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Planning Step-Stress Life Tests for the Generalized Rayleigh Distribution under Progressive Type-II Censoring with Binomial Removals

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Планирование испытаний на долговечность при ступенчатой нагрузке на основе обобщенного распределения Рэлея при прогрессивном цензурировании типа II с биномиальными выборками

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Рассмотрены параметр оценки и оптимальное проектирование частично ускоренных испытаний на долговечность при ступенчатой нагрузке на основе обобщенного рэлеевского распределения при прогрессивном цензурировании типа II с биномиальными выборками. В качестве фактора ускорения используются максимальные оценки вероятности параметров масштаба и формы, которые согласуются между собой. Построены приближенные доверительные интервалы параметров модели и рассчитаны границы вероятности. Разработаны оптимальные планы испытаний для улучшения статистического анализа. Предложены результаты моделирования и числовой пример.

Ключевые слова: статистический анализ, оптимальный план испытаний, ступенчатая нагрузка, максимальная вероятность, обобщенное рэлеевское распределение, случайные выборки.

Introduction. Burr [1] introduced twelve families of distributions for modeling lifetime data. Among those families, Burr type X and Burr type XII have received the most attention. The Burr-type X distribution is also known as the generalized Rayleigh distribution (GRD). According to Burr [1], the probability density function (pdf), cumulative distribution function (cdf) and hazard function of the two-parameter GRD are defined, respectively, as below:

$$f(y;\,\alpha,\,\theta)=(2\theta/\alpha^2)ye^{-(y/\alpha)^2}(1-e^{-(y/\alpha)^2})^{\theta-1}, \tag{1}$$

$$F(y;\,\alpha,\,\theta)=(1-e^{-(y/\alpha)^2})^{\theta}, \tag{2}$$

$$h(y;\,\alpha,\,\theta)=\frac{(2\theta/\alpha^2)\,ye^{-(y/\alpha)^2}(1-e^{-(y/\alpha)^2})^{\theta-1}}{1-(1-e^{-(y/\alpha)^2})^{\theta}}, \tag{3}$$

$$y>0, \qquad \alpha>0, \qquad \theta>0,$$

where θ and α are the shape and scale parameters, respectively. If $\theta = 1$, the GRD reduces to the traditional Rayleigh distribution. As indicated by [2–5], the GRD has been studied in many papers. Also, Surles and Padgett [6] showed that the two-parameter GRD can be used quite effectively in modeling strength data and also modeling general lifetime data.

As shown by Burr [1], if $\theta \leq 1/2$, the GRD has a decreasing pdf and a bathtub-type hazard function. But, when $\theta > 1/2$, the pdf is a right-skewed unimodal function and the hazard function is an increasing function. The two-parameter GRD has several properties commonly happened in the two-parameter gamma, Weibull and generalized exponential distributions. However, when $\theta > 1/2$, the hazard function behaves more close to the hazard function of Weibull with shape parameter greater than 1. Similar to the generalized exponential distribution and Weibull distribution, the GRD has a closed form of cdf and is very popular for dealing with censored data. Readers can refer to [4] and [7] for more detailed information about the comparison among these distributions.

In reference to the literature, there is no research work about optimum partially accelerated life test plans for the GRD under progressive type-II censoring scheme with random removals.

Because of continual improvement in manufacturing design, one often deals with products that are highly reliable with a substantially long life span. In these situations, the standard life testing methods may require time-consuming and prohibitively expensive testing time to obtain enough failure data necessary to make the desired inference. In order to assure rapid failure and then to shorten the testing period, all or some of test units may be subjected to stress conditions more severe than normal ones. Such accelerated life testing (ALT) or partially accelerated life testing (PALT) results in shorter lives than would be observed under normal operating conditions. In ALT test units are run only at accelerated conditions, while in PALT they are run at both normal (use) and accelerated conditions.

As [8] indicates, the stress can be applied in various ways, commonly used methods are step-stress and constant-stress. Under step-stress PALT (SSPALT), a test item is first run at use condition and, if it does not fail for a specified time, then it is run at accelerated condition until failure occurs or the observation is censored. But the constant-stress PALT runs each item at either use condition or accelerated condition only, i.e., each unit is run at a constant-stress level until the test is terminated. Accelerated test stresses involve higher than usual temperature, voltage, pressure, load, humidity, …, etc., or some combination of them. The objective of a PALT is to collect more failure data in a limited time without necessarily using high stresses to all test units.

As shown from the literature, for example see [9–19], PALT has been studied under step-stress scheme by several authors. In ALT or PALT, tests are often stopped before all units fail. The estimate from the censored data is less accurate than those from complete data. However, this is more than offset by the reduced test time and expense. The most common censoring schemes are type-I and type-II censoring. Consider n units placed on life test. In conventional type-I censoring, the experiment continues up to a prespecified time, T. Any failures that occur after that time are not observed. The termination point T of the experiment is assumed to be s-independent of the failure times. But in conventional type-II censoring, the experimenter terminates the experiment after a prespecified number of units $m \leq n$ fail. In this scenario, only the smallest lifetimes are observed. In type-I censoring, the number of failures observed is random and the endpoint of the experiment is fixed. While the number of failures is fixed in type-II censoring and the endpoint is random.

According to [20–24], numerous articles in the literature have dealt with inference under type-I and type-II censoring for various parametric families of distributions. Conventional type-I and type-II censoring schemes do not allow removal of units at points other than the terminal point of the experiment. This paper considers a generalized censoring scheme which is progressive type-II censoring to save more time and cost associated with testing. It allows for the surviving units to be removed from the test at each failure time. This type of censoring will be described in the next Section.

The main purpose of this paper is to study the optimal design problem of SSPALT for units whose lifetimes follow the generalized Rayleigh distribution under progressive type-II censoring scheme with binomial removals. The rest of this paper is organized as follows: In Section 1 the test procedure and its assumptions are presented. Point and interval estimations of the model parameters are considered in Section 2. In Section 3, optimum SSPALT plans are developed under different progressive censoring schemes. In Section 4 simulation studies are conducted for illustrative purposes. Moreover, in Section 5 data analysis via a numerical example is provided. Finally, Conclusions involve some important notes and potential effort required in this track.

1. **Test Procedure and Its Assumptions**. As indicated by many authors, see for example [25], the type-II progressive censoring scheme has received considerable interest among the statisticians. It is a generalization of type-II censoring. Although progressively type-II censored sampling is effective in time and cost, it is not very popular in lifetime experiment. It may be due to the complicated calculation of the likelihood function.

According to [26], if an experimenter desires to remove live units at points other than the final termination point of the life test, the traditional type II censoring scheme will not be of use to the experimenter. Type II censoring does not allow for units to be lost or removed from the test at points other than the final termination point. This allowance will be desirable, as in the case of studies of wear, in which the study of the actual aging process requires units to be fully disassembled at different stages in the experiment. Intermediate removal may also be desirable when a compromise between reduced time of experimentation and the observation of at least some extreme lifetimes is sought, or when some of the surviving units in the experiment that are removed early on can be used for some other tests. As in the case of accidental breakage of experimental units or loss of contact with individuals under study, the loss of test units at points other than the termination point may also be unavoidable. These reasons lead us directly into the area of progressive censoring.

This censoring scheme can be described as follows. Suppose that n units are placed on a life test and the experimenter decides beforehand the quantity m, the number of units to be failed. Now at the time of the first failure, R_1 of the remaining $n-1$ surviving units are randomly removed from the experiment. Continuing on, at the time of the second failure, R_2 of the remaining $n-R_1-2$ units are randomly withdrawn from the experiment. Finally, at the time of the mth failure, all the remaining $R_m = n-m-R_1-\ldots-R_m-1$ surviving units are removed from the experiment. Some of the earlier work on progressive censoring was conducted by [27–29]. Recently, several articles have been published on estimating the parameters for different distribution functions, see, for example, [30–35]. A recent account on progressive censoring schemes can be found in the excellent review article introduced by [36]. In progressive type-II censoring, if $R_1 = R_2 = \ldots = R_m - 1 = R_m = 0$, then $n = m$ which is the complete sampling case. But if $n-m-R_1-\ldots-R_m-1 = 0$, then $R_m = n-m$ which is the case of conventional type-II right censoring scheme.

In many reliability experiments, the pattern of removal at each failure is random. We assume that any test unit being dropped out from the life test is independent of the others but with the same removal probability (as a binomial parameter), p. Then, the number of test units removed at each failure time has a binomial distribution, see for example [37]. As indicated in [38], when such conditions are not satisfied or can’t be assumed, one can use another distribution. In some cases, it can be assumed that the pattern of removal at each failure is fixed. But the random assumption is more realistic than the fixed one.

Now, the following assumptions are considered:

(i) n identical and independent items are put on the life test;

(ii) the lifetime of each item has the GRD;

(iii) the test is finished at the mth failure, where m is pre-specified ($m \le n$);

(iv) each of the n units is first run under design condition. If it does not fail or eliminate from the test by a pre-specified time τ, it is put under severe condition;

(v) at the ith failure a random number of the remaining items, R_i, $i = 1, 2, ..., m-1$, are randomly chosen and eliminated from the test. Finally, at the mth failure, the living units $R_m = n - m - \sum_{i=1}^{m-1} R_i$ are all eliminated from the test and the test is finished;

(vi) assume that an individual item being eliminated from the test is independent of the others but with the same removal probability p. Then, the number of items eliminated at each failure time follows a binomial distribution. That is, $R_1 \sim \text{binomial}(n-m, p)$.

For $i = 2, 3, \ldots, m-1$, $R_i \sim \text{binomial}(n - m - \sum_{j=1}^{i-1} r_j, p)$ and $r_m = n - m - r_1 - r_2 - \ldots - r_{m-1}$;

(vii) the lifetime, say Y, of an item tested under SSPALT can be expressed by

$$Y = \begin{cases} T & \text{if } T \le \tau, \\ \tau + (T-\tau)/\beta & \text{if } T > \tau, \end{cases} \tag{4}$$

where T is the lifetime of the item under use condition, τ is the stress change time and β is the acceleration factor, $\beta > 1$. Therefore, the pdf of Y can be given by

$$f(y) = \begin{cases} 0, & y \le 0, \\ f_1(y) \equiv f(y; \alpha, \theta), & 0 < y \le \tau, \\ f_2(y), & y > \tau, \end{cases}$$

where

$$f_2(y) = f(y; \beta, \alpha, \theta) = \beta(2\theta/\alpha^2)[\tau + \beta(y-\tau)]e^{-([\tau+\beta(y-\tau)]/\alpha)^2}(1 - e^{-([\tau+\beta(y-\tau)]/\alpha)^2})^{\theta-1},$$

which is found by the transformation-variable procedure using $f_1(y)$ and the model set in (4).

Let $(y_i, r_i, \delta_{1i}, \delta_{2i})$, $i = 1, 2, \ldots, m$, denote the observation obtained from a progressively type-II censored sample with random removals in a SSPALT. Here $y_{(1)} \le \ldots \le y_{(m)}$.

Given the pre-fixed number of removals $R = (R_1 = r_1, \ldots, R_{m-1} = r_{m-1})$, the conditional likelihood function of the observations $\mathbf{y} = \{(y_i, r_i, \delta_{1i}, \delta_{2i}), i = 1, 2, \ldots, m\}$ takes the form

$$L_1(y;\ \alpha, \beta, \delta_{1i}, \delta_{2i} | \mathbf{R} = \mathbf{r}) = \prod_{i=1}^{m} \{[f_1(y_i)(S_1(y_i))^{r_i}]^{\delta_{1i}} [f_2(y_i)(S_2(y_i))^{r_i}]^{\delta_{2i}}\}, \tag{5}$$

where

$$S_1(y) = 1 - (1 - e^{-(y/\alpha)^2})^{\theta}, \qquad S_2(y) = 1 - (1 - e^{-([\tau+\beta(y-\tau)]/\alpha)^2})^{\theta}.$$

As shown earlier, the number of items eliminated at each failure time follows a binomial distribution. Then,

$$P(R_1 = r_1) = \binom{n-m}{r_1} p^r (1-p)^{n-m-r_1}.$$

For $i = 2, 3, \ldots, m-1$,

$$P(R_i = r_i | R_{i-1} = r_{i-1}, \ldots, R_1 = r_1) = \binom{n-m-\sum_{j=1}^{i-1} r_j}{r_i} p^{r_i} (1-p)^{n-m-\sum_{j=1}^{i} r_j},$$

where $0 \le r_i \le n-m-(r_1 + \ldots + r_{i-1})$. Moreover, assume that R_i is independent of Y_i for all i. Then the full likelihood function can be established as

$$L(y;\ \alpha,\ \beta,\ p) = L_1(y;\ \alpha,\ \beta | \mathbf{R} = \mathbf{r}) P(\mathbf{R} = \mathbf{r}).$$

Here, $\mathbf{R} = (R_1, R_2, \ldots, R_m)$ and $\mathbf{r} = (r_1, r_2, \ldots, r_m)$, and

$$P(\mathbf{R} = \mathbf{r}) = P(R_{m-1} = r_{m-1}, R_{m-2} = r_{m-2}, \ldots, R_1 = r_1) =$$

$$= P(R_{m-1} = r_{m-1} | R_{m-2} = r_{m-2}, \ldots, R_1 = r_1) P(R_{m-2} = r_{m-2} | R_{m-3} = r_{m-3}, \ldots, R_1 = r_1) \times \ldots$$

$$\ldots \times P(R_2 = r_2 | R_1 = r_1) P(R_1 = r_1).$$

That is,

$$P(\mathbf{R} = \mathbf{r}) = \frac{(n-m)!}{(n-m-\sum_{i=1}^{m-1} r_i)! \prod_{i=1}^{m-1} r_i!} p^{\sum_{i=1}^{m-1} r_i} (1-p)^{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i) r_i}.$$

2. **Parameter Estimation**. This section deals with the process of finding the maximum likelihood estimates (MLE) of the parameters θ, α, and β based on progressively type-II censored data with binomial removals. Both point and interval estimations of the parameters are considered.

2.1. ***Point Estimation***. In this subsection, the MLEs of the model parameters are considered. From (5) we obtain the natural logarithm of the conditional likelihood function, $\ln L_1$, as follows:

$$\ln L_1 = 2m \ln \theta - 2m \ln \alpha + m_a \ln \beta + \sum_{i=1}^{m_u} \ln y_i - (1/\alpha^2) \sum_{i=1}^{m_u} y_i^2 +$$

$$+ (\theta - 1) \sum_{i=1}^{m_u} \ln[1 - e^{-(y_i/\alpha)^2}] + \sum_{i=1}^{m_u} r_i \ln\{1 - (1 - e^{-(y_i/\alpha)^2})^\theta\} + \sum_{i=1}^{m_a} \ln \psi_i - (1/\alpha^2) \sum_{i=1}^{m_a} \psi_i^2 +$$

$$+ (\theta - 1) \sum_{i=1}^{m_a} \ln[1 - e^{-(\psi_i/\alpha)^2}] + \sum_{i=1}^{m_a} r_i \ln\{1 - (1 - e^{-(\psi_i/\alpha)^2})^\theta\},$$

where $\psi_i = \tau + \beta(y_i - \tau)$.

Since $P(R;\ p)$ does not depend on the parameters θ, α, and β, hence the MLEs of θ, α, and β can be derived by maximizing $\ln L_1$ directly. Similarly, since $L_1(y;\ \theta, \alpha, \beta | R = r)$ does not include the binomial parameter p, the MLE of p can be obtained by maximizing $P(R;\ p)$ directly. In particular, the MLEs of θ, α, and β can be obtained by solving the following equations:

$$\frac{\partial \ln L}{\partial \theta}=\frac{2m}{\theta}+\sum_{i=1}^{m_u}\ln[1-e^{-(y_i/\alpha)^2}]-\sum_{i=1}^{m_u}\frac{r_i[1-e^{-(y_i/\alpha)^2}]^{\theta}\ln(1-e^{-(y_i/\alpha)^2})}{1-[1-e^{-(y_i/\alpha)^2}]^{\theta}}+$$

$$+\sum_{i=1}^{m_a}\ln[1-e^{-(\psi_i/\alpha)^2}]-\sum_{i=1}^{m_a}\frac{r_i[1-e^{-(\psi_i/\alpha)^2}]^{\theta}\ln(1-e^{-(\psi_i/\alpha)^2})}{1-[1-e^{-(\psi_i/\alpha)^2}]^{\theta}}=0,$$

$$\frac{\partial \ln L}{\partial \alpha}=-\frac{2m}{\alpha}+\frac{2}{\alpha^3}\left[\sum_{i=1}^{m_u}y_i^2-(\theta-1)\sum_{i=1}^{m_u}\frac{y_i^2e^{-(y_i/\alpha)^2}}{1-e^{-(y_i/\alpha)^2}}+\right.$$

$$\left.+\,\theta\sum_{i=1}^{m_u}\frac{r_i\,y_i^2e^{-(y_i/\alpha)^2}[1-e^{-(y_i/\alpha)^2}]^{\theta-1}}{1-[1-e^{-(y_i/\alpha)^2}]^{\theta}}\right]+\frac{2}{\alpha^3}\left[\sum_{i=1}^{m_a}\psi_i^2-(\theta-1)\sum_{i=1}^{m_a}\frac{\psi_i^2e^{-(\psi_i/\alpha)^2}}{1-e^{-(\psi_i/\alpha)^2}}+\right.$$

$$\left.+\,\theta\sum_{i=1}^{m_a}\frac{r_i\psi_i^2e^{-(\psi_i/\alpha)^2}[1-e^{-(\psi_i/\alpha)^2}]^{\theta-1}}{1-[1-e^{-(\psi_i/\alpha)^2}]^{\theta}}\right]=0,$$

$$\frac{\partial \ln L}{\partial \beta}=\frac{m_a}{\beta}+\sum_{i=1}^{m_a}\frac{(y_i-\tau)}{\psi_i}-\frac{2}{\alpha^2}\sum_{i=1}^{m_a}(y_i-\tau)\psi_i+$$

$$+\frac{2(\theta-1)}{\alpha}\sum_{i=1}^{m_a}\frac{(y_i-\tau)\psi_ie^{-(\psi_i/\alpha)^2}}{1-e^{-(\psi_i/\alpha)^2}}-\frac{2\theta}{\alpha}\sum_{i=1}^{m_a}\frac{r_i[1-e^{-(\psi_i/\alpha)^2}]^{\theta-1}(y_i-\tau)\psi_ie^{-(\psi_i/\alpha)^2}}{1-[1-e^{-(\psi_i/\alpha)^2}]^{\theta}}=0.$$

Now, we have a nonlinear system of equations. It is very difficult to obtain a closed form solution. The Newton–Raphson algorithm is applied to obtain the MLEs of the unknown parameters numerically.

Independently, the MLE of the binomial parameter p can be found by solving the following equation:

$$\frac{\partial \ln L}{\partial p}=\frac{\sum_{i=1}^{m-1}r_i}{p}-\frac{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i}{1-p}=0.$$

Therefore, we find immediately

$$p=\frac{\sum_{i=1}^{m-1}r_i}{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i-1)r_i}.$$

2.2. ***Interval Estimation***. Based on the asymptotic distributions of the MLEs of the elements of the vector of unknown parameters $\Omega=(\theta,\,\alpha,\,\beta)$, the approximate confidence intervals of the parameters can be derived. It is known that the asymptotic distribution of the MLEs of Ω is given by [39],

$$((\hat{\theta}-\theta),\,(\hat{\alpha}-\alpha),\,(\hat{\beta}-\beta))\to N(0,\,\mathbf{I}^{-1}(\theta,\,\alpha,\,\beta)),$$

where $\mathbf{I}^{-1}(\theta, \alpha, \beta)$ is the variance-covariance matrix of the unknown parameters $\Omega = (\theta, \alpha, \beta)$. The elements of the 3×3 matrix $\mathbf{I}^{-1}$, $I_{ij}(\theta, \alpha, \beta)$, $i, j = 1, 2, 3$; can be approximated by $I_{ij}(\hat{\theta}, \hat{\alpha}, \hat{\beta})$, where

$$I_{ij}(\hat{\Omega}) = -\frac{\partial^2 \ln L(\Omega)}{\partial\Omega_i \partial\Omega_j}\bigg|_{\Omega=\hat{\Omega}}.$$

Now, we get the following:

$$\frac{\partial^2 \ln L}{\partial\theta^2} = -\frac{2m}{\theta^2} - \sum_{i=1}^{m_u} \frac{r_i[\psi_{ui}^{\theta}(\ln\psi_{ui})^2(1-\psi_{ui}^{\theta}) + (\psi_{ui}^{\theta}\ln\psi_{ui})^2]}{(1-\psi_{ui}^{\theta})^2} -$$

$$-\sum_{i=1}^{m_a} \frac{r_i[\psi_{ai}^{\theta}(\ln\psi_{ai})^2(1-\psi_{ai}^{\theta}) + (\psi_{ai}^{\theta}\ln\psi_{ai})^2]}{(1-\psi_{ai}^{\theta})^2},$$

where $\psi_{ui} = 1 - e^{-\alpha y_i}$ and $\psi_{ai} = 1 - e^{-\alpha\psi_i}$,

$$\frac{\partial^2 \ln L}{\partial\alpha^2} = \frac{2m}{\alpha^2} + \frac{2}{\alpha^6}\Bigg\{-(\theta-1)\sum_{i=1}^{m_u} \frac{2y_i^4 e^{-(y_i/\alpha)^2}\psi_{ui}^2 + 2y_i^2 e^{-2(y_i/\alpha)^2}}{\psi_{ui}^2} +$$

$$+\theta\sum_{i=1}^{m_u} \frac{r_i y_i^2}{(1-\psi_{ui}^{\theta})^2}\Big[(1-\psi_{ui}^{\theta})[-2y_i^2 e^{-(y_i/\alpha)^2}\psi_{ui}^{\theta-1} - (\theta-1)2y_i^2 e^{-2(y_i/\alpha)^2}\psi_{ui}^{\theta-2}] +$$

$$+ 2\theta y_i^2 e^{-2(y_i/\alpha)^2}\psi_{ui}^{2(\theta-1)}\Big] -$$

$$-3\alpha^2\left[\sum_{i=1}^{m_u} y_i^2 - (\theta-1)\sum_{i=1}^{m_u}\frac{y_i^2 e^{-(y_i/\alpha)^2}}{\psi_{ui}} + \theta\sum_{i=1}^{m_u}\frac{r_i y_i^2 e^{-(y_i/\alpha)^2}\psi_{ui}^{\theta-1}}{1-\psi_{ui}^{\theta}}\right]\Bigg\} +$$

$$+\frac{2}{\alpha^6}\Bigg\{-(\theta-1)\sum_{i=1}^{m_a} \frac{2\psi_i^4 e^{-(\psi_i/\alpha)^2}\psi_{ai}^2 + 2\psi_i^2 e^{-2(\psi_i/\alpha)^2}}{\psi_{ai}^2} +$$

$$+\theta\sum_{i=1}^{m_a} \frac{r_i \psi_i^2}{(1-\psi_{ai}^{\theta})^2}\Big[(1-\psi_{ai}^{\theta})[-2\psi_i^2 e^{-(\psi_i/\alpha)^2}\psi_{ai}^{\theta-1} - (\theta-1)2\psi_i^2 e^{-2(\psi_i/\alpha)^2}\psi_{ai}^{\theta-2}] +$$

$$+ 2\theta\psi_i^2 e^{-2(\psi_i/\alpha)^2}\psi_{ai}^{2(\theta-1)}\Big] -$$

$$-3\alpha^2\left[\sum_{i=1}^{m_a} \psi_i^2 - (\theta-1)\sum_{i=1}^{m_a}\frac{\psi_i^2 e^{-(\psi_i/\alpha)^2}}{\psi_{ai}} + \theta\sum_{i=1}^{m_a}\frac{r_1\psi_i^2 e^{-(\psi_i/\alpha)^2}\psi_{ai}^{\theta-1}}{1-\psi_{ai}^{\theta}}\right]\Bigg\},$$

$$\frac{\partial^2 \ln L}{\partial \beta^2} = -\frac{m_a}{\beta^2} - \sum_{i=1}^{m_a} \frac{(y_i-\tau)^2}{\psi_i^2} - \frac{2}{\alpha^2}\sum_{i=1}^{m_a}(y_i-\tau)^2 -$$

$$-\frac{2(\theta-1)}{\alpha}\sum_{i=1}^{m_a}\frac{(y_i-\tau)^2 e^{-(y_i/\alpha)^2}[\psi_{ai}(1-(2/\alpha)\psi_i^2)-(2/\alpha)\psi_i^2]}{\psi_{ai}^2} -$$

$$-\frac{2\theta}{\alpha}\sum_{i=1}^{m_a}\frac{r_i(y_i-\tau)^2}{(1-\psi_{ai}^\theta)^2}\{[\psi_{ai}^{\theta-1}e^{-(\psi_i/\alpha)^2}[1-(2/\alpha)\psi_i^2]-$$

$$-(2(\theta-1)/\alpha)\psi_{ai}^{\theta-2}\psi_i^2 e^{-2(\psi_i/\alpha)^2}](1-\psi_{ai}^\theta)+2(\theta/\alpha)\psi_{ai}^{2(\theta-1)}\psi_i^2 e^{-2(\psi_i/\alpha)^2}\},$$

$$\frac{\partial^2 \ln L}{\partial\theta\partial\alpha} = -\frac{2}{\alpha^3}\left\{\sum_{i=1}^{m_u}\frac{y_i^2 e^{-(y_i/\alpha)^2}}{\psi_{ui}} - \sum_{i=1}^{m_u}\frac{r_i y_i^2 e^{-(y_i/\alpha)^2}\psi_{ui}^{\theta-1}}{1-\psi_{ui}^\theta} -\right.$$

$$\left.-\theta\sum_{i=1}^{m_u}\frac{r_i y_i^2 e^{-(y_i/\alpha)^2}\psi_{ui}^{\theta-1}\ln\psi_{ui}}{(1-\psi_{ui}^\theta)^2}\right\} - \frac{2}{\alpha^3}\left\{\sum_{i=1}^{m_a}\frac{\psi_i^2 e^{-(\psi_i/\alpha)^2}}{\psi_{ai}} -\right.$$

$$\left.-\sum_{i=1}^{m_a}\frac{r_i\psi_i^2 e^{-(\psi_i/\alpha)^2}\psi_{ai}^{\theta-1}}{1-\psi_{ai}^\theta} - \theta\sum_{i=1}^{m_a}\frac{r_i\psi_i^2 e^{-(\psi_i/\alpha)^2}\psi_{ai}^{\theta-1}\ln\psi_{ai}}{(1-\psi_{ai}^\theta)^2}\right\},$$

$$\frac{\partial^2 \ln L}{\partial\alpha\partial\beta} = \frac{2}{\alpha^3}\left\{2\sum_{i=1}^{m_a}(y_i-\tau)\psi_i - 2(\theta-1)\sum_{i=1}^{m_a}\frac{(y_i-\tau)\psi_i e^{-(\psi_i/\alpha)^2}}{\psi_{ai}^2}[\psi_{ai}(1-(\psi_i^2/\alpha))-\right.$$

$$-(\psi_i^2/\alpha)e^{-(\psi_i/\alpha)^2}] + \theta\sum_{i=1}^{m_a}\frac{r_i}{(1-\psi_{ai}^\theta)^2}[(1-\psi_{ai}^\theta)\times$$

$$\times\{[2(y_i-\tau)\psi_i e^{-(\psi_i/\alpha)^2}(1-(\psi_i^2/\alpha))]\psi_{ai}^{\theta-1} +$$

$$\left.+(2(\theta-1)/\alpha)(y_i-\tau)\psi_i^3 e^{-2(\psi_i/\alpha)^2}\psi_{ai}^{\theta-2}\} + (2\theta/\alpha)(y_i-\tau)\psi_i^3 e^{-2(\psi_i/\alpha)^2}\psi_{ai}^{2(\theta-1)}]\right\},$$

and

$$\frac{\partial^2 \ln L}{\partial\theta\partial\beta} = \frac{2}{\alpha}\sum_{i=1}^{m_a}\frac{(y_i-\tau)\psi_i e^{-(\psi_i/\alpha)^2}}{\psi_{ai}} - \frac{2}{\alpha}\left\{\sum_{i=1}^{m_a}\frac{r_i(y_i-\tau)\psi_i e^{-(\psi_i/\alpha)^2}\psi_{ai}^{\theta-1}}{1-\psi_{ai}^\theta} +\right.$$

$$\left.+\theta\sum_{i=1}^{m_a}\frac{r_i(y_i-\tau)\psi_i e^{-(\psi_i/\alpha)^2}\psi_{ai}^{\theta-1}\ln\psi_{ai}}{(1-\psi_{ai}^\theta)^2}\right\}.$$

Thus, the approximate $100(1-\gamma)\%$ two-sided confidence bounds for θ, α, and β are, respectively, given by

$$\hat{\theta} \pm Z_{\gamma/2}\sqrt{I_{11}^{-1}(\hat{\theta}, \hat{\alpha}, \hat{\beta})}, \qquad \hat{\alpha} \pm Z_{\gamma/2}\sqrt{I_{22}^{-1}(\hat{\theta}, \hat{\alpha}, \hat{\beta})}, \qquad \hat{\beta} \pm Z_{\gamma/2}\sqrt{I_{33}^{-1}(\hat{\theta}, \hat{\alpha}, \hat{\beta})},$$

where $Z_{\gamma/2}$ is the upper $(\gamma/2)$th percentile of a standard normal distribution.

3. **Optimum Test Plans**. The core intention of this section is to decide the optimal stress-change time τ^* based on progressive type-II censoring with binomial removals using three different progressive censoring schemes. In step-stress setting, the experimenter is often interested in estimating the mean life at use condition with high precision. The mean lifetime is an important characteristic in reliability analysis. Practically, the optimum test plans are important for improving the quality of the statistical inference. One selection optimality criterion is the D-optimality criterion. It is used to determine the optimal value of τ.

The D-optimality criterion is based on the determinant of Fisher's information matrix F. It has been extensively used in the context of planning life test. If one is more interested in estimation with high precision, a more reasonable criterion should be D-optimality, which takes into account the overall parameter space. According to Bai et al. [40], it can be constructed in terms of the generalized asymptotic variance (GAV) of the MLEs of the model parameters. This GAV is proportional to reciprocal of the determinant of Fisher-information matrix. So that maximizing this determinant is equivalent to minimizing GAV. The criterion function is then expressed by

$$GAV(\hat{\alpha}, \hat{\lambda}, \hat{\beta}) = \frac{1}{|\mathbf{F}|}. \tag{6}$$

Hence, the optimal stress-change time τ^* is determined such that the GAV is minimized.

4. **Simulation Studies**. To demonstrate theoretical results introduced in this article, simulation studies are conducted. In order to evaluate the performance of the MLEs, the mean square error (MSE), the average confidence interval lengths of the model parameters and their coverage probabilities are obtained under three different progressive censoring schemes. Also, the influences of the sample size n, the observation size m, and the binomial parameter p on the accuracy of the parameter estimates are discussed.

The simulation studies are implemented based on the following algorithm:

(i) determine the value of n;

(ii) determine the value of m;

(iii) determine the values of the parameters θ, α, β, and p;

(iv) determine the value of τ;

(v) generate a random sample with size n and observation size m from the random variable Y given by (4) and sort it. The generalized Rayleigh random variable can be easily generated. For example, if U represents a uniform random variable from [0, 1], then $Y = \alpha[-\ln(1-U^{1/\theta})]^{1/2}$ has generalized Rayleigh distribution with the distribution function given by (2). The true parameters values are set to be $\theta = 2$, $\alpha = 1.5$, and $\beta = 2.5$;

(vi) generate a random number R_1 from binomial $(n-m, p)$;

(vii) for $i = 2, 3, \ldots, m-1$, generate a random number R_i from binomial $(n-m-$ $-\sum_{l=1}^{i-1} r_l, p)$;

(viii) set R_m according to the following relation:

$$R_m = \begin{cases} n - m - \sum_{l=1}^{m-1} r_l, & \text{if} \quad n - m > \sum_{l=1}^{m-1} r_l, \\ 0, & \text{otherwise;} \end{cases}$$

(ix) use the progressively type-II censored sample created in steps 5–8 to calculate the MLEs of the parameters θ, α, and β. The Newton–Raphson method is used to simultaneously solve the nonlinear equations to get the MLEs of the model parameters;

(x) steps 5–9 are replicated 10,000 times.

(xi) calculate the MSE related to the MLEs of the parameters;

(xii) construct the confidence intervals width (CIW) for each parameter with confidence level $1-\gamma = 0.95$;

(xiii) steps 1–12 are implemented with different values of n, m, and p.

Tables 1–3 give the MSE of the MLEs of θ, α, and β. Also, these tables includes both the CIW of the model parameters and their coverage probabilities against the values of n when $n = 30, 50, 75$, and 100 and various m when $p = 0.10$, 0.25, and 0.50, respectively.

From the results presented in Tables 1–3, it can be concluded that:

1. For fixed m/n and p, the MSE associated with the parameter estimates decreases as n increases.

2. For fixed n and p, as m decreases the MSE increases.

3. The effect of m on the precision of the MLEs of the parameters is influenced by the value of removal probability p. As p increases, for fixed n and m/n, the MSE of the parameter estimates increases.

4. For fixed m/n and p, the CIW decreases as n increases. But for fixed n and m/n, the CIW increases as p increases.

5. For fixed m/n and small p, the CP is very close to the nominal level as n increases. But for fixed n and m/n, the CP is not good as p is large.

6. For fixed n and p, as m decreases, the CP is considerably lower than the nominal level.

Therefore, it can be said that as the sample size n increases and the effective sample proportion m/n increases, the act of the MLEs in terms of MSE become better unless p is large. A higher value of p leads to higher values of MSE. When p increases, the experiment is terminated more quickly. But it is important to note that with a highly larger p the experiments will be less informative and lead to larger standard errors in estimates. These results coincide with the note of Wu et al. [41]. Regarding the effect of τ on MSE of the parameter estimates, it can be said that a small value of τ gives a better estimate in the sense of having smaller MSE.

As coincided with the note of Wu and Huang [42], the design of an optimal life test already enables us to obtain estimations of high degree of precision. They said that in order to obtain a precise estimate of mean life, one needs to design an optimal life test. So, in this section, the optimal choice of τ is explored. Optimum test plans have been developed here numerically under different values of n, m, and p. The numerical results of the optimal stress change-time τ^* under different progressive censoring schemes, as well as the optimal GAV of the MLEs of the model parameters are given in Table 4. The optimal GAV is numerically obtained with τ^* in place of τ.

From the results shown in Table 4, it can be detected that:

1. For fixed m/n and p, both the optimal stress change-time τ^* and the optimal GAV of the MLEs of the model parameters decrease as the sample size n increases. That is, good estimates of the model parameters are obtained.

2. For fixed n and p, as m decreases both τ^* and the optimal GAV of the MLEs of the model parameters increase. That is, inefficient estimates of the model parameters are obtained.

Table 1

Results for $p = 0.10$: Average Values of MSE, CIW, and CP when θ, α, β, and τ Set at 1.5, 2, 2.5, and 5, Respectively

n	m	$MSE_{\hat{\theta}}$	$MSE_{\hat{\alpha}}$	$MSE_{\hat{\beta}}$	$CIW(\theta)$	$CIW(\alpha)$	$CIW(\beta)$	$CP(\theta)$	$CP(\alpha)$	$CP(\beta)$
30	30	0.001217	0.002219	1.893067	1.275569	1.475701	2.425164	0.9452	0.9445	0.9431
	25	0.001462	0.002946	2.642556	1.296151	1.960679	2.657831	0.9438	0.9428	0.9422
50	50	0.000485	0.001656	1.013526	0.508048	1.102341	1.268735	0.9463	0.9457	0.9443
	45	0.000536	0.001914	1.131995	0.561169	1.273663	1.363604	0.9445	0.9438	0.9427
	40	0.000618	0.002246	1.330531	0.647662	1.495169	1.602377	0.9381	0.9374	0.9364
	35	0.000796	0.002439	1.906093	0.833581	1.623662	1.702679	0.9353	0.9347	0.9336
	30	0.001126	0.005353	2.512575	1.179544	3.563141	1.886014	0.9332	0.9326	0.9315
	25	0.001703	0.005636	3.720213	1.784301	3.751335	2.127013	0.9293	0.9286	0.9272
75	75	0.000347	0.001141	0.735891	0.363671	0.759266	0.837441	0.9558	0.9551	0.9541
	70	0.000381	0.001879	0.826313	0.399084	1.043733	1.054683	0.9539	0.9531	0.9522
	65	0.000425	0.002037	0.944028	0.445394	1.107462	1.248992	0.9474	0.9465	0.9457
	60	0.000594	0.002251	1.177313	0.622461	1.152765	1.345457	0.9446	0.9437	0.9429
	55	0.000739	0.003069	1.483242	0.775012	1.318365	1.526344	0.9425	0.9419	0.9408
	50	0.000852	0.003569	1.727252	0.892832	1.421446	1.731061	0.9383	0.9374	0.9368
	45	0.001207	0.004066	2.687445	1.264673	2.375571	1.941812	0.9372	0.9363	0.9355
	40	0.001512	0.004731	3.981686	1.584076	2.706533	1.980412	0.9367	0.9356	0.9349
	35	0.002256	0.005156	4.898563	2.363855	3.148681	2.264005	0.9313	0.9305	0.9296
	30	0.003028	0.005656	6.279956	3.172238	3.431621	2.537966	0.9318	0.9307	0.9298
	25	0.003561	0.005944	7.913035	3.730682	3.764314	2.961408	0.9267	0.9259	0.9251
100	100	0.000159	0.00667	0.326703	3.730682	0.443878	0.602627	0.9511	0.9504	0.9496
	95	0.000168	0.000874	0.377281	0.166852	0.581887	0.791056	0.9524	0.9517	0.9506
	90	0.000185	0.001126	0.398866	0.175706	0.749315	0.934575	0.9481	0.9472	0.9463
	85	0.000246	0.001281	0.497412	0.194094	0.852714	1.164924	0.9467	0.9462	0.9447
	80	0.000366	0.001405	0.731985	0.258110	0.935346	1.301911	0.9441	0.9435	0.9424
	75	0.000511	0.001879	1.022658	0.383422	1.250733	1.552325	0.9426	0.9421	0.9409
	70	0.000752	0.002016	1.395732	0.535291	1.342018	1.672194	0.9423	0.9411	0.9403
	65	0.000931	0.002165	2.017223	0.787952	1.440661	1.869879	0.9384	0.9376	0.9366
	60	0.001186	0.002412	2.574228	0.975916	1.605491	1.906743	0.9379	0.9374	0.9363
	55	0.001443	0.003345	3.029566	1.242199	2.226314	1.956241	0.9354	0.9347	0.9338
	50	0.001719	0.004525	3.550372	1.509162	3.011972	2.169941	0.9284	0.9276	0.9268
	45	0.002103	0.004791	4.369404	1.801324	3.188484	2.310687	0.9277	0.9269	0.9262
	40	0.002419	0.005329	5.348727	2.203132	3.547133	2.455186	0.9261	0.9253	0.9244
	35	0.002761	0.005468	6.010147	2.534113	3.639716	2.847099	0.9257	0.9246	0.9239
	30	0.003592	0.006128	7.634965	2.892334	4.078836	3.179888	0.9251	0.9244	0.9234
	25	0.004183	0.006242	8.796061	3.763372	4.154546	3.517201	0.9246	0.9238	0.9231

3. For fixed n and m/n, as p increases both the optimal stress change-time τ^* and the optimal GAV of the MLEs of the model parameters increase.

5. **Data Analysis: An Illustrative Example**. To show the applicability of the methodology presented in this article, a numerical example is displayed. Generalized Rayleigh model is used to fit the data set. To confirm the power of the model, we compute the Kolmogorov–Smirnov (K–S) distance between the empirical distribution function and

Table 2

Results for $p = 0.25$: Average Values of MSE, CIW, and CP when θ, α, β, and τ Set at 1.5, 2, 2.5, and 5, Respectively

n	m	$MSE_{\hat{\theta}}$	$MSE_{\hat{\alpha}}$	$MSE_{\hat{\beta}}$	$CIW(\theta)$	$CIW(\alpha)$	$CIW(\beta)$	$CP(\theta)$	$CP(\alpha)$	$CP(\beta)$
30	30	0.001704	0.003547	2.082374	1.658241	1.962682	2.885945	0.9357	0.9256	0.9148
	25	0.002047	0.004714	2.906812	1.684996	2.607703	3.162819	0.9344	0.9239	0.9139
50	50	0.000679	0.002652	1.114879	0.660462	1.466114	1.509795	0.9368	0.9268	0.9167
	45	0.000752	0.003062	1.245195	0.729523	1.693972	1.622689	0.9351	0.9249	0.9144
	40	0.000865	0.003594	1.463584	0.841961	1.988575	1.906829	0.9287	0.9187	0.9083
	35	0.001114	0.003902	2.096702	1.083655	2.15947	2.026188	0.9259	0.9163	0.9056
	30	0.001576	0.008565	2.763833	1.533407	4.738978	2.244357	0.9239	0.9139	0.9036
	25	0.002384	0.009018	4.092231	2.319591	4.989276	2.531145	0.9231	0.9131	0.8994
75	75	0.000486	0.001826	0.809482	0.472772	1.009824	0.996555	0.9462	0.9364	0.9255
	70	0.000533	0.003006	0.908944	0.518809	1.388165	1.255073	0.9444	0.9342	0.9236
	65	0.000595	0.003259	1.038431	0.579012	1.472924	1.486371	0.9379	0.9276	0.9173
	60	0.000832	0.003602	1.295044	0.809199	1.533177	1.601094	0.9352	0.9248	0.9146
	55	0.001035	0.004912	1.631566	1.007516	1.753425	1.816349	0.9331	0.9231	0.9126
	50	0.001193	0.005713	1.899977	1.160682	1.890523	2.059963	0.9289	0.9187	0.9087
	45	0.001692	0.006506	2.956193	1.644075	3.159509	2.310756	0.9278	0.9176	0.9074
	40	0.002117	0.007572	4.379855	2.059299	3.599689	2.356694	0.9273	0.9169	0.9069
	35	0.003158	0.008252	5.388419	3.073012	4.187746	2.694166	0.9221	0.9119	0.9017
	30	0.004239	0.009054	6.907952	4.123909	4.564056	3.020182	0.9225	0.9121	0.9019
	25	0.004985	0.009516	8.704339	4.849887	5.006538	3.524076	0.9174	0.9074	0.8973
100	100	0.000223	0.001067	0.359373	4.849887	0.590358	0.717126	0.9416	0.9314	0.9211
	95	0.000235	0.001398	0.415008	0.216908	0.773911	0.941357	0.9429	0.9327	0.9221
	90	0.000259	0.001802	0.438753	0.228418	0.996589	1.112144	0.9386	0.9283	0.9179
	85	0.000344	0.002050	0.547153	0.252322	1.134113	1.386261	0.9372	0.9273	0.9164
	80	0.000512	0.002248	0.805184	0.335543	1.244016	1.549274	0.9347	0.9246	0.9141
	75	0.000715	0.003006	1.124924	0.498449	1.663475	1.847267	0.9332	0.9233	0.9127
	70	0.001053	0.003226	1.535305	0.695878	1.784884	1.989911	0.9329	0.9223	0.9121
	65	0.001303	0.003464	2.218945	1.024338	1.916079	2.225156	0.9291	0.9188	0.9085
	60	0.001662	0.003859	2.831651	1.268691	2.135303	2.269024	0.9285	0.9187	0.9082
	55	0.002021	0.005352	3.332523	1.614859	2.960998	2.327927	0.9262	0.9162	0.9058
	50	0.002407	0.007241	3.905409	1.961911	4.005923	2.582235	0.9191	0.9091	0.8991
	45	0.002944	0.007666	4.806344	2.341721	4.240684	2.749718	0.9184	0.9084	0.8984
	40	0.003387	0.008526	5.883632	2.864072	4.717687	2.921671	0.9168	0.9068	0.8967
	35	0.003865	0.008749	6.611162	3.294347	4.840822	3.388048	0.9164	0.9061	0.8962
	30	0.005029	0.009805	8.398462	3.760034	5.424852	3.784067	0.9158	0.9059	0.8957
	25	0.005856	0.009987	9.675667	4.892384	5.525546	4.185469	0.9154	0.9053	0.8954

the fitted distribution function when the parameters estimates are obtained by the maximum likelihood method. The result of K–S test is $D = 0.0691$ with $p = 0.483$. This result observably shows that the generalized Rayleigh model provides excellent fit to the data set. Thus, it can be used successfully for modeling this data set. Assuming generalized Rayleigh distribution under progressive type-II censoring scheme with binomial removals we use $n = 36$, $\theta = 2.2$, $\alpha = 1.25$, $\beta = 2.5$, $\tau = 9$, $m = 21$, and $P = 0.20$. The MSE of the MLEs of

Table 3

Results for $p = 0.50$: Average Values of MSE, CIW, and CP when θ, α, β, and τ Set at 1.5, 2, 2.5, and 5, Respectively

n	m	$MSE_{\hat{\theta}}$	$MSE_{\hat{\alpha}}$	$MSE_{\hat{\beta}}$	$CIW(\theta)$	$CIW(\alpha)$	$CIW(\beta)$	$CP(\theta)$	$CP(\alpha)$	$CP(\beta)$
30	30	0.002386	0.004966	2.498849	2.155713	2.355218	3.174545	0.9168	0.9071	0.8965
	25	0.002866	0.006631	3.488174	2.190495	3.129244	3.479101	0.9157	0.9054	0.8956
50	50	0.000951	0.003713	1.337855	0.858601	1.759337	1.660775	0.9181	0.9083	0.8984
	45	0.001053	0.004287	1.494234	0.948382	2.032766	1.784958	0.9164	0.9064	0.8961
	40	0.001211	0.005032	1.756301	1.094549	2.386291	2.097512	0.9101	0.9003	0.8901
	35	0.001563	0.005463	2.516042	1.408752	2.591364	2.228807	0.9074	0.8982	0.8875
	30	0.002206	0.011991	3.316641	1.993429	5.686774	2.468793	0.9054	0.8956	0.8855
	25	0.003338	0.012625	4.910677	3.015468	5.987131	2.784264	0.9046	0.8948	0.8814
75	75	0.000682	0.002556	0.971378	0.614604	1.211789	1.096211	0.9181	0.9177	0.9072
	70	0.000746	0.004208	1.090733	0.674452	1.665798	1.380582	0.9164	0.9155	0.9051
	65	0.000833	0.004563	1.246117	0.752716	1.767509	1.635008	0.9101	0.9091	0.8994
	60	0.001165	0.005043	1.554053	1.051959	1.839812	1.761203	0.9074	0.9063	0.8963
	55	0.001449	0.006877	1.957879	1.309771	2.104113	1.997984	0.9054	0.9046	0.8943
	50	0.001676	0.007998	2.279972	1.508887	2.268628	2.265959	0.9046	0.9003	0.8905
	45	0.002369	0.009108	3.547432	2.137298	3.791411	2.541832	0.9181	0.8992	0.8893
	40	0.002964	0.010601	5.255826	2.677089	4.319627	2.592363	0.9164	0.8986	0.8888
	35	0.004421	0.011553	6.466103	3.994916	5.025295	2.963583	0.9101	0.8937	0.8837
	30	0.005935	0.012676	8.289542	5.361082	5.476867	3.322241	0.9074	0.8939	0.8839
	25	0.006979	0.013322	10.44521	6.304853	6.007846	3.876484	0.9054	0.8893	0.8794
100	100	0.000312	0.001494	0.431248	6.304853	0.708432	0.788839	0.9228	0.9128	0.9027
	95	0.000329	0.001957	0.498011	0.281983	0.928693	1.035493	0.9241	0.9142	0.9037
	90	0.000363	0.002523	0.526504	0.296943	1.195907	1.223358	0.9198	0.9097	0.8995
	85	0.000482	0.002872	0.656584	0.328019	1.360936	1.524887	0.9185	0.9088	0.8981
	80	0.000717	0.003147	0.966221	0.436206	1.492819	1.704201	0.9162	0.9061	0.8958
	75	0.001001	0.004208	1.349909	0.647984	1.996172	2.031994	0.9145	0.9048	0.8944
	70	0.001474	0.004516	1.842366	0.904641	2.141861	2.188902	0.9142	0.9039	0.8939
	65	0.001824	0.004853	2.662734	1.331639	2.299295	2.447672	0.9105	0.9004	0.8903
	60	0.002327	0.005403	3.397981	1.649298	2.562364	2.495926	0.9099	0.9003	0.8931
	55	0.002829	0.007493	3.999028	2.099317	3.553198	2.560723	0.9077	0.8979	0.8877
	50	0.003372	0.010137	4.686491	2.550484	4.807108	2.840459	0.9007	0.8909	0.8811
	45	0.004122	0.010732	5.767613	3.044237	5.088821	3.024691	0.9011	0.8902	0.8804
	40	0.004742	0.011936	7.060358	3.723294	5.661224	3.213838	0.8985	0.8887	0.8788
	35	0.005411	0.012249	7.933394	4.282651	5.808986	3.726853	0.8981	0.8883	0.8783
	30	0.007041	0.013727	10.07815	4.888044	6.509822	4.162474	0.8975	0.8878	0.8778
	25	0.008198	0.013982	11.61082	6.460099	6.630655	4.604016	0.8971	0.8872	0.8775

θ, α, and β are respectively 0.001941, 0.002811, and 1.005113. The CIW of θ, α, and β are 1.450012, 2.381473, and 2.796281 with CP 0.9360, 0.9287, and 0.9150, respectively. Using the same settings of the life-testing experiment with different items, the results of the test design are as follows. The optimal stress change-time τ^* and the optimal GAV of the MLEs of the model parameters are respectively 8.3441 and 0.041532.

Table 4

Average Values of Optimal τ and Optimal GAV; Considering Three Different Cases of p, θ, α, and β

n	m	$\tau^{*(1)}$	Optimal $GAV^{(1)}$	$\tau^{*(2)}$	Optimal $GAV^{(2)}$	$\tau^{*(3)}$	Optimal $GAV^{(3)}$
30	30	6.1328	0.024364	8.1213	0.040607	11.1873	0.073092
	25	7.3656	0.034012	11.3372	0.056686	14.8639	0.102035
50	50	2.4441	0.013044	4.3481	0.021741	8.3569	0.039133
	45	2.7062	0.014569	4.8563	0.024282	9.6556	0.043707
	40	3.1123	0.017125	5.7082	0.028541	11.3351	0.051374
	35	4.0169	0.024531	8.1771	0.040886	12.3094	0.073594
	30	5.6694	0.032338	10.7793	0.053897	27.0121	0.097014
	25	8.5787	0.047882	15.9605	0.079803	28.4399	0.143645
75	75	1.7527	0.009472	3.1572	0.015786	5.7562	0.028415
	70	1.9172	0.010635	3.5449	0.017725	7.9125	0.031904
	65	2.1408	0.012152	4.0499	0.020253	8.3957	0.036449
	60	2.9941	0.015152	5.0507	0.025254	8.7391	0.045456
	55	3.7239	0.019089	6.3631	0.031816	9.9945	0.057268
	50	4.3073	0.022231	7.4099	0.027052	10.7763	0.066689
	45	6.0883	0.034587	11.5292	0.057645	18.0091	0.103761
	40	7.6175	0.051243	17.0811	0.085406	20.5183	0.153732
	35	11.3621	0.063047	21.0156	0.105078	23.8751	0.189141
	30	15.2531	0.080824	26.9414	0.134707	28.0156	0.242473
	25	17.9362	0.101842	33.9472	0.169736	36.5378	0.305525
100	100	0.8018	0.004205	1.4016	0.007008	3.3651	0.012614
	95	0.8455	0.004856	1.6185	0.008093	4.4113	0.014567
	90	0.9329	0.005233	1.7111	0.008556	5.6806	0.015443
	85	1.2387	0.006402	2.1339	0.010674	6.4644	0.019205
	80	1.8427	0.009421	3.1402	0.015701	7.0909	0.028262
	75	2.5726	0.013162	4.3872	0.021936	9.4818	0.039485
	70	3.7882	0.017963	5.9877	0.029939	10.1741	0.053889
	65	4.6877	0.025962	8.6539	0.043278	10.9224	0.077885
	60	5.9804	0.033132	11.0434	0.055217	12.1717	0.099391
	55	7.2705	0.038993	12.9976	0.064988	16.8788	0.116978
	50	8.6661	0.045694	15.2312	0.076156	22.8340	0.137081
	45	10.5942	0.056236	18.7454	0.093727	24.1722	0.168709
	40	12.1873	0.068838	22.9461	0.114731	26.8916	0.206515
	35	13.9062	0.077355	25.7849	0.128925	27.5932	0.232064
	30	18.0951	0.98264	32.7547	0.163774	35.9226	0.294792
	25	21.0697	0.113206	37.7354	0.188677	39.4965	0.339619

(1) When $p = 0.10$ and θ, α, and β set at 1.5, 2, and 2.5, respectively;
(2) when $p = 0.25$ and θ, α, and β set at 1.5, 2, and 2.5, respectively;
(3) when $p = 0.50$ and θ, α, and β set at 1.5, 2, and 2.5, respectively.

Conclusions. The issue of progressive type-II censoring has received care in the past few years. The GRD can be used quite effectively in modeling strength data and also modeling general lifetime data. MLEs of parameters of the GRD were discussed using progressively type-II censored data with binomial removals. The performance of the MLEs was assessed numerically. In addition, the roles of sample size n, failure size m, and removal probability p toward the accuracy of estimations were explored via simulation studies. It can be concluded that as both the sample size n and the effective sample fraction m/n increase, the performance of the MLEs become better unless p is large.

Moreover, statistically optimum step-stress partially accelerated life test plans have been developed. The optimality criterion adopted is the minimization of the GAV of the MLEs of the model parameters. That is, τ^* is obtained such that the GAV is minimized. Thus, the optimal design of the life tests can be considered as a technique to improve the quality of the statistical inference. This issue agrees with the annotation of Wu and Huang [38]. In order to obtain a precise estimate of mean life, one needs to design an optimal life test. As a future work, the Bayesian inference in the case of SSPALT under the same censoring schemes proposed in this paper will be considered. Also, the optimum test plans will be explored under constant-stress PALT using progressively type-II censored data which is an extension to the work of Ismail [43].

Р е з ю м е

Розглянуто параметр оцінки і оптимальне проектування частково прискорених випробувань на довговічність при ступеневому навантаженні на основі узагальненого релеївського розподілу при прогресивному цензуруванні типу II з біноміальними виборками. Як фактор прискорення використовуються максимальні оцінки імовірності параметрів масштабу і форми, які узгоджуються між собою. Побудовано наближені довірчі інтервали параметрів моделі і розраховано границі імовірності. Розроблено оптимальні плани випробувань із метою покращання статистичного аналізу. Запропоновано результати моделювання і числовий приклад.

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