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A NEW FAMILY OF EXPECTILES AND ITS PROPERTIES

This paper considers a risk measure called expectile. We propose a new expression defining expectile, using maximization of CVaR by changing confidence level. This expression is specified for continuous and finite discrete distribution. It is proved that the optimal value of the confidence level is equal to the CDF of expectile value. We also consider a new family of expectiles defined by two parameters. Comparison of different new expectiles with quantile for a set of distributions shows that proposed expectiles are closer to the quantile than the standard expectile. Two variants of expectile linearization are proposed and it is shown how to use them with linear loss function. Finally, we build three fundamental risk quadrangles where expectile is a statistic and risk.

Keywords: *Expectile, EVaR, Quantile, Conditional Value-at-Risk, CVaR, Kusuoka representation, Fundamental Risk Quadrangle, Portfolio Safeguard Package.*

Introduction. Expectile is a characteristic of a random variable calculated using the asymmetric least square method [1, 2]. The level of asymmetry is defined by the parameter in the interval $(0, 1)$. Expectile as a function of parameter is considered as a generalization of quantile (Value-at-Risk or VaR) [1, 3]. On the other hand, expectile can be compared with Conditional Value-at-Risk (CVaR) [4] which also depends on parameter varying in $(0, 1)$. VaR, CVaR, and expectile are used as risk measures in various applications. Which function will be used depends on the application and the task at hand. However, expectile has a wider set of advantages than VaR and CVaR. Expectile is a coherent risk measure on half of the interval $(0, 1)$, i.e. it satisfies the properties of translation invariance, positive homogeneity, monotonicity, and subadditivity [5]. Another advantage of expectile is the elicitability. This property is important for financial risk management, forecasting, hypotheses testing. Comparing expectile with VaR and CVaR, the authors [5–7] notice that CVaR is a coherent risk measure but lacks of elicitability; VaR is an elicitable risk measure but lacks of coherency. It is proved [5] that only elicitable law-invariant coherent risk measure is expectile.

A popular topic in financial applications is portfolio optimization. Expectile can be used as an objective in portfolio optimization problems [6, 7]. In the paper [8] a portfolio optimization is considered as a problem of maximization expected portfolio return subject to the risk of the portfolio expressed by expectile or Omega functions.

Expectile and its properties have been studied by many authors (see overview in [9]). As a rule, expectile is compared with quantile [1, 3]. Our goal is to compare expectile with CVaR by introducing the same parameter – confidence level. To do this we first represent expectile using a sum of mean and CVaR with varying confidence level and varying coefficient before CVaR.

We then propose some novelties in the definition of expectile as a function of parameter. Expectile is equal to mean when its parameter is 0.5 while CVaR is equal to mean when its parameter is 0.

We suggest changing the dependence of expectile on its parameter so that expecile is also equal to mean for a zero parameter. This results in the interval $(-1, 1)$ for a new parameter. But, we do not consider the subinterval $(-1, 0)$. Instead, we calculate expectile on the left-tail of distribution as lower CVaR using a random variable with changed sign. Then, we add a second parameter that changes the dependence of expectile on the confidence level. This results in a family of expectiles depending on two parameters. To demonstrate the usefulness of such novelty we show and compare VaR, CVaR, and expectile curves for different distributions.

VaR and CVaR are considered as statistics within the framework of the fundamental risk quadrangle concept [10, 11]. There are regular VaR and CVaR quadrangles. CVaR is a risk function in quantile quadrangle. We build the regular risk quadrangle with a new error function where expectile is both a risk and statistic.

Further, we use the following notations.

Let X be a random variable with a finite expected (mean) value $E[X]$.

The cumulative distribution function is denoted by $F_X(x) = prob\{X \le x\}$.

Typically, the quantile or $VaR_\alpha(X)$ is the inverse function to $F_X(x)$ defined on $\alpha \in (0, 1)$. But for cases when VaR is a result of optimization, it is defined as the interval using the lower and upper VaR [10, 12]. We define the lower and upper VaR as follows:

$$VaR_\alpha^-(X) = \begin{cases} \sup\{x, F_X(x) < \alpha\} & for\ \ 0 < \alpha \le 1 \\ \inf\{x, F_X(x) \ge \alpha\} & for\ \ \alpha = 0 \end{cases} \quad \text{and} \quad VaR_\alpha^+(X) = \begin{cases} \inf\{x, F_X(x) > \alpha\} & for\ \ 0 \le \alpha < 1 \\ \sup\{x, F_X(x) \le \alpha\} & for\ \ \alpha = 1 \end{cases}.$$

The VaR is an interval if the lower and upper quantiles do not coincide:

$$VaR_\alpha(X) = \left[VaR_\alpha^-(X), VaR_\alpha^+(X)\right],$$

otherwise, VaR is a singleton $VaR_\alpha(X) = VaR_\alpha^-(X) = VaR_\alpha^+(X)$.

The superquantile or $CVaR_\alpha(X)$ [4] with a confidence level $\alpha \in (0, 1)$ can be defined in many ways. In financial applications the most popular definition of CVaR is

$$CVaR_\alpha(X) = \frac{1}{1-\alpha} \int_\alpha^1 VaR_\beta^-(X) d\beta .$$

We extend the definition of CVaR for $\alpha = 0$ as $CVaR_0(X) = \lim_{\varepsilon \to +0} CVaR_\varepsilon(X) = E[X]$ and for $\alpha = 1$ as $CVaR_1(X) = VaR_1^-(X)$ if a finite value of $VaR_1^-(X)$ exists.

The functions $(z)^+ \ge 0$ and $(z)^- \ge 0$, used below for the scalar variable z, are defined as $(z)^+ = \max\{0, z\}$, $(z)^- = \max\{0, -z\}$.

The Partial Moment function with a threshold C is defined as $E[(X - C)^+]$.

Definitions of VaR, CVaR, Mean, Partial Moment functions used in this paper are the same as mathematical definitions in the description of the optimization package Portfolio Safeguard [13].

This paper is organized as follows. Section 1 provides different expressions defining expectile. We start with the standard equations defining expectile, then we propose simple formulas for discrete finite and continuous distributions that include optimization by one parameter. Section 2 introduces a family of expectiles depending on two parameters. We compare such expectiles with VaR and CVaR for various finite discrete and continuous distributions. Section 3 describes using the expectile in convex optimization problems and, in particular, using two variants of linearization of expectile. Section 4 introduces three variants of risk quadrangle where expectile is a statistic. In one of them expectile is both a statistic and a risk function.

1. Definitions of Expectile

The name "expectile" was introduced in [1] for the minimizer in the asymmetric least square method. The expectile function can be defined in different ways. We begin with the commonly used definition of the expectile [14]. The expectile function $e_q(X)$ with a scalar parameter $0<q<1$ for a random variable X is defined as

$$e_q(X) = \arg\min_{C\in R^1} \left\{ qE[((X-C)^+)^2] + (1-q)E[((X-C)^-)^2] \right\} \tag{1}$$

or by the first order condition as a solution C_q^* of the equation

$$qE[(X-C)^+] = (1-q)E[(X-C)^-], \tag{2}$$

then $e_q(X) = C_q^*$. The cases $q=0$ and $q=1$ need separate consideration.

Taking into account that the solution of equation (2) depends only on the ratio $q/(1-q)$ we can write a more general equation depending on two positive coefficients $q_1 > q_2 > 0$

$$q_1 E[(X-C)^+] = q_2 E[(X-C)^-]. \tag{3}$$

The case $q_1 = q_2$ defines the mean value $E[X]$.

To simplify formula (3) we use equality $E[(X-C)^-] = E[(X-C)^+] - E[X-C]$ and obtain

$$(q_1 - q_2)E[(X-C)^+] = -q_2 E[X] + q_2 C. \tag{4}$$

Using the coefficient $K = \frac{q_2}{q_1 - q_2} > 0$ formula (4) is rewrote as

$$KC - KE[X] = E[(X-C)^+]. \tag{5}$$

The left and right-hand sides of equation (5) contain two simple functions of a random variable X and the variable C. The first one is a linear function of C and $E[X]$, the second one is the Partial Moment function $E[(X-C)^+]$ with a variable threshold C. A similar formula was used in [1], where some properties of this formula were discussed.

The left-hand side of (5) is a linear increasing function of C when the coefficient K is positive, the right-hand side of (5) is a positive decreasing convex function of the threshold C. Elements of the subgradients of the right-hand side of (5) are bounded and lie in the interval $[-1, 0]$. Therefore, equation (5) has a unique solution C_K^* that defines expectile.

We denote such defined expectile as $e_K(X)$ with subscript K, meaning that $e_K(X)$ is equal to the solution of equation (5) for $K>0$.

There are many variants of expectile representation using equations. See, for example, [2, 3, 9, 15, 16].

Our goal is to derive a formula for calculating expectile without variable C on the right-hand side.

First, let us consider a **finite discrete distribution** of a random variable X with N atoms X_j and probabilities p_j, $j \in J = \{1,...,N\}$.

In this case, the Partial Moment function on the right-hand side of (5) is equal to the solution of the following optimization problem

$$E[(X-C)^+] = \max_{\gamma,\pi_j} \left\{ \sum_{j\in J} \pi_j (X_j - C) \mid \sum_{j\in J} \pi_j + \gamma = 1,\ 0 \le \pi_j \le p_j, j \in J \right\}. \qquad (6)$$

Let us denote the feasible set of vectors $\bar{v} = (\gamma, \pi_1, ..., \pi_N)$ in (6) as V. Each vector $\bar{v} \in V$ defines a linear function of C in the right-hand side of (6). Then the Partial Moment (6) is the maximum of these linear functions.

Each linear function defined by vectors $\bar{v} \in V$ intersects the linear function on the left-hand side of (5) at the point

$$C^*_{K\bar{v}} = E[X] + \frac{\sum_{j=J} \pi_j (X_j - E[X])}{K + (1-\gamma)}. \qquad (7)$$

Since the Partial Moment is a decreasing function, only the maximum value of $C^*_{K\bar{v}}$, where $\bar{v} \in V$, gives the solution of equation (5). Hence, expectile is equal to

$$e_K(X) = \max_{\bar{v}\in V} C^*_{K\bar{v}} = E[X] + \max_{\gamma,\pi_j} \left\{ \frac{\sum_{j=J} \pi_j (X_j - E[X])}{K + (1-\gamma)} \mid \sum_{j=1,...,N} \pi_j + \gamma = 1,\ 0 \le \pi_j \le p_j, j = 1,...,N \right\}.$$

Separating variables γ and π_j we get

$$e_K(X) = E[X] + \max_{0\le\gamma\le 1} \left\{ \frac{\max_{\pi_j} \left\{ \sum_{j=J} \pi_j (X_j - E[X]) \mid \sum_{j=J} \pi_j + \gamma = 1,\ 0 \le \pi_j \le p_j, j = 1,...,N \right\}}{K + (1-\gamma)} \right\}.$$

The maximization problem in the numerator corresponds to the dual definition of the CVaR function for finite distribution. Therefore, we have

$$e_K(X) = E[X] + \max_{0\le\gamma\le 1} \frac{(1-\gamma) \cdot CVaR_\gamma(Y)}{K + (1-\gamma)}, \qquad (8)$$

where $Y = X - E[X]$ and $K = \frac{q_2}{q_1 - q_2} > 0$ (or $K = \frac{1-q}{2q-1} > 0$).

Formula (8) is not the easiest way to calculate expectile in the case of a finite discrete distribution. The optimization problem (6) has an obvious solution: $\pi_j = p_j$ for $j \in \{1,...N\}$ such that $X_j > C$, otherwise $\pi_j = 0$. The variable γ depends on the variables π_j.

The right-hand side in (6) has not greater than $N+1$ linear pieces. Every linear piece is defined by the interval of C for which subset $J_C = \{ j \in \{1,...N\} \mid X_j > C \}$ is fixed. We can enumerate all pieces using an ordered set of atoms. Let us sort the atoms X_j, $j = 1,...,N$ in descending order $(j_1, ..., j_\sigma, ..., j_N)$. Then t-th linear piece is defined by the first t atoms. The linear function describing t-th linear piece is $\sum_{\sigma=1}^{t} p_{j_\sigma} (X_{j_\sigma} - C)$. The $N+1$-th piece corresponds to $J_C = \varnothing$ and has the fixed value $E[X]$.

Intersections of these linear functions with the linear function on the left-hand side of (5) are described by a formula like (7). The intersection that has the maximal value C gives the solution for the equation (5). Hence,

$$e_K(X) = E[X] + \max_{t=1,\dots,N} \frac{\sum_{\sigma=1}^{t} p_{j_\sigma}(X_{j_\sigma} - E[X])}{K + \sum_{\sigma=1}^{t} p_{j_\sigma}}. \tag{9}$$

We omit the $N+1$-th linear piece here because it is not needed to calculate the maximum.

Formula (9) is also correct for $K=0$ in the case of a finite discrete distribution, that corresponds to $q=1$ in (2) or to $q_2=0$ in (3). In this case, expectile is equal to the maximal atom's value $e_{K=0}(X) = X_{j_1}$.

In the case of an **arbitrary random variable** X, we can approximate the Partial Moment on the right-hand side of (5) with prescribed accuracy by a piecewise linear function with a finite set of pieces. Such approximation produces corresponding finite discrete distribution and the formula (9) can be used for approximated calculation of the expectile.

To find the maximum in (8) or (9) it is not necessary to consider the whole domain $[0,1]$ where $CVaR_\gamma$ is defined or all atoms of the distribution. It suffices to consider the interval $[F_X(E[X]), 1]$ of confidence level γ or atoms with a value greater than the mean $X_j > E[X]$.

Thus, the following formula gives the exact expectile value in the general case

$$e_K(X) = E[X] + \sup_{F_X(E[X])<\gamma<1} \frac{(1-\gamma)\cdot CVaR_\gamma(Y)}{K+(1-\gamma)}, \tag{10}$$

where $Y = X - E[X]$ and $K>0$. Formula (10) is transformed Kusuoka representation [17] of expectile. The general Kusuoka representation for law-invariant coherent risk measures is discussed, for example, in [18, 19]. Formula (10) has a narrower interval for the variable γ and simpler notation (see for comparison Proposition 9 in [3] and section 3.2.1. in [7]).

The optimal value of the parameter γ in (10). To find the optimal value γ^* in (10) we first find zero value of the derivative by γ of the fraction in (10) for the points γ where $CVaR_\gamma(Y)$ is smooth. In this case, the derivative is equal to zero for the γ^* such that

$$\frac{(1-\gamma^*)}{K+(1-\gamma^*)} CVaR_{\gamma^*}(Y) = VaR_{\gamma^*}(Y).$$

The points γ where CVaR is not smooth correspond to the discontinuity of VaR. To deal with such cases we use VaR defined as an interval $VaR_\gamma = [VaR_\gamma^-, VaR_\gamma^+]$ (see for example [11, 20]). Then, general optimality condition is

$$\frac{(1-\gamma^*)}{K+(1-\gamma^*)} CVaR_{\gamma^*}(Y) \in VaR_{\gamma^*}(Y).$$

Combining this relation with (10) we obtain

$$e_K(X) \in VaR_{\gamma^*}(X) \quad \text{and} \quad \gamma^* = F_X(e_K(X)).$$

This result is intuitively obvious, because the expectile (optimal value C^* in (1)) divides the whole interval of a random variable into two subintervals with different weights. The same division should produce the optimal value γ^* in the formula (10) or the optimal t^* in the formula (9). The optimal value γ^* gives a solution for equations defining expectile through a partial moment and probability in [9, 15].

2. A new family of expectiles

As a rule, expectile is compared with quantile (VaR) on the whole domain of VaR and whole interval $(0, 1)$ of expectile parameter. A comparison for many continuous distributions was made in [7, 14]. But we deliberately have not considered expecile with parameter values $0 < q < 0.5$ or $0 < q_1 < q_2$ or $K < -1$, since our goal is to compare expectile with CVaR. Expectile with parameter in the interval $0.5 \le q \le 1$ and $CVaR_\alpha(X)$ with confidence level in the interval $0 \le \alpha \le 1$ have similar features. They equal to $E[X]$ at the left endpoint of its intervals and equal to the maximal value of X (if such finite value exists) at the right endpoint. Expectile changes its properties at the point $q = 0.5$, so we suggest working with expectile on the left tail of distribution as with lower CVaR. This means using the following simple equalities to deal with expectile in the left tail of the distribution (see [3, 14]). These relations are

$$e_q(X) = -e_{1-q}(-X) \text{ for } 0 < q < 0.5 \quad \text{and} \quad e_K(X) = -e_{-1-K}(-X) \text{ for } K < -1.$$

Note that $K \notin [-1, 0]$ for positive coefficients q_1, q_2. If one of these coefficients is equal to zero expectile can be estimated using limit operation.

To simplify comparison with CVaR we change the parameter in formula (1) on α, where $0 < \alpha < 1$ and $q = (1+\alpha)/2$. Then we have

$$e_\alpha(X) = \arg\min_{C \in R^1} \left\{ \frac{1+\alpha}{2} E[((X-C)^+)^2] + \frac{1-\alpha}{2} E[((X-C)^-)^2] \right\}. \tag{11}$$

We will distinguish expectile defined by formula (11) from expectiles defined by formulas (1) and (5) by subscript α. Formula (11) "stretches" expectile (1) as a function of the parameter two times left.

Now we can compare Cvar and expectile writing formulas with the same parameter $0 < \alpha < 1$

$$CVaR_\alpha(X) = E[X] + CVaR_\alpha(Y),$$

$$e_\alpha(X) = E[X] + \sup_{F_X(E[X]) < \gamma < 1} \frac{(1-\gamma) \cdot CVaR_\gamma(Y)}{K_\alpha + (1-\gamma)},$$

where $K_\alpha = \frac{1-\alpha}{2\alpha}$.

To compare quantile (VaR) with this expectile in the usual manner we will "stretch" VaR twice to the left using the confidence level $\beta_\alpha = 1-(1-\alpha)/2 = (1+\alpha)/2$

$$VaR_{\beta_\alpha}(X) = E[X] + VaR_{\beta_\alpha}(Y).$$

These three risk measures are well studied and described in many works [1, 4, 5, 7, 21 – 24]. VaR is an elicitable risk measure but lacks of coherency and considers only a percentile of the distribution. CVaR is a coherent risk measure but lacks of elicitability and considers only the right tail (in our interpretation) of the distribution. Expectile is a coherent and elicitable risk measure that takes into account the whole distribution and assigns greater weight to the right tail. Expectile the only coherent risk measure that is elicitable [5].

Our next goal is to consider a new family of expectile functions using a power parameter $\beta > 0$ for the coefficients in formula (11), namely

$$e_{\alpha\beta}(X) = \underset{C\in R^1}{\arg\min}\left\{\left(\frac{1+\alpha}{1-\alpha}\right)^{\beta} E[((X-C)^+)^2] + \left(\frac{1-\alpha}{1+\alpha}\right)^{\beta} E[((X-C)^-)^2]\right\}. \quad (12)$$

Since, the solution in (1), (3), (11), and (12) depends only on the ratio of the coefficients before mean operators, these formulas give the same solutions if $\frac{q}{1-q} = \frac{q_1}{q_2} = \left(\frac{1+\alpha}{1-\alpha}\right)^{2\beta}$. So, formulas (11) and (12) are equivalent for $\beta = 0.5$.

Now we show four examples with simple uniform discrete distributions to compare CVaR and expectiles $e_{\alpha\beta}(X)$ for different β as functions of α.

Each example contains 5 atoms with the probability 0.2. The minimal atom's value is 30, the maximal is 100. In Fig. 1*a* atoms' values vary uniformly from 30 to 100. Fig. 1*b* contains three larger atoms in the middle of the distribution. Fig. 1*c* contains four large atoms and one small. Fig. 1*d* contains four small atoms and one large. (The legend entries for the expectiles are arranged in the same order as the curves).

These examples show that the following expectiles are closest to CVaR: in the first case with $\beta = 1$, in the second and third cases with $\beta = 1.5$, and in the last case with $\beta = 0.5$. Thus, using different β may be useful in approximation VaR and CVaR function by expectle.

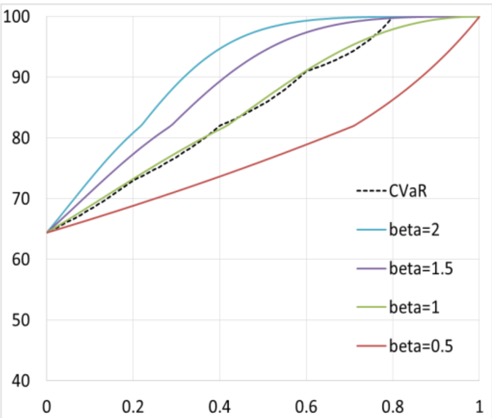

FIG. 1*a*. Atoms 30, 46, 64, 82, 100, mean 64.4.

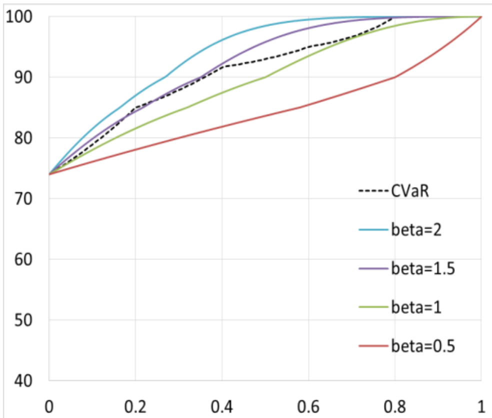

FIG. 1*b*. Atoms 30, 65, 85, 90, 100, mean 74.

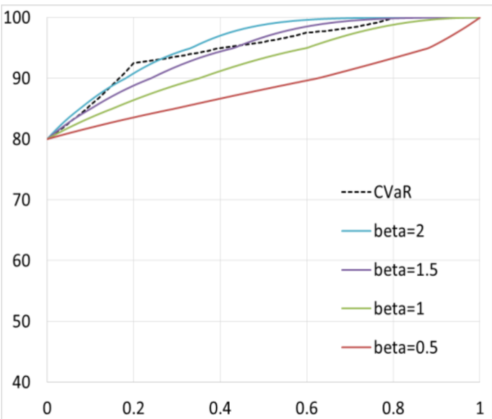

FIG. 1*c*. Atoms 30, 85, 90, 95, 100, mean 80.

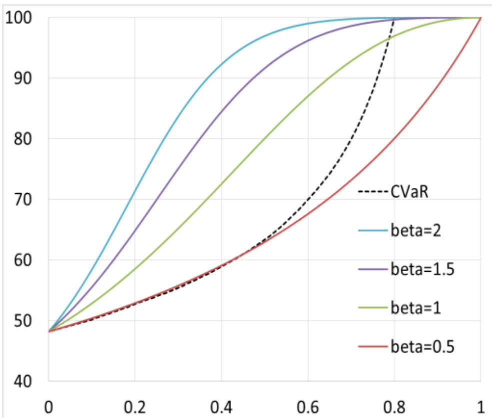

FIG. 1*d*. Atoms 30, 34, 37, 40, 100, mean 48.2.

To compare VaR, CVaR, and expectile in continuous case we use Standard Normal, Uniform on $[0,1]$, and Exponential with $\lambda=1$ distributions. The first two distributions are symmetric, so we only show the right tail of distributions (Fig. 2*a* and Fig. 2*b*). Exponential distribution is not symmetric. We split it into left and right tails (Fig. 3*a* and Fig. 3*b*) in the median and calculate expectiles and CVaR functions on the left tail as $-f_\alpha(-X)$. VaR function is stretched on all figures twice to the left of point $\alpha=1$ to compare it with other functions as in [7, 14, 16].

(The legend entries for the expectiles on these and other figures are arranged in the same order as the curves).

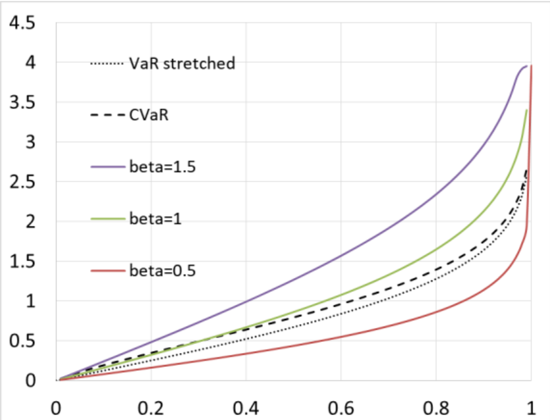

FIG. 2*a*. Standard Normal Distribution

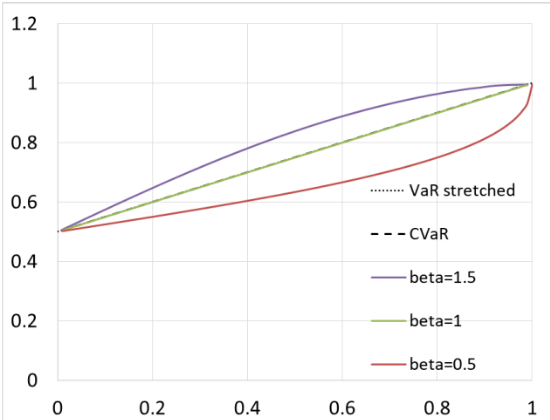

FIG. 2*b*. Uniform distribution on [0, 1]

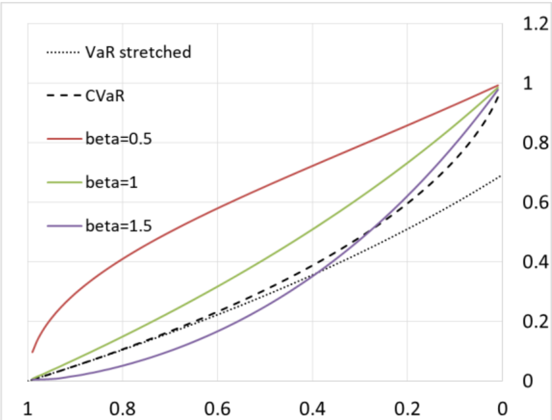

FIG. 3*a*. Exponential distribution (λ=1) on the left tail

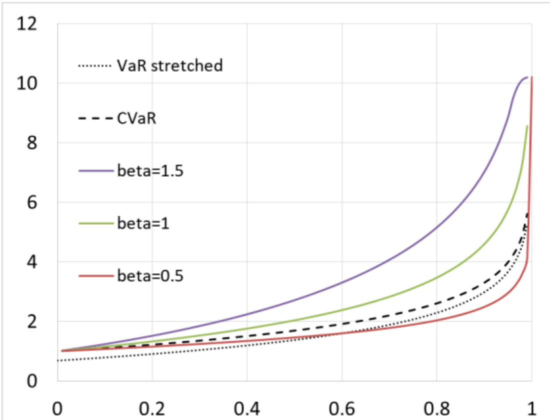

FIG. 3*b*. Exponential distribution (λ=1) on the right tail

These examples show that CVaR function gives the best approximation for VaR (quantile). In the case of Normal distribution, expectiles with $\beta=1$ and $\beta=0.5$ seem to give closer approximations. CVaR and expectile with $\beta=1$ give exact approximations for uniform distribution.

In the case of Exponential distribution, expectile with $\beta=0.5$ gives the best approximation on the right tail and with $\beta=1.5$ on the left tail of the distribution. But over the entire interval $(0,1)$, the best approximation is given by expectile with $\beta=1$.

Bellini and other authors [14, 21] note that for the most common distributions, the expectile is closer to the center of the distribution than the corresponding quantile, and the two curves typically intersect in a unique point, which corresponds to the center of symmetry in the case of symmetric distribution.

In the case of asymmetric distribution expectiles with different β intersect quantile in different points. Taking into account that value $E[X]$ is common for different expectiles at the endpoint of the domain and that confidence level $F_X(E[X])$ is used in definition (10) of expectile we propose another way for comparison quantile and expectiles. We split the domain of quantile into two non-equal intervals: left $\left[0, F_X(E[X])\right]$ and right $\left[F_X(E[X]), 1\right]$. In this case, quantile and expectile have the same value at the endpoints of its intervals. Then we compare quantile on these intervals with "left" and "right" expectiles and CVaRs. Below we compare Exponential distribution with $\lambda = 1$ (Fig. 4*a* and Fig. 4*b*) and Gamma distribution with shape 3 and scale 1 (Fig. 5*a* and Fig. 5*b*) on its left and right intervals with expectiles.

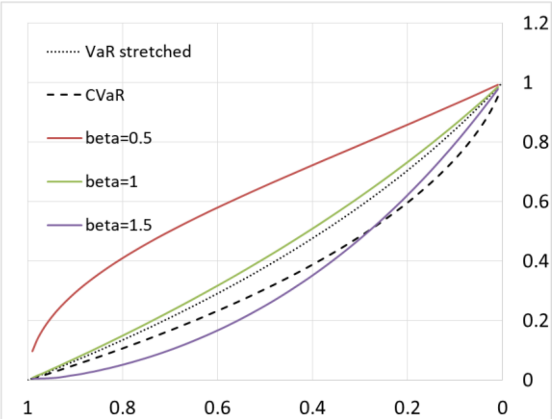

FIG. 4*a*. Exp. distribution (λ=1) on the interval (0, 0.6321]

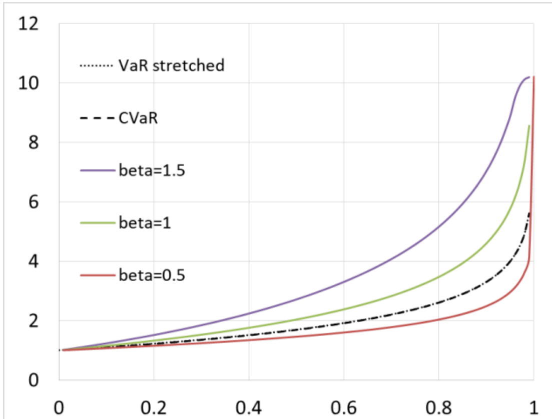

FIG. 4*b*. Exp. distribution (λ=1) on the interval [0.6321, 1)

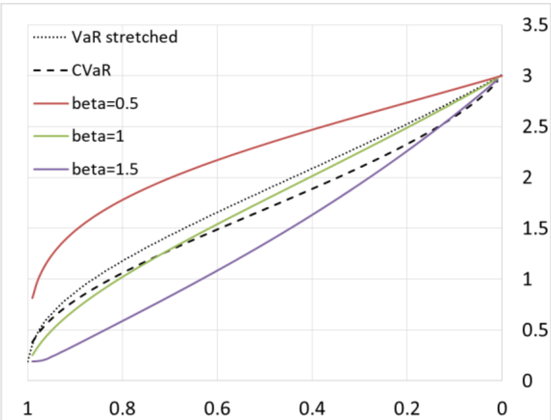

FIG. 5*a*. Gamma distribution (3,1) on the interval (0, 0.577]

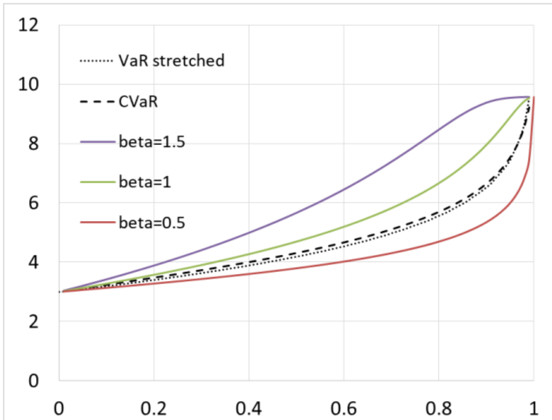

FIG. 5*b*. Gamma distribution (3,1) on the interval [0.577, 1)

We see that quantile is close to expectile with $\beta = 1$ in Fig. 4*a* and coincides with CVaR in Fig. 4*b*. for Exponential distribution.

For the Gamma distribution, quantile is also close to expectile with $\beta = 1$ in Fig. 5*a* and is very closed to CVaR in Fig. 5*b*.

3. Expectile linearization

We consider here a random loss function $L(x)$, where $x \in R^n$ is a vector of decision variables, and expectile $e_\alpha(L(x))$ defined by formula (11) for $\alpha \in (0,1)$.

Lemma. Expectile $e_\alpha(L(x))$ is a convex function of x if $L(x)$ is a convex.

Proof. To prove lemma we use equation (5) with a random loss function $L(x)$ and $K = \frac{1-\alpha}{2\alpha}$.

$$KC - KE[L(x)] = E[(L(x) - C)^+]. \tag{13}$$

Consider two arbitrary points x_1 and x_2. Let's denote values of expectile corresponding to these points as C_1 and C_2, values of means as $E_1 = E[L(x_1)]$ and $E_2 = E[L(x_2)]$, random variables as $L_1 = L(x_1)$ and $L_2 = L(x_2)$. Then

$$KC_1 = KE_1 + E[(L_1 - C_1)^+] \text{ and } KC_2 = KE_2 + E[(L_2 - C_2)^+].$$

The linear combination of these two equalities for $\lambda \in [0,1]$ is

$$K(\lambda C_1 + (1-\lambda)C_2) = K(\lambda E_1 + (1-\lambda)E_2) + \lambda E[(L_1 - C_1)^+] + (1-\lambda)E[(L_2 - C_2)^+]. \tag{14}$$

Since the mean $E[L(x)]$ is a convex function of x then

$$\lambda E_1 + (1-\lambda)E_2 \ge E[L(\lambda x_1 + (1-\lambda)x_2)]. \tag{15}$$

The following inequality is true for any realization $L^\omega(x)$ of a loss function

$$\lambda(L_1^\omega - C_1)^+ + (1-\lambda)(L_2^\omega - C_2)^+ \ge \left(\lambda L_1^\omega + (1-\lambda)L_2^\omega - \lambda C_1 - (1-\lambda)C_2\right)^+.$$

Then following inequalities are true for Partial Moment function on the right-hand side of (13)

$$\lambda E[(L_1 - C_1)^+] + (1-\lambda)E[(L_2 - C_2)^+] \ge E\left[\left(\lambda L_1 + (1-\lambda)L_2 - \lambda C_1 - (1-\lambda)C_2\right)^+\right] \ge$$

$$\ge E\left[\left(L(\lambda x_1 + (1-\lambda)x_2) - \lambda C_1 - (1-\lambda)C_2\right)^+\right]. \tag{16}$$

Then substituting (15) and (16) into (14) we have

$$K(\lambda C_1 + (1-\lambda)C_2) \ge KE_3 + E\left[\left(L_3 - \lambda C_1 - (1-\lambda)C_2\right)^+\right], \tag{17}$$

where $x_3 = \lambda x_1 + (1-\lambda)x_2$, $L_3 = L(x_3)$, $E_3 = E[L_3]$.

Expectile for the point x_3 is defined by equation $KC_3 = KE_3 + E[(L_3 - C_3)^+]$. Comparing this equation with (17) and taking into account that $K > 0$ and Partial Moment $E[(X - C)^+]$ is decreasing function of C we derive that $C_3 \le \lambda C_1 + (1-\lambda)C_2$. Hence, expectile $e_\alpha(L(x))$ is a convex function of x.

Lemma is proved.

Since expectile of a convex loss function is convex it can be linearized in a convex optimization problem when the loss function is linear with a finite set of scenarios $j = 1,\dots,N$. Each scenario j is a linear function $L^j(x)$. Different variants of expectile linearization are shown in papers [6, 7, 25]. We propose variants corresponding to our representation (9) that contain the minimum number of additional variables and can be used in linear optimization problems.

The Partial Moment function on the right-hand side of (13) can be expressed in a way dual to (6) as

$$E[(L(x)-C)^{+}]=\min_{u_j\ge 0}\left\{\sum_{j=1}^{N}u_j p_j \mid u_j \ge L^j(x)-C,\ j=1,\dots,N\right\}.$$

Then expectile $e_K(L(x))$ for $K>0$ is calculated as

$$e_K(L(x))=\min_{C,u_j} C, \tag{18}$$

$$C\ge E[L(x)]+\frac{1}{K}\sum_{j=1}^{N}p_j u_j, \tag{19}$$

$$u_j\ge L^j(x)-C,\ u_j\ge 0,\ j=1,\dots,N. \tag{20}$$

To solve a linear optimization problem with expectile using linear programming methods expectile is replaced with the variable C. The variables C, u_j and constraints (19), (20) are added to the optimization problem. After solving the optimization problem value of expectile should be calculated since the optimal value C^* may be greater than expectile at the optimal point. For example, in case if expectile enters in a constraint that is not active at the optimal point.

It can be helpful to use a linear maximization problem to calculate expectile. Such problem can be obtained by reducing the linear-fractional problem (9) to a linear problem or by transforming a problem dual to (18) – (20). We formulate a linear maximization problem as

$$e_K(L(x))=E[L(x)]+\max_{w_j}\sum_{j=1}^{N}\left(L^j(x)-E[L(x)]\right)p_j w_j,$$

$$Kw_j+\sum_{i=1}^{N}p_i w_i\le 1,\ \ j=1,\dots,N,$$

$$w_j\ge 0,\ \ j=1,\dots,N.$$

If we know the optimal value $e_K(L(x))$ of the objective then the optimal values of variables are restored as follows. Let J_+ be a subset of $J=\{1,\dots N\}$ such that $L^j(x)>e_K(L(x))$ for $j\in J_+$ and $L^j(x)\le e_K(L(x))$ for $j\in J\setminus J_+$. Then $w_j=1/(K+1-\gamma^*)$ for $j\in J_+$ and $w_j=0$ for $j\in J\setminus J_+$, where $\gamma^*=1-\sum_{j\in J_+}p_j$. This result corresponds to the formulation of the optimization problem in (9) and γ^* is the optimal confidence level in (8).

The LP formulation in [6] is derived from the dual representation of expectile [3] for continuous distribution using Radon – Nikodym derivatives. Comparing our result with the result in [6] we can say that discrete analogs φ_j of Radon – Nikodym derivatives are equal to $\varphi_j=1+w_j-\sum_{i=1}^{N}p_i w_i$. Taking into account that the optimal values of $w_j,\ j=1,\dots,N$ have two variants, we can specify that $\varphi_j=(K+1)\big/(K+1-\gamma^*)$ for $j\in J_+$ and $\varphi_j=K\big/(K+1-\gamma^*)$ for $j\in J\setminus J_+$. Variable m in [6, section 2.2] equals to the sum of these values $m=(2K+1)\big/(K+1-\gamma^*)$.

4. The fundamental risk quadrangle and expectile

The definition of the fundamental risk quadrangle was given in the paper Rockafellar and Uryasev [10]. This concept links together four functions of a random variable X: risk $R(X)$, deviation $D(X)$, regret $V(X)$, and error $\mathcal{E}(X)$. These functions are related using mean value $E[X]$ and the optimal value of some scalar parameter called statistic $S(X)$. This value estimates certain characteristic of a random variable. The concept of the fundamental risk quadrangle combines estimation and optimization tasks for random value. To estimate different characteristics of a random variable different risk quadrangles are used. For example, there are mean (average-based) quadrangle, quantile (VaR) quadrangle, superquantile (CVaR) quadrangle, and so on.

The paper [10] focuses on the regular risk quadrangles. The four functions to be elements of the regular risk quadrangle should be regular measures of risk, deviation, regret, and error. The regular risk quadrangles may be scaled, mixed, reverted, and so on according to theorems from [10]. The regular risk quadrangle has a set of "good" properties for estimation and optimization.

According to [10] a regular measure of risk is closed convex functional with values in $(-\infty,\infty]$ such that $R(X) > E[X]$ for nonconstant X and $R(X) = X$ for constant X, i.e. X having one value with probability 1.

A regular measure of deviation is closed convex functional with values in $[0,\infty]$ such that $D(X) > 0$ for nonconstant X and $D(X) = 0$ for constant X.

A regular measure of error is closed convex functional with values in $[0,\infty]$ such that $\mathcal{E}(0) = 0$ for constant $X = 0$, otherwise $\mathcal{E}(X) > 0$, satisfying the convergence condition: if $\lim\limits_{k\to\infty} \mathcal{E}(X^k) = 0$ then $\lim\limits_{k\to\infty} E[X^k] = 0$, where $\{X^k\}$ is a sequence of random variables.

And a regular measure of regret is closed convex functional with values in $(-\infty,\infty]$ such that $V(0) = 0$ for constant $X = 0$, otherwise $V(X) > E[X]$, satisfying the convergence condition: if $\lim\limits_{k\to\infty}\left(V(X^k) - E[X^k]\right) = 0$ then $\lim\limits_{k\to\infty} E[X^k] = 0.$

For the regular risk quadrangle risk, deviation, regret, error, and statistic are related as follows:

$$V(X) = \mathcal{E}(X) + E[X], \tag{21}$$

$$R(X) = \min_{C\in R^1}\{C + V(X - C)\} = \min_{C\in R^1} \mathcal{E}(X - C) + E[X] = D(X) + E[X], \tag{22}$$

$$S(X) = \arg\min_{C\in R^1}\{C + V(X - C)\} = \arg\min_{C\in R^1} \mathcal{E}(X - C). \tag{23}$$

Our goal is to build quadrangles with expectile function and to analyze its properties.

The first risk quadrangle is prompted by the definition (1) of expectile function.

We define an error function with parameter q as $\mathcal{E}_q(X) = qE[((X)^+)^2] + (1-q)E[((X)^-)^2]$.

Expectile function is a statistic in such quadrangle $S(X) = e_q(X)$.

The deviation is equal to $D_q(X) = qE[((X - e_q(X))^+)^2] + (1-q)E[((X - e_q(X))^-)^2]$.

The regularity property holds for this quadrangle but it seems that deviation and risk functions do not have attractive expressions and properties.

The second risk quadrangle can be constructed using equation (5). As shown above, equation (5) has a unique solution that is equal to expectile. The equation (5) is equivalent to

$$0 = E[X - e_K(X)] + \frac{1}{K} E[(X - e_K(X))^+].$$

The last equation prompts formulas which define possible error functions

$$\mathcal{E}_K(X) = \left(E[X] + \frac{1}{K} E[(X)^+] \right)^2 \text{ or } \mathcal{E}_K(X) = \left| E[X] + \frac{1}{K} E[(X)^+] \right|.$$

Expectile is a statistic in this quadrangle, but the quadrangle is not regular because it has zero deviation.

The third risk quadrangle is constructed as a solution of equation (5) for $K > 0$ in the form

$$e_K(X) = \min_{C \in R^1} \left(\max \left\{ C, E[X] + \frac{1}{K} E[(X - C)^+] \right\} \right). \quad (24)$$

Since equation (5) has a unique solution, and the minimization problem in (24) is convex, it has a single solution. The optimization problem in (24) can be reformulated as

$$e_K(X) = \min_{C \in R^1} \left(\max \left\{ -\left(E[X - C] \right), \frac{1}{K} E[(X - C)^+] \right\} \right) + E[X]. \quad (25)$$

This notation coincides with the definition of risk through the deviation plus the mean in (22).

The error and regret functions in this quadrangle are equal to

$$\mathcal{E}_K(X) = \max \left\{ -E[X], \frac{1}{K} E[(X)^+] \right\} \text{ and } V_K(X) = \left(E[X] + \frac{1}{K} E[(X)^+] \right)^+. \quad (26)$$

Expectile in this quadrangle is both a risk and statistic. The regularity property holds for functions of this quadrangle, so this quadrangle is regular.

5. Conclusions

After considering different definitions and representations of expectile, we can divide them into two types. The first defines expectile as the solution of an equation. Such equations cannot be solved analytically; therefore, effective procedures are needed to solve these equations. The second defines expectile as a solution of an optimization problem with one variable parameter. Expectile is equal to the optimal value of objective or the optimal parameter value. We formulated two new representations of expectile of the second type. In the first representation, an expression is maximized by the confidence level of CVaR. This representation is related to other known representations through a transformation but has a simpler formulation and a narrower interval for the variable confidence level. The second representation defines expectile as a risk function of the new risk quadrangle. Expectile, in this case, is a result of minimization of the error function.

The next conclusion is the follows. The dependence of expectile on its parameter can be formulated in different ways. Moreover, the two parameters can be used. This is equivalent to changing variables in equation defining expectile. The two parameters of expectile and unequal partition of quantile domain on the left and right tail allow approximate quantiles by expectiles with more accuracy.

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Нове сімейство експектилів та його властивості

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Вступ. У статті розглядається міра ризику, що називається експектиль. Експектиль – це характеристика випадкової величини, яка обраховується з використанням асиметричного методу найменших квадратів. Рівень асиметрії задається параметром, що змінюється в інтервалі (0, 1). Експектиль використовується у фінансовому аналізі, портфельній оптимізації, в інших задачах оцінки так само, як квантиль (Value-at-Risk або VaR) та суперквантиль (Conditional Value-at-Risk або CVaR). Але експектиль має ряд переваг. Експектиль – це одноразово і когерентна, і сприйнятлива (elicitable) міра ризику, що враховує весь розподіл випадкової величини, але надає більшу вагу правому хвосту.

Мета роботи. Як правило, експектиль порівнюється із квантилем. Наша мета – порівняти експектиль із суперквантилем (CVaR), використовуючи однаковий параметр – рівень довіри. Для цього спочатку дається нове представлення експектиля через зважену суму середнього та CVaR. Потім розглядається нове сімейство експектилей, яке задається двома параметрами. Такі експектилі порівнюються з квантилем та CVaR для різних неперервних та скінчених дискретних розподілів. Ще одна мета – побудувати регулярний ризик-квадрат, де експектиль є функцією ризику.

Результати. Запропоновано та обгрунтувано два нові вирази, що визначають експектиль. Перший вираз використовує максимізацію, в якій змінюється рівень довіри CVaR та коефіцієнт перед CVaR. Цей вираз конкретизовано для неперервних та скінченних дискретних розподілів. Другий вираз використовує мінімізацію нової функції помилок у новому ризик-квадраті. Використання двох параметрів у визначенні експектиля змінює його залежність від рівня довіри та генерує нове сімейство експектилів. Порівняння нових експектилів з квантилем та CVaR для ряду розподілів показує, що запропоновані експектилі можуть бути ближчі до квантиля, ніж стандартний експектиль. Запропоновано два варіанти лінеаризації експектиля та показано, як їх використовувати з лінійною функцією втрат.

Ключові слова: експектиль, EVaR, квантиль, суперквантиль, CVaR, представлення Кусуокі, фундаментальний ризик квадрат, пакет Portfolio Safeguard.

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Новое семейство экспектилей и его свойства

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Введение. В статье рассматривается мера риска, которая называется экспектиль. Экспектиль – это характеристика случайной величины, вычисляемая с использованием асимметричного метода наименьших квадратов. Уровень асимметрии задается параметром, изменяющимся в интервале (0, 1). Экспектиль используется в финансовом анализе, портфельной оптимизации, в других задачах оценки так же, как квантиль (Value-at-Risk или VaR) и суперквантиль (Conditional Value-at-Risk или CVaR). Но экспектиль имеет ряд преимуществ. Экспектиль является одновременно и когерентной, и восприимчивой (elicitable) мерой риска, учитывающей все распределение случайной величины, но дает больший вес правому хвосту.

Цель работы. Как правило, экспектиль сравнивается с квантилем. Наша задача – сравнить экспектиль с суперквантилем (CVaR), используя одинаковый параметр – уровень доверия. Для этого мы сначала даем новое представление экспектиля через взвешенную сумму среднего и CVaR. Потом рассматриваем новое семейство экспектилей, которое задается двумя параметрами. Такие экспектили сравниваются с квантилем и CVaR для разных непрерывных и конечных дискретных распределений. Еще одна цель – построить регулярный риск-квадрат, где экспектиль является функцией риска.

Результаты. Предложено и обосновано два новых выражения, определяющие экспектиль. Первое выражение использует максимизацию, в которой меняется уровень доверия CVaR и коэффициент перед CVaR. Это выражение конкретизировано для непрерывных и конечных дискретных распределений. Второе выражение использует минимизацию новой функции ошибки в новом риск-квадрате. Использование двух параметров в определении экспектиля меняет его зависимость от уровня доверия и генерирует новое семейство экспектилей. Сравнение новых экспектилей с квантилем и CVaR для ряда распределений показывает, что предложенные экспектили могут быть ближе к квантилю, чем стандартный экспектиль. Предложено два варианта линеаризации экспектиля и показано, как их использовать с линейной функцией потерь.

Ключевые слова: экспектиль, EVaR, квантиль, суперквантиль, CVaR, представление Кусуоки, фундаментальный риск-квадрат, пакет Portfolio Safeguard.