}\ln A-cA^{2c-1}(1+A^c)^{-2}\ln A],$$

$$f'_{13} = \sum_{i=1}^{n} \delta_{2i} c\{(y_i - \tau)(1-\beta)[(c-1)A^{c-2}(1+A^c)^{-1} - cA^{2(c-1)}(1+A^c)^{-2}] -$$

$$- A^{c-1}(1+A^c)^{-1}\} + (n-n_0)c[(y_{(r)} - \tau)(1-\beta)[(c-1)D^{c-2}(1+D^c)^{-1} -$$

$$- cD^{2(c-1)}(1+D^c)^{-2}] - D^{c-1}(1+D^c)^{-1}].$$

Р е з ю м е

Проведено оптимальне планування частково прискорених ресурсних досліджень при покроковій зміні напружень із використанням цензурованих у часі даних із розподілу Бурра XII типу. У рамках концепції максимальної імовірності оцінено коефіцієнт прискорення досліджень і параметри розподілу. Як критерій оптимізації частково прискорених ресурсних досліджень, що плануються, використано мінімізацію узагальненої асимптотичної дисперсії для показників максимальної імовірності параметрів розподілу. Ефективність запропонованого методу показано на прикладі чисельних розрахунків.

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