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On groups, whose non-normal subgroups are either contranormal or core-free

Presented by Corresponding Member of the NAS of Ukraine V.P. Motornyi

We investigate the influence of some natural types of subgroups on the structure of groups. A subgroup H of a group G is called contranormal in G, if $G = H^G$. A subgroup H of a group G is called core-free in G, if $Core_G(H) = \langle 1\rangle$. We study the groups, in which every non-normal subgroup is either contranormal or core-free. In particular, we obtain the structure of some monolithic and non-monolithic groups with this property.

Keywords: *normal subgroup, contranormal subgroup, core-free subgroup, monolithic group.*

Let G be a group. The following two normal subgroups are associated with any subgroup H of group G: H^G, the *normal closure* of H in a group G, the least normal subgroup of G including H, and $\mathrm{Core}_G(H)$, the *normal core* of H in G, the greatest normal subgroup of G which is contained in a subgroup H. We have

$$H^G = \langle H^x \mid x \in G\rangle$$

and

$$\mathrm{Core}_G(H) = \bigcap_{x\in G} H^x .$$

A subgroup H is normal, if and only if $H = H^G = \mathrm{Core}_G(H)$. In this sense, the subgroups H, for which $\mathrm{Core}_G(H) = \langle 1\rangle$, are the complete opposites to normal subgroups. A subgroup H of a group G is called *core-free* in G, if $\mathrm{Core}_G(H) = \langle 1\rangle$.

A subgroup H is normal, if and only if $H = H^G$. In this sense, the subgroups H, for which $H^G = G$, are the complete opposites to the normal subgroups. A subgroup H of a group G is called *contranormal* in G, if $G = H^G$. J.S. Rose has introduced the term "*a contranormal subgroup*" in paper [1].

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For each subgroup H of a group G, we have the following two extreme and opposite situations:

$H^G = H$ or $H^G = G$,

and, respectively,

$\mathrm{Core}_G(H) = H$ or $\mathrm{Core}_G(H) = \langle 1 \rangle$.

The following extreme cases immediately appear.

The first case: every proper subgroup of G is normal. Such group is called a *Dedekind group*. A Dedekind group G has the following structure: it is either Abelian, or $G = Q_8 \times D \times P$ where Q_8 is a quaternion group of order 8, D is an elementary Abelian 2-group and P is an Abelian $2'$-group [2].

The second case: every proper subgroup of G is core-free. In this case, group G does not include proper non-trivial normal subgroups, that is, G is a simple group. However, a simple group include normal subgroups: they are G and the trivial subgroup.

The third case: every proper non-trivial subgroup of G is contranormal. In this case, group G does not include proper non-trivial normal subgroups, so that again G is a simple group.

In the last two cases, we came only to simple groups. Note again that a simple group has the only three following types of subgroups: normal, core-free, and contranormal.

Therefore, the following question naturally appears: what can we say about the groups, whose subgroups are either normal, core-free or contranormal?

In the process of this study, the need of a separate study of the following two types of groups naturally arises: the groups, whose subgroups are either normal or core-free, and the groups, whose subgroups are either normal or contranormal.

Note that groups having only two types of subgroups, which are also antagonistic in some sense to each other, have been considered by many authors. Here, we provide the citation on some of them, whose subjects are to some extent related to this topic [3-13].

Groups, whose subgroups are either normal or core-free, have been studied in [14, 15].

The study of groups, whose subgroups are either normal or contranormal was initiated in paper [5]. In Theorem 2 of [5], basis structural features of such groups were shown.

A group G is called *quasisimple*, if a central factor-group $G/\zeta(G)$ is simple and $G = [G,G]$.

Let G be a quasisimple group, and let H be a non-trivial subgroup of G. If $\zeta(G)$ does not include H, then $H\zeta(G)/\zeta(G)$ is non-trivial. Since $G/\zeta(G)$ is simple, $H^G\zeta(G)/\zeta(G) = G/\zeta(G)$, that is $H^G\zeta(G) = G$. If we suppose that $H^G \neq G$, then G/H^G is Abelian, what is impossible. Hence, $H^G = G$. Thus, every subgroup of a quasisimple group is either normal or contranormal.

Theorem A. *Let G be a group, whose non-normal subgroups are contranormal. If G is not soluble, then G is simple or quasisimple group.*

Let D be a Prüfer 2-group, $D = \langle d_n \mid d_1^2 = 1, d_{n+1}^2 = d_n, n \in \mathbb{N} \rangle$, and Q be a quaternion group of order 8, $Q = \langle u_1 \rangle \langle u_2 \rangle$ where $|u_1| = |u_2| = 4$,

$\langle c \rangle = \langle u_1 \rangle \cap \langle u_2 \rangle$

and $c = [u_1, u_2]$. Consider the semidirect product $H = D \leftthreetimes Q$, where $C_Q(D) = \langle u_1 \rangle$ and $d^{u_2} = d^{-1}$ for all elements $d \in D$. Put $v = d_2 u_1^{-1}$, then

$$v^{u_2} = (d_2 u_1^{-1})^{u_2} = d_2^{u_2}(u_1^{-1})^{u_2} = d_2^{-1}(u_1^{u_2})^{-1} = d_2^{-1}(u_1^{-1})^{-1} = d_2^{-1}u_1 = (d_2 u_1^{-1})^{-1} = v^{-1}.$$

It shows that the subgroup $V = \langle v \rangle$ is normal in H. Put $Q_\infty = H/V$, $A = DV/V$, $y = u_2V$, then $A \cong D$ is clearly a normal Prüfer 2-subgroup of Q_∞, $|y| = |u_2| = 4$. We have $d_2V = u_1V$, so that

$$y^2 = (u_2V)^2 = u_2^2V = u_1^2V = d_2^2V \in A.$$

It follows that $\langle y \rangle \cap A = \Omega_1(A)$, and $a^y = a^{-1}$ for all elements $a \in A$. The constructed above group Q_∞ is called the *infinite generalized quaternion group*.

Theorem B. *Let G be a soluble group, whose non-normal subgroups are contranormal. Suppose that G is not a Dedekind group.*

If G is a p-group for some prime p, then p = 2, and G is a group of one of the following types of groups:

(i) *$G = D \rtimes \langle g \rangle$, where D is a normal divisible Abelian 2-subgroup, element g has order 2 or 4, $d^g = d^{-1}$ for every element $d \in D$;*

(ii) *$G = D\langle g \rangle$, where D is a normal divisible Abelian 2-subgroup, element g has order 2 or 4, $d^g = d^{-1}$ for every element $d \in D$, $D = A \times B$, where A is a Prüfer 2-subgroup, $g^2 \in \Omega_1(A)$, and $\langle A,g \rangle$ is an infinite generalized quaternion group.*

If G is a periodic group and $|\Pi(G)| \geqslant 2$, then G is a group of one of the following types:

(iii) *$G = S \rtimes \langle g \rangle$, where g is a p-element for some prime p, S is an Abelian Sylow p'-subgroup of G, $C_{\langle g \rangle}(S) = \langle g^p \rangle$, and every subgroup of S is G-invariant;*

(iv) *$G = D \rtimes \langle g \rangle$, where D is a normal Abelian subgroup, $D = S \times K$, where S is a Sylow $2'$-subgroup of G, K is a divisible 2-subgroup, element g has order 2 or 4, $d^g = d^{-1}$ for every element $d \in D$;*

(v) *$G = D\langle g \rangle$, where D is a normal Abelian subgroup, $D = S \times A \times B$, where S is a Sylow $2'$-subgroup of G, $A \times B$ is a divisible 2-subgroup, element g has order 2 or 4, $d^g = d^{-1}$ for every element $d \in D$, $g^2 \in \Omega_1(A)$, and $\langle A,g \rangle$ is an infinite generalized quaternion group.*

If G is a non-periodic group, then G is a group of one of following types:

(vi) *$G = D \rtimes \langle g \rangle$, an element g has order 2 or 4, and $x^g = x^{-1}$ for each element $x \in D$, $C_{\langle g \rangle}(D) = \langle g^2 \rangle$, $2 \notin \Pi(D)$, $D^2 = D$, and every subgroup of D is G-invariant;*

(vii) *$G = D \rtimes \langle g \rangle$, an element g has order 2 or 4, and $x^g = x^{-1}$ for each element $x \in D$, $C_{\langle g \rangle}(D) = \langle g^2 \rangle$, $D^2 = D$, $D = S \times B$, $2 \notin \Pi(B)$, S is a divisible Sylow 2-subgroup of D;*

($viii$) *$G = D\langle g \rangle$, an element g has order 2 or 4, and $x^g = x^{-1}$ for each element $x \in D$, $C_{\langle g \rangle}(D) = \langle g^2 \rangle$, $D^2 = D$, $D = S \times B$, where $2 \notin \Pi(B)$, $S = A \times C$ is a divisible Sylow 2-subgroup of D, $\langle g \rangle \cap (A \times B) = \langle 1 \rangle$, $g^2 \in \Omega_1(C)$ and $\langle C,g \rangle$ is an infinite generalized quaternion group.*

Further, at the study of groups, whose subgroups either are normal, core-free or contranormal, it is natural to assume that they include proper contranormal and proper core-free subgroups. In this paper, we considered such groups imposing an additional restriction that they are periodic and locally soluble. Their description decomposes to few natural parts.

Theorem C. *Let G be a group, whose subgroups are either normal or contranormal or core-free. If G is locally soluble, then G is a soluble group.*

Let G be a group and A be a normal subgroup of G. The intersection $\mathrm{Mon}_G(A)$ of all non-trivial G-invariant subgroups of A is called the *G-monolith* of A. If $\mathrm{Mon}_G(A)$ is not trivial, then subgroup A is called *G-monolithic*. If $A = G$, then we will say that A is the *monolith* of group G and denote it by $\mathrm{Mon}(G)$.

The following theorem is devoted to the monolithic case.

Theorem D. *Let G be a soluble periodic monolithic group, whose non-normal subgroups are contranormal or core-free. Suppose that G includes proper contranormal and core-free subgroups. Then G is a group of one of following types:*

(*i*) $G = D \leftthreetimes \langle v \rangle$, *where D is a normal Prüfer 2-subgroup,* $v^2 = 1$, $d^v = d^{-1}$ *for all elements* $d \in D$;

(*ii*) $G = M \leftthreetimes S$, *M is an elementary Abelian p-subgroup, p is a prime, S is a locally cyclic p'-subgroup,* $C_G(M) = M$, *and every complement to M in G is conjugate with S. In particular, if M is finite, then G is finite, and* $G = M \leftthreetimes S$, *where S is a cyclic Sylow p'-subgroup of G;*

(*iii*) $G = D \leftthreetimes \langle g \rangle$, *where D is a normal cyclic p-subgroup,* $|g| = q$ *is a prime,* $q < p$, $C_G(D) = D$;

(*iv*) $G = D \leftthreetimes \langle g \rangle$, *where D is a normal Prüfer p-subgroup,* $|g| = q$ *is a prime,* $q < p$, $C_G(D) = D$;

(*v*) $G = D \leftthreetimes \langle g \rangle$, *where D is an extraspecial p-subgroup, p is a prime,* $|g| = q$ *is a prime,* $q < p$, $q \neq 2$, *moreover,* $M = [D,D] = \zeta(D)$ *is a monolith of G, and every subgroup of D/M is G-invariant;*

(*vi*) $G = M \leftthreetimes K$, *where M is a finite elementary Abelian p-subgroup, p is an odd prime, K is a quaternion group of order 8, M is a minimal normal subgroup of G,* $C_G(M) = M$;

(*vii*) $G = M \leftthreetimes B$, *where M is a minimal normal elementary Abelian p-subgroup, p is an odd prime,* $B = K \leftthreetimes \langle u \rangle$ *where K is a normal Prüfer 2-subgroup,* $u^2 = 1$, $a^u = a^{-1}$ *for each* $a \in K$;

(*viii*) $G = M \leftthreetimes B$, *where M is a minimal normal elementary Abelian p-subgroup, p is an odd prime, B is an infinite generalized quaternion group;*

(*ix*) $G = M \leftthreetimes V$, *where M is a minimal normal elementary Abelian p-subgroup, p is a prime,* $V = D_1 \leftthreetimes \langle g \rangle$ *where D_1 is a locally cyclic p'-subgroup,* $|g| = p$, *every subgroup of D_1 is $\langle g \rangle$-invariant,* $C_V(D_1) = D_1$;

(*x*) $G = M \leftthreetimes V$, *where M is a minimal normal elementary Abelian p-subgroup, p is a prime,* $V = D_1 \leftthreetimes \langle g \rangle$, *where D_1 is a locally cyclic subgroup, g is a q-element, q is an odd prime,* $p, q \notin \Pi(D_1)$, $C_{\langle g \rangle}(D_1) = \langle g^q \rangle$, *and every subgroup of D_1 is $\langle g \rangle$-invariant;*

(*xi*) $G = M \leftthreetimes V$, *where M is a minimal normal elementary Abelian p-subgroup, p is a prime,* $V = D_1 \leftthreetimes \langle g \rangle$, *where D_1 is a locally cyclic subgroup, g is a 2-element,* $2, p \notin \Pi(D_1)$, $C_{\langle g \rangle}(D_1) = \langle g^2 \rangle$, $x^g = x^{-1}$ *for each element* $x \in D_1$;

(*xii*) $G = M \leftthreetimes V$, *where M is a minimal normal elementary Abelian p-subgroup, p is a prime,* $V = (S \times K) \leftthreetimes \langle g \rangle$, *where $S \times K$ is a locally cyclic subgroup, moreover, K is a Prüfer 2-subgroup, S is a $2'$-subgroup,* $|g| = 2$, $x^g = x^{-1}$ *for each element* $x \in S \times K$;

(*xiii*) $G = M \leftthreetimes V$, *where M is a minimal normal elementary Abelian p-subgroup, p is a prime,* $V = S \leftthreetimes (K\langle g \rangle)$, *where S is a locally cyclic $2'$-subgroup, K is a Prüfer 2-subgroup,* $|g| = 4$, $g^2 \in \Omega_1(K)$, $\langle K,g \rangle$ *is an infinite generalized quaternion group,* $C_V(S) = S \times K$, $x^g = x^{-1}$ *for each element* $x \in S \times K$.

Finally, the last theorem considers the non-monolithic case.

Theorem E. *Let G be a soluble periodic non-monolithic group, whose non-normal subgroups are either contranormal or core-free. Suppose that G includes proper contranormal and core-free subgroups. Then G is a group of one of the following types:*

(*i*) $G = A \leftthreetimes \langle v \rangle$, *where A is a normal divisible 2-subgroup,* $v^2 = 1$, $a^v = a^{-1}$ *for all elements* $a \in A$;

(*ii*) $G = M \leftthreetimes (\langle c \rangle \times \langle g \rangle)$, *where M is a normal subgroup, having prime order* $p \neq 2$, $|c| = s$ *is a prime,* $|g| = q$ *is a prime,* $q \neq s$, *q divides* $p - 1$, $C_G(M) = M \times \langle c \rangle$;

(iii) $G = M \leftthreetimes (\langle c \rangle \times \langle g \rangle)$, *where* M *is a normal subgroup of prime order* $p \neq 2$, $|c| = |g| = q$ *is a prime,* q *divides* $p - 1$, $C_G(M) = M \times \langle c \rangle$;

(iv) $G = M \leftthreetimes \langle g \rangle$, *where* M *is a normal subgroup of prime order* $p \neq 2$, g *is an element of order* q, q *is a prime,* q *divides* $p - 1$, $C_G(M) = M$;

(v) $G = [G,G] \leftthreetimes \langle g \rangle$, *where* $[G,G]$ *is a normal cyclic* p*-subgroup, where* p *is an odd prime,* $\langle g \rangle$ *is a cyclic* q*-subgroup,* q *is a prime,* $C_G([G,G]) = [G,G]$;

(vi) $G = [G,G] \leftthreetimes \langle g \rangle$, *where* $[G,G]$ *is a normal Prüfer* p*-subgroup, where* p *is an odd prime,* $\langle g \rangle$ *is a cyclic* q*-subgroup,* q *is a prime,* $C_G([G,G]) = [G,G]$;

(vii) $G = (\langle a_1 \rangle \times \langle a_2 \rangle) \leftthreetimes \langle g \rangle$, *where* $|a_1| = |a_2| = q$, $|g| = p$, p *is a prime,* $p < q$, $C_G([G,G]) = [G,G]$, $a_1^g = a_1^m$, $a_2^g = a_2^s$, $1 \leqslant m, s < q$, $m \neq s$;

(viii) $G = [G,G] \leftthreetimes \langle g \rangle$, *where* g *is an element of order* p, p *is a prime,* $p < q$, $[G,G]$ *is an Abelian Sylow* q*-subgroup of* G, $C_G([G,G]) = [G,G]$, *and every subgroup of* $[G,G]$ *is* G*-invariant;*

(ix) $G = S \leftthreetimes \langle g \rangle$, *where* g *is an element of order* 2, S *is an Abelian* $2'$*-subgroup,* $C_G(S) = S$, *and* $x^g = x^{-1}$ *for every element* $x \in S$;

(x) $G = S \leftthreetimes P$, *where* P *is a Sylow* 2*-subgroup of* G *and* S *is an Abelian* $2'$*-subgroup,* $P = P_1 \leftthreetimes \langle g \rangle$, *where* D *is a normal divisible Abelian* 2*-subgroup,* $[S,P_1] = \langle 1 \rangle$, $|g| = 2$, *and* $x^g = x^{-1}$ *for every element* $x \in S \times P_1$;

(xi) $G = S \leftthreetimes \langle g \rangle$, *where* $|g| = p$, *where* p *is the least prime of the set* $\Pi(G)$, S *is an Abelian Sylow* p'*-subgroup of* G, $C_G(S) = S$, *and every subgroup of* S *is* G*-invariant.*

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ПРО ГРУПИ, ПІДГРУПИ ЯКИХ АБО НОРМАЛЬНІ,
АБО КОНТРАНОРМАЛЬНІ, АБО ВІЛЬНІ ВІД ЯДРА

Досліджено вплив деяких типів підгруп на структуру груп. Підгрупу H групи G називаємо контранормальною в G, якщо $G = H^G$. Підгрупу H групи G називаємо вільною від ядра в G, якщо $\mathrm{Core}_G(H) = \langle 1 \rangle$. Вивчено групи, в яких кожна підгрупа або нормальна, або контранормальна, або вільна від ядра. Точніше, одержано будову деяких монолітичних та немонолітичних груп з цією властивістю”.

Ключові слова: *нормальна підгрупа, контранормальна підгрупа, вільна від ядра підгрупа, монолітична група.*