UDC 517.9

I. A. Vestfrid (Haifa, Israel)

ISOMETRIC EMBEDDING OF SOME METRIC SPACES IN l_p-SPACES

ІЗОМЕТРИЧНЕ ВКЛАДЕННЯ ДЕЯКИХ МЕТРИЧНИХ ПРОСТОРІВ У l_p-ПРОСТОРИ

We present a generalization of the Fichet result who proved in 1988 that every ulltrametric space consisting of n points is embedded isometrically in l_p^{n-1}, $p \geq 1$.

Наведено узагальнення результату Фічета, який у 1988 р. довів, що кожний ультраметричний простір, що складається з n точок, ізометрично вкладений у l_p^{n-1}, $p \geq 1$.

Embeddability is a central theme in modern mathematics, common to various different fields such as set theory, topology, algebra and functional analysis. For instance, solving the inverse problem for the ε-entropy (see [1, p. 690] for formulations) Timan and Kreĭnovič [2 – 4] showed that this problem can be partially reduced to an isometric embedding of some ultrametric compact sets into Banach spaces with a previously chosen norm. These papers also contain results on such embedding into the spaces l_p and L_p for $p \geq 1$, as well as into C. The author [5 – 7] showed that *every* separable ultrametric space is isometrically embeddable in l_1, l_2 and c_0.

Recall that a metric space (R, ρ) is said to be *ultrametric* if the triangle inequality is satisfied in the stronger form

$$\rho(x, y) \leq \max\{\rho(x, z), \rho(y, z)\}.$$

An important class of ultrametric spaces is obtained from non-Archimedean valuations over fields. For instance, the p-adic valuation $|\cdot|_p$ over the p-adic field $\mathbb{Q}_p$ satisfies $|x+y|_p \leq \max\left\{|x|_p, |y|_p\right\}$ and the corresponding distance $\rho(x, y) = |x-y|_p$ is ultrametric.

The same problem on isometric embedding of ultrametric spaces, as well as of some other metric spaces which are their generalizations, also arises in taxonomy, psychology, theoretical physics, theoretical chemistry etc. So for example, for needs of data analysis Fichet [8] proved that every finite ultrametric space R embeds isometrically in $l_p^{|R|-1}$, whenever $p \geq 1$. (Here and later on we denote by $|A|$ the cardinality of the set A.)

Here we continue this investigation and give the following theorem.

Theorem 1. *Every countable metric space* $R = R(a_0, a_1, a_2, \ldots)$ *with a metric* ρ, *which satisfies the following conditions* $\forall\, 0 \leq j \leq m \leq k \leq i$:

1. $\rho^p(a_j, a_0) - \rho^p(a_m, a_0) \leq \rho^p(a_j, a_i) - \rho^p(a_m, a_i)$;

2. $\rho^p(a_m, a_k) + \rho^p(a_j, a_i) \le \rho^p(a_j, a_k) + \rho^p(a_m, a_i)$

is isometrically embeddable in l_p, $p \ge 1$. *If moreover* R *consists of* $n+1$ *points, then* R *is isometrically embeddable in* l_n^p.

As we show bellow by some results from [7], this theorem extends Fichet's one.

We begin the proof of the theorem with the following statement.

Lemma 1. *Let* $\varphi(t)$ *be a nonnegative continuous even function such that:*

(a) $\varphi(0) = 0$;

(b) $\forall\, 0 \le t_1 \le t_2 \le t_3 \le t_4 \quad \varphi(t_3 - t_1) - \varphi(t_3 - t_2) \le \varphi(t_4 - t_1) - \varphi(t_4 - t_2)$;

(c) $\forall\, t_1, t_2 > 0 \quad \varphi(t_1) + \varphi(t_2) < \varphi(t_1 + t_2)$.

Let $D = (d_{ij})$, $i = 0, 1, 2, \ldots$, $j = 0, 1, 2, \ldots$; *be a real symmetric matrix such that* $\forall\, 0 \le i \le k \le m \le n$:

1. $d_{ik} > 0$ *for* $i \ne k$, $d_{ii} = 0$;
2. $d_{i0} - d_{k0} \le d_{mi} - d_{mk}$;
3. $d_{i0} - d_{10} < d_{1i}$ *for* $i > 1$, $d_{mk} - d_{nk} \le d_{mi} - d_{ni}$.

Then the system

$$\begin{aligned} \varphi(x_1^1) &= d_{10}, \\ \varphi(x_2^1) + \varphi(x_2^2) &= d_{20}, \\ \varphi(x_1^1 - x_2^1) + \varphi(x_2^2) &= d_{21}, \\ &\cdots\cdots \\ \sum_{s=1}^{k} \varphi(x_k^s) &= d_{k,0}, \\ &\cdots\cdots \\ \sum_{s=1}^{i} \varphi(x_i^s - x_k^s) + \sum_{s=i+1}^{k} \varphi(x_k^s) &= d_{k,i}, \quad i < k, \\ &\cdots\cdots \\ x_i^i &\ge 0 \quad \text{for all } i \end{aligned} \tag{I}$$

has one and only one real solution $\{x_i^s\}$, $i = 1, 2, \ldots$, $s = 1, \ldots, i$, *and this solution satisfies the following condition:*

$$\text{for each } i, \quad x_i^i > x_{i+1}^i \ge x_{i+2}^i \ge \ldots \ge 0. \tag{*}$$

Proof. Note that the function $\varphi(t)$ strictly increases on $[0, \infty)$. Hence for every $A > 0$, the function

$$\psi(t; A) = \varphi(t) - \varphi(A - t)$$

strictly increases on t on the segment $[0, A]$. Besides, by conditions (b) and (c), it is monotonic nondecreasing on the whole real axis, and for all t_1, t_2 such that $t_1 < 0 < < A < t_2$,

$$\psi(t; A) < \psi(0; A) = -\varphi(A) < 0 < \varphi(A) = \psi(A; A) \le \psi(t_2; A). \tag{1}$$

It follows from (c) that $\varphi(t)$ is not bounded. Hence, the equation $\varphi(x) = d_{10}$ has a unique nonnegative solution x_1^1. It follows from (I) and conditions 2 and 3 that

$$\begin{aligned} \psi(x_2^1; x_1^1) &= \varphi(x_2^1) - \varphi(x_1^1 - x_2^1) = d_{20} - d_{12}, \\ \psi(0; x_1^1) &= -d_{10} \le d_{20} - d_{12}, \\ \psi(x_1^1; x_1^1) &= d_{10} > d_{20} - d_{12}. \end{aligned} \tag{2}$$

Since $\psi(t; x_1^1)$ strictly increases on $[0, x_1^1]$ and in virtue of (1), the equation

$$\psi(t; x_1^1) = d_{20} - d_{12}$$

has one and only one real solution x_2^1; besides $0 \le x_2^1 < x_1^1$. It follows from (2) that if $\varphi(x_2^1) \ge d_{20}$, then $\varphi(x_1^1 - x_2^1) \ge d_{12}$ and

$$d_{20} + d_{12} \le \varphi(x_2^1) + \varphi(x_1^1 - x_2^1) < \varphi(x_1^1) = d_{10}.$$

But this contradicts condition 2. Hence, $\varphi(x_2^1) < d_{20}$ and, consequently, there exists one and only one $x_2^2 > 0$ such that

$$\varphi(x_2^2) = d_{20} - \varphi(x_2^1).$$

Proceed by induction and suppose that the lemma assertion is valid for the system

$$\begin{gathered}
\varphi(x_1^1) = d_{10}, \\
\varphi(x_2^1) + \varphi(x_2^2) = d_{20}, \\
\varphi(x_1^1 - x_2^1) + \varphi(x_2^2) = d_{21}, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\sum_{s=1}^{n-1} \varphi(x_{n-1}^s) = d_{n-1,0}, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\sum_{s=1}^{i} \varphi(x_i^s - x_{n-1}^s) + \sum_{s=i+1}^{n-1} \varphi(x_{n-1}^s) = d_{n-1,i}, \quad i < n-1, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\
\sum_{s=1}^{n-2} \varphi(x_{n-2}^s - x_{n-1}^s) + \varphi(x_{n-1}^{n-1}) = d_{n-1,n-2}, \\
x_i^i \ge 0 \quad \text{for all} \quad 1 \le i \le n-1.
\end{gathered} \tag{II}$$

We shall prove that the system

$$\begin{gathered}
\sum_{s=1}^{n} \varphi(x_n^s) = d_{n,0}, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\sum_{s=1}^{i} \varphi(x_i^s - x_n^s) + \sum_{s=i+1}^{n} \varphi(x_n^s) = d_{n,i}, \quad i < n, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\sum_{s=1}^{n-1} \varphi(x_{n-1}^s - x_n^s) + \varphi(x_n^n) = d_{n,n-1}, \\
x_n^n \ge 0,
\end{gathered} \tag{III}$$

has one and only one real solution $\{x_n^s\}$, $s = 1, \ldots, n$, and this solution satisfies the following conditions:

$$0 \le x_n^s \le x_{n-1}^s \quad \text{for} \quad s < n-1, \qquad 0 \le x_n^{n-1} < x_{n-1}^{n-1}, \quad x_n^n > 0.$$

Indeed, it follows from (III) that

$$\varphi(x_n^1) - \varphi(x_1^1 - x_n^1) = d_{n0} - d_{n1}.$$

By properties of the function $\psi(t; x_1^1)$, the equation

$$\psi(t; x_1^1) \equiv \varphi(t) - \varphi(x_1^1 - t) = d_{n0} - d_{n1}$$

has one and only one real root x_n^1, and it is nonnegative. In addition, by condition 3, we have

$$\psi(x_n^1; x_1^1) \equiv d_{n0} - d_{n1} \le d_{n-1,0} - d_{n-1,1},$$

and by (II),

$$\psi(x_{n-1}^1; x_1^1) = d_{n-1,0} - d_{n-1,1}.$$

Hence, $x_n^1 \le x_{n-1}^1$.

Now suppose that for a given $i \le n-1$, we have already $x_n^1, \ldots, x_n^{i-1}$, satisfying system (III), have the uniqueness of such solution and that $0 \le x_n^k \le x_{n-1}^k$ for any $k \le \le i-1$. Then it follows from (III) that

$$\sum_{s=1}^{i-1} \left[\varphi(x_n^s) - \varphi(x_i^s - x_n^s)\right] + \psi(x_n^i; x_i^i) = d_{n0} - d_{ni}.$$

By virtue of conditions 2 and 3 and properties of $\varphi(t)$, we have

$$\sum_{s=1}^{i-1} \left[\varphi(x_n^s) - \varphi(x_i^s - x_n^s)\right] + \psi(0; x_i^i) =$$

$$= \sum_{s=1}^{i-1} \left[\varphi(x_n^s) - \varphi(x_i^s - x_n^s) + \varphi(x_i^s)\right] - d_{i0} \le$$

$$\le \sum_{s=1}^{i-1} \left[\varphi(x_n^s) - \varphi(x_{i-1}^s - x_n^s) + \varphi(x_{i-1}^s)\right] - d_{i0} =$$

$$= d_{n0} - d_{n,i-1} + d_{i-1,0} - d_{i0} \le d_{n0} - d_{ni}$$

and

$$\sum_{s=1}^{i-1} \left[\varphi(x_n^s) - \varphi(x_i^s - x_n^s)\right] + \psi(x_i^i; x_i^i) =$$

$$= \sum_{s=1}^{i-1} \left[\varphi(x_n^s) - \varphi(x_i^s - x_n^s) - \varphi(x_{i-1}^s - x_i^s)\right] + d_{i,i-1} >$$

$$> \sum_{s=1}^{i-1} \left[\varphi(x_n^s) - \varphi(x_{i-1}^s - x_n^s)\right] + d_{i,i-1} = d_{n0} - d_{n,i-1} + d_{i,i-1} \ge d_{n0} - d_{ni}.$$

By properties of $\psi(t; x_i^i)$, the equation

$$\sum_{s=1}^{i-1} \left[\varphi(x_n^s) - \varphi(x_i^s - x_n^s)\right] + \psi(t; x_i^i) = d_{n0} - d_{ni}$$

has one and only one real root x_n^i, and it is nonnegative. By (II), (III), condition 3 and properties of $\varphi(t)$, we obtain

$$0 \ge d_{n-1,i} - d_{n,i} - d_{n-1,i-1} + d_{n,i-1} =$$

$$= \varphi(x_n^i) - \varphi(x_{n-1}^i) + \varphi(x_i^i - x_{n-1}^i) - \varphi(x_i^i - x_n^i) +$$

$$+\sum_{s=1}^{i-1}\left[\varphi\left(x_{i-1}^s-x_n^s\right)-\varphi\left(x_{i-1}^s-x_{n-1}^s\right)-\varphi\left(x_i^s-x_n^s\right)+\varphi\left(x_i^s-x_{n-1}^s\right)\right] \geq$$

$$\geq \psi\left(x_n^i; x_i^i\right) - \psi\left(x_{n-1}^i; x_i^i\right).$$

Hence, $x_n^i \leq x_{n-1}^i$.

It follows from the equality

$$\sum_{s=1}^{n-1}\varphi\left(x_n^s\right) - \sum_{s=1}^{n-1}\varphi\left(x_{n-1}^s - x_n^s\right) = d_{n0} - d_{n,n-1}$$

that if $\sum_{s=1}^{n-1}\varphi\left(x_n^s\right) \geq d_{n0}$, then $\sum_{s=1}^{n-1}\varphi\left(x_{n-1}^s - x_n^s\right) \geq d_{n,n-1}$ and

$$d_{n0} + d_{n,n-1} \leq \sum_{s=1}^{n-1}\left[\varphi\left(x_n^s\right)+\varphi\left(x_{n-1}^s - x_n^s\right)\right] < \sum_{s=1}^{n-1}\varphi\left(x_{n-1}^s\right) = d_{n-1,0}.$$

But this contradicts condition 2. Hence

$$\sum_{s=1}^{n-1}\varphi\left(x_n^s\right) < d_{n0}.$$

Consequently, there exists one and only one nonnegative root x_n^n of the equation

$$\varphi(t) = d_{n0} - \sum_{s=1}^{n-1}\varphi\left(x_n^s\right)$$

and $x_n^n > 0$. It completes the proof.

***Remark* 1.** Consider the function $\varphi(t) = t^2$. Then by Lemma 1, any symmetric matrix $D = \left(d_{ij}\right)$, which satisfies conditions $1-3$, is a matrix of squared distances for a linear independent system of points in a real Hilbert space. Consequently, the matrix $B = \left(b_{ij}\right)$, where $b_{ij} = d_{i0} + d_{j0} - d_{ij}$, is positive semi-definite [9], and, by (*), it is positive definite. Thus, we have another proof of Lemma 1 from [7].

***Remark* 2.** Obviously, if we omit in the above lemma condition (c), require $\varphi(t)$ be not identically zero and strengthen condition 2 as follows:

$$d_{i0} - d_{k0} < d_{mi} - d_{mk}, \tag{2'}$$

then the lemma assertion remains valid. Furthermore, if we replace all strict inequalities by nonstrict inequalities, then system (I) has a real solution, but possibly more than one.

***Proof of Theorem* 1.** Consider the function $\varphi(t) = |t|^p$, which satisfies conditions (a) and (b). Then the proof follows immediately from Remark 2.

Now we shall show that Theorem 1 implies Fichet's result mentioned above.

Indeed, it is shown in [7] that one can enumerate any finite ultrametric space R so that $R = (x_0, x_1, \ldots, x_n)$ and

$$\forall\ 0 \leq i < k < j \leq n \quad \rho(x_i, x_j) = \max\left\{\rho(x_i, x_k),\ \rho(x_k, x_j)\right\}.$$

We then enumerate this anew:

$$a_0 = x_0, \qquad a_i = x_{n-i}, \quad i = 1, \ldots, n.$$

Then we have

$$\forall\ 1 \leq j < k < i \leq n \quad \rho(a_j, a_i) = \max\left\{\rho(a_j, a_k),\ \rho(a_k, a_i)\right\}, \tag{3}$$